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Degree tolerant coloring: Graphs from graphs

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Abstract
Degree tolerant coloring is a study on a new coloring regime which sets conditions in respect of $deg(v)$ and $deg(u)$ where, $v,u \in V(G)$ and $vu \in E(G)$. Results for *graphs from graphs* or (derivative graphs) such the line graph, the middle graph, the total graph, the power graph and others, are presented in this paper.

Keywords
Degree tolerant coloring, degree tolerant chromatic number, proper coloring, defect coloring.

AMS Subject Classification
05C15, 05C38, 05C75, 05C85.

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Contents

1. Introduction and preliminaries

It is assumed that the reader is familiar with most of the classical concepts in graph theory and has good knowledge of most classical graph structures such as trees, path graphs, cycle graphs, complete graph, path in a graph, cycle in a graph, and their properties. Throughout only finite, undirected, simple connected graphs will be considered. For more on general notation and concepts in graphs see [1, 2, 6]. It is also assumed that the reader is familiar with the concept of graph coloring. Recall that in a proper coloring of G all edges are good i.e. $\forall\, uv \in E(G)$, $c(u) \neq c(v)$. The set of colors assigned in a graph coloring is denoted by $\mathscr{C}$ and a subset of colors assigned to a subset of vertices $X \subseteq V(G)$ is denoted by $c(X)$. The number of times a color $c_i \in \mathscr{C}$ is assigned is denoted by, $\theta(c_i)$. In an improper (or defect) coloring it is permitted that for some $uv \in E(G)$, $c(u) = c(v)$ (see[3]).

Since any graph G has the parameters, $\delta(G)$ and $\Delta(G)$, an integer degree condition related to an integer k, $\delta(G) \leq k \leq \Delta(G)$ has been introduced in [4]. Recall from [4] that a *degree tolerant coloring* abbreviated as, DT-coloring of a graph G has the following conditions:
(i) If $uv \notin E(G)$ then, either $c(u) = c(v)$ or $c(u) \neq c(v)$;
(ii) If $uv \in E(G)$ and $deg(u) = deg(v)$ then, $c(u) = c(v)$ else, $c(u) \neq c(v)$.
Alternative formulation for condition (ii) is: if $uv \in E(G)$ then, $c(u) = c(v)$ if and only if $deg(u) = deg(v)$. The minimum number of colors which yields a DT-coloring is called the *degree tolerant chromatic number* of G and is denoted by, $\chi_{dt}(G)$. For ease of reference we recall some results from [4].

Theorem 1.1. *[4] Any graph permits a DT-coloring.*

Theorem 1.2. *[4] For $n \in \mathbb{N}$ there exists a graph G with, $\chi_{dt}(G) = n$.*

Theorem 1.3. *[4] For $k \in \mathbb{N}$ there exists a minimal graph G of order $n = 2k-1$ (or $\nu(G) = 2k-1$) and size, $\varepsilon(G) = k(k-1)$ for which, $\chi_{dt}(G) = k$. Also, this minimal graph is unique.*

Theorem 1.4. *[4] For a graph G of order $n \geq 1$ it follows that,*

$$\chi_{dt}(G) \leq \lfloor \tfrac{n+1}{2} \rfloor.$$

Theorem 1.5. *[4] For a graph G of size $\varepsilon(G) = q \geq 1$ it follows that,*

$$\chi_{dt}(G) \leq \lfloor \tfrac{1+\sqrt{(1+4q)}}{2} \rfloor.$$

The next section will present results for certain *graphs from graphs*.

2. Graphs from Graphs

Recall the formal definition of a graph i.e. a graph G is, an ordered triple $(V(G), E(G), \iota_G)$ consisting of an non-empty set $V(G)$ of vertices, a set $E(G)$, disjoint from $V(G)$, of edges and an *incidence function*, ι_G that associates with each edge of G an unordered pair of vertices of G (vertices in an unordered pair are not necessarily distinct). Since it is agreed to consider simple connected graphs the formal definition is specialised to read, a simple connected graph G is, an ordered triple $(V(G), E(G), \iota_G)$ consisting of an non-empty set $V(G)$ of vertices, a set $E(G)$, disjoint from $V(G)$, of edges and an *incidence function*, ι_G that associates with some unordered pairs of vertices (vertices are distinct) an unique edge of G such that, between any two distinct vertices u and v there exists a uv-path. It is immediate from the specialised definition that a simple connected graph G has been derived from at least one other, more general graph G^s. It sets the notion of a *graph from a graph* at intuitive level. Another example is that, the graph G obtained by deleting all loops and multiple edges from a graph G^s, if such exist. The graph G is called a *spanning subgraph* of G^s.
If $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$ and $X = \{\{v_i, v_j\}_{i \neq j} : \forall\, v_i, v_j \in V(G)\}$, then label each $v_i v_j = e_k$ to obtain the set $X' = \{e_i : i = 1, 2, 3, \dots, \frac{n(n-1)}{2}\}$. Let $\{X_1, X_2\}$ be any 2-partition of X'. Hence, with $E(G) = X'$ having an incidence function implicit, the notation can be abbreviated to read, a graph $G = G(V, E)$ and it yields a complete graph K_n, $n \geq 1$. Similarly, $G = G(V, X'_1)$ yields a simple graph (not necessarily connected). The graph $\overline{G} = H(V, X'_2)$ is called the complement graph of G. Or differently put, $\overline{G} = H(V, X' \backslash X'_1)$. Again it sets the notion that $\overline{G}$ is a graph from a graph, G. There are numerous defined ways to obtain a graph from a graph.

2.1 Unary operations on graph

The vertices and the edges of a graph are called the *elements* of a graph. Unary operations on a graph G are those operations which define a local change to G or put differently, it defines some change to the elements of G. Examples are, merging of vertices, splitting of vertices (both connected or disconnected splitting), contraction of edges, edge deletion etc. More advanced unary operations are deemed more complex such as obtaining the complement graph, the line graph, power graphs etc. Following any unary operation the resultant graph will by *convention* be reduced to a maximal simple graph (not necessarily connected) by eliminating loops and multiple edges, if any. The next results are for some of the well known unary operations.

The contraction of an edge $e = uv \in E(G)$ is denoted by, G/e and results in a new vertex denoted by, u_v.

Proposition 2.1. For a simple connected graph G and $e = uv \in E(G)$ we have:

(a) If either u or v was distinctly colored amongst all other vertices in G (say, $\theta(c(u)) = 1$), and it is permissible to color u_v with a color in $\mathscr{C} \backslash \{c(u)\}$ then, $\chi_{dt}(G/e) = \chi_{dt}(G) - 1$. Else, $\chi_{dt}(G/e) = \chi_{dt}(G)$;
(b) If both u and v were distinctly colored amongst all other vertices in G (say, $\theta(c(u)) = \theta(c(v)) = 1$ and $c(u) = c(v)$), and it is permissible to color u_v with a color in $\mathscr{C} \backslash \{c(u)\}$ then, $\chi_{dt}(G/e) = \chi_{dt}(G) - 1$. Else, $\chi_{dt}(G/e) = \chi_{dt}(G)$;
(c) If both u and v were distinctly colored amongst all other vertices in G (say, $\theta(c(u) = \theta(c(v)) = 1$ and $c(u) \neq c(v)$), and it is permissible to color u_v with a color in $\mathscr{C} \backslash \{c(u), c(v)\}$ then, $\chi_{dt}(G/e) = \chi_{dt}(G) - 2$. Else, $\chi_{dt}(G/e) = \chi_{dt}(G) - 1$.
(d) If both u and v were not distinctly colored amongst all other vertices in G and it is permissible to color u_v with a color in $\mathscr{C}$ then, $\chi_{dt}(G/e) = \chi_{dt}(G)$. Else, $\chi_{dt}(G/e) = \chi_{dt}(G) + 1$.

Proof. Let $e = uv$ in G. In G/e, let the merged vertex be u_v. Now $max\{deg(u), deg(v)\} - 1 \leq deg(u_v) \leq deg(u) + deg(v) - 2$. Also, with regards to the open open neighborhood we have $N(u_v) = N(u) \cup N(v)$. (a) If either u or v was distinctly colored amongst all other vertices in G, we assume without loss of generality that say, $\theta(c(u)) = 1$. If there exist a vertex $w \in N(u_v)$ with $deg(w) = deg(u_v)$ then color, $c(u_v) = c(w)$ as prescribed by condition (ii). Clearly a minimum DT-coloring is yielded. If a vertex $w \in V(G) \backslash N[u_v]$ exists with $deg(w) = deg(u_v)$ then color, $c(u_v) = c(w)$ as prescribed by condition (i). Clearly a minimum DT-coloring is yielded. If a vertex $w \in V(G) \backslash N[u_v]$ exists with $c(w) \notin c(N(u_v))$ then let $c(u_v) = c(w)$ as permitted by condition (i). Clearly a minimum DT-coloring is yielded. If any of the three options are permissible then, $\chi_{dt}(G/e) = \chi_{dt}(G) - 1$. Else, $\chi_{dt}(G/e) = \chi_{dt}(G)$.
(b), (c) and (d) follow by similar reasoning as in (a). □

The merging of two or more vertices does not require adjacency between the vertices. However, if two vertices are adjacent and merged then a loop is created. By our reduction convention this loop is removed to result in a simple graph. The merging of vertices $u, v \in V(G)$ is denoted by $G \sim (u, v)$ and it results in a new vertex denoted by, $u_{\sim v}$. These observations lead to the next immediate corollary.

Corollary 2.1. For a simple connected graph G and $u, v \in V(G)$ we have:

(a) If either u or v was distinctly colored amongst all other vertices in G (say, $\theta(c(u)) = 1$), and it is permissible to color u_v with a color in $\mathscr{C} \backslash \{c(u)\}$ then, $\chi_{dt}(G \sim (u, v)) = \chi_{dt}(G) - 1$. Else, $\chi_{dt}(G \sim (u, v)) = \chi_{dt}(G)$;
(b) If both u and v were distinctly colored amongst all other vertices in G (say, $\theta(c(u)) = \theta(c(v)) = 1$ and $c(u) = c(v)$), and it is permissible to color u_v with a color in $\mathscr{C} \backslash \{c(u)\}$ then, $\chi_{dt}(G \sim (u, v)) = \chi_{dt}(G) - 1$. Else, $\chi_{dt}(G \sim (u, v)) = \chi_{dt}(G)$;
(c) If both u and v were distinctly colored amongst all other vertices in G (say, $\theta(c(u) = \theta(c(v)) = 1$ and $c(u) \neq c(v)$), and it is permissible to color u_v with a color in $\mathscr{C} \backslash \{c(u), c(v)\}$ then, $\chi_{dt}(G \sim (u, v)) = \chi_{dt}(G) - 2$. Else, $\chi_{dt}(G \sim (u, v)) = \chi_{dt}(G) - 1$.
(d) If both u and v were not distinctly colored amongst all

other vertices in G and it is permissible to color u_v with a color in $\mathscr{C}$ then, $\chi_{dt}(G\sim(u,v))=\chi_{dt}(G)$. Else, $\chi_{dt}(G\sim(u,v))=\chi_{dt}(G)+1$.

The k-power graph G^k of graph G is obtained by adding all edges such that two distinct vertices at a distance of at most k in G, are adjacent in G^k. It is well known that, $1\leq k\leq diam(G)$.

Theorem 2.1. *For a graph G we have, $\chi_{dt}(G^k)\leq\chi_{dt}(G)$.*

Proof. Case 1: Consider any two non-adjacent vertices $u,v\in V(G)$ for which $uv\in E(G^k)$. If $c(u)=c(v)$ in a DT-coloring of G and $deg(u)=deg(v)$ in G^k then, $c(u)=c(v)$ remains. If however, $deg(u)\neq deg(v)$ let either $c(u)=c(w)$, $uw\notin E(G^k)$ or $c(v)=c(w)$, $vw\notin E(G^k)$. Such vertex w exists else, $deg(u)=deg(v)$ which is a contradiction. For this case the result holds.
Case 2: Consider any two non-adjacent vertices $u,v\in V(G)$ for which $uv\in E(G^k)$. If $c(u)\neq c(v)$ in a DT-coloring of G and $deg(u)=deg(v)$ in G^k then, $c(u)$ can be colored $c(v)$ or the other way around, as is permissible. Similar reasoning as in Case 1 for the possibility that, $deg(u)\neq deg(v)$ settles the result for this case.
Through immediate induction over all pairs of distinct vertices of G it follows that, $\chi_{dt}(G^k)\leq\chi_{dt}(G)$. □

Corollary 2.2. For a graph G of order n we have, $\lim\limits_{k\to diam(G)}\chi_{dt}(G^k)=1$.

Proof. Since, $\lim\limits_{k\to diam(G)}\chi_{dt}(G^k)=G^{diam(G)}=K_n$ and $\chi_{dt}(K_n)=1$, the result holds. □

In a graph G of size $q\geq 1$ with edge set $E(G)=\{e_i:i=1,2,3,\dots,q\}$, the subdivision operation is executed by inserting a new vertex u_i on the edge e_i, $\forall\ i$. Put differently, delete the edge $e_i=vw$ and insert the vu_iw-path. The new edges are labeled e'_i,e''_i, respectively. The new graph is denoted by, $G^{\div}$.

Lemma 2.1. A non-regular connected graph G of order $n\geq 3$ has at least one pair of distinct vertices say u,v such that, $uv\in E(G)$ and $deg(u)\neq deg(v)$.

Proof. Any connected graph G has a spanning tree T. Consider such T for a non-trivial non-regular graph G. Choose any longest path P along the edges, $e_1,e_2,e_3,\dots,e_t$. Choose any endpoint of P and consider the internal vertex adjacent to this endpoint say vertex v_1. Compare $deg_G(v_1)$ vis-a-vis $deg_G(u)$, $\forall\ u\in N(v_1)$. If $deg_G(v_1)\neq deg_G(u)$ is found, the result holds. Else, the path-vertex adjacent to v_1 say, v_2 has $deg_G(v_2)=deg_G(v_1)$. Now repeat the comparison for $deg_G(v_2)$ and its neighborhood $N(v_3)$. Exhaust the comparison consecutively along the path P. Hereafter, branch from any v_i along a new path in T. Iteratively T can be covered by this degree comparison test. Hence, either all vertices $v\in V(G)$ have equal degree which implies G is regular. This is a contraction. Thus, at least one pair of distinct vertices say, u,v exists such that, $uv\in E(G)$ and $deg(u)\neq deg(v)$. □

Theorem 2.2. *For a graph G of size $q\geq 1$ we have:*
Cases 1 and 2:

$$\chi_{dt}(G^{\div})=\begin{cases}1, & G=C_n,\ n\geq 3;\\ 2, & G \text{ is } k\text{-regular, } k=1; \text{ or}\\ k\geq 3. & \end{cases}$$

Case 3. $\chi_{dt}(G^{\div})=$ either $\chi_{dt}(G)$ or $\chi_{dt}(G)+1$, if G is not regular.

Proof. Case 1: For C_n, $n\geq 3$ on vertices $v_1,v_2,v_3,\dots,n$, $deg(v_i)=2$, $\forall\ i$. Also in $C_n^{\div}$, $deg(v_i)=2$ and $deg(u_j)=2$, $j=1,2,3,\dots,n$. Hence, the result.
Case 2(a): For K_2, $deg(v_1)=deg(v_2)=1$. In $K_2^{\div}$, $deg(v_1)=deg(v_2)=1$ and $deg(u_1)=2$. By condition (ii) it follows that $\chi_{dt}(K_2^{\div})=2$.
(b) For any k-regular graph G, $k\geq 3$, $n\geq 4$ and size q, it follows that $deg(v_i)=k>2$, $i=1,2,3,\dots,n$. The aforesaid vertex degrees remain the same in $G_n^{\div}$ whilst, $deg(u_j)=2$, $j=1,2,3,\dots,q$. Thus by condition (ii), $\chi_{dt}(K_n^{\div})=2$.
Case 3: Let G of order $n\geq 3$ and size $q\geq 2$ be non-regular. There exist at least two vertices v_i and v_j such that, $deg(u)\neq deg(v)$. Because G is connected some v_iv_j-path exists in G. Also because G is non-trivial and non-regular, by Lemma 2.5 we have, $\chi_{dt}(G)\geq 2$.
Subcase 3(a): If for edge $e_l=v_iv_j$ in G the DT-coloring is $c(v_i)=c(v_j)$ say, c_1. Color $c(u_l)=c_2$ which is permissible.
Subcase 3(b): If for edge $e_l=v_iv_j$ in G the DT-coloring is $c(v_i)\neq c(v_j)$ say, $c(v_i)=c_1$ and $c(v_j)=c_2$. If $\chi_{dt}(G)\geq 3$ then let, $c(u_l)=c_3$. Similarly, all other subdivision vertices can be colored such that, $\chi_{dt}(G)=\chi_{dt}(G)$. If $\chi_{dt}(G)=2$ then let, $c(u_l)=c_1$. If it is possible to recolor a DT-coloring for $G^{\div}$ with colors c_1,c_2 then, $\chi_{dt}(G)=\chi_{dt}(G)$. If not, color all subdivision vertices the color c_3. The aforesaid is permissible by condition (ii) and clearly a DT-coloring is obtained. Hence, $\chi_{dt}(G)=\chi_{dt}(G)+1$. □

An important corollary follows from Lemma 2.1 with regards to the Nordhaus-Gaddum type bounds in Theorem 3.4 of [4].

Corollary 2.3. For a non-regular connected graph G of order $n\geq 3$ it holds that,

$$4\leq\chi_{dt}(G)+\chi_{dt}(\overline{G})\leq 2di(G),$$
$$4\leq\chi_{dt}(G)\cdot\chi_{dt}(\overline{G})\leq di(G)^2.$$

Proof. Lemma 2.5 implies that $\chi_{dt}(G)\geq 2$. Since $\overline{G}$ is also of order $n\geq 3$ and non-regular, the improved lower bounds are settled. □

2.2 New graphs from subgraphs of graph

In this subsection certain *derivative graphs* which are conventionally obtained through unary operations, will be obtained from an *intersection graph* perspective. Results for the respective degree tolerant chromatic number will be presented. It is agreed that, as much as the vertices u, v of the edge $e = uv$ are said to be adjacent, similarly, if the edges e_1, e_2 share a common end-vertex in G it is said, e_1 and e_2 are *adjacent*. Also, an edge $e = uv$ is said to be *adjacent* to its end-vertices (instead of "incident with").

Definition 2.1. Let $\mathfrak{C}$ be a non-empty set of non-empty subgraphs of G. Then, let each element (subgraph) in $\mathfrak{C}$ be represented by a unique vertex say v_i. Hence, $v_i \in \mathfrak{C}$, $i = 1,2,3,\dots,|\mathfrak{C}|$, has well-defined meaning. Define the derivative graph $G(\mathfrak{C})$ on the vertex set $\mathfrak{C}$ with edge set, $E(G(\mathfrak{C})) = \{v_iv_j$: if and only if $v_i \neq v_j$ and v_i, v_j satisfy some adjacency condition$\}$.

For a graph G let $\mathfrak{C} = E(G)$. Then the line graph of G denoted by, $L(G)$ is defined by, $L(G) = G(\mathfrak{C})$ subject to: if and only if v_i, v_j are adjacent in G. As a research tool the *expanded line graph* denoted by, $L^{\cdot\cdot}(G)$ has been defined in [5]. We recall the construction thereof.

Construction of expanded line graph[5].
(a) Label the edges of the graph G as $e_1, e_2, e_3, \dots, e_{\varepsilon(G)}$.
(b) Replace each vertex $v \in V(G)$ with a complete graph K_t, $t = d_G(v)$ such that each distinct vertex of the complete graph is inserted into a distinct edge adjacent to vertex v. Hence each edge $e_i \in E(G)$ will have two new vertices inserted. The complete graph $K_{d_G(v)}$ is called the v-clique of vertex v.
(c) For each edge e_i, label the new inserted vertices $u_{i,1}$ and $u_{i,2}$.
(d) Connect the pairs of vertices $u_{i,1}, u_{i,2}$ with a broken edge.

The graph obtained is called the expanded line graph of G and is denoted by $L^{\cdot\cdot}(G)$. Clearly by contracting all broken edges hence, by merging all vertices $u_{i,1}$ and $u_{i,2}$ for $1 \leq i \leq \varepsilon(G)$ we obtain the line graph $L(G)$. We recall figures 1 and 2 from [5] which depict an example. Note that any vertex in $L^{\cdot\cdot}(G)$ has some solid edges (possibly none) adjacent to it. If a broken edge is adjacent to a vertex it is always exactly one. The *reduced expanded* line graph denoted by, $L_r^{\cdot\cdot}(G)$ is obtained by eliminating (removing) all vertices which have only a broken edge adjacent to it. The solid vertex degree denoted by, $deg_s(u_{i,j})$ in $L_r^{\cdot\cdot}(G)$ is the number of solid edges adjacent to the vertex $u_{i,j}$. The weight of a broken edge denoted by, $w(u_{i,1}u_{i,2})$ is defined by, $w(u_{i,1}u_{i,2}) = deg_s(u_{i,1}) + deg_s(u_{i,2})$. The parameter $di(L_r^{\cdot\cdot}(G))$ is the number of distinct broken edge weights. For figure 2 we have, $di(L_r^{\cdot\cdot}(G)) = 2$ because the distinct broken edge weights are 4 and 5.

Proposition 2.2. For a graph G we have that, $\chi_{dt}(L(G)) = di(L_r^{\cdot\cdot}(G))$.

Proof. When a broken edge $u_{i,1}u_{i,2}$ is contracted the new vertex say, u_i has $deg(u_i) = w(u_{i,1}u_{i,2})$ in $L(G)$. Since every vertex u_i, $i = 1,2,3,\dots,\varepsilon(G)$ is a vertex in a clique, condition (ii) settles the result. □

Corollary 2.4. For a regular graph G we have, $\chi_{dt}(L(G)) = 1$.

Proof. If G is regular then $L(G)$ is regular. Thus, the result. □

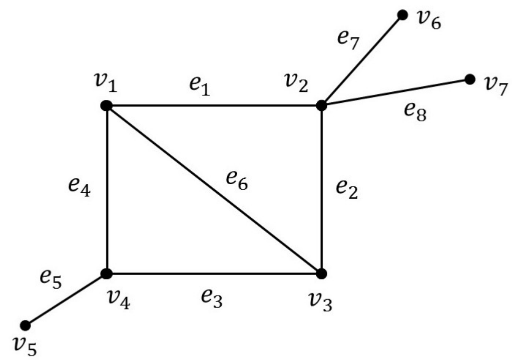

Fig. 1. Graph G.

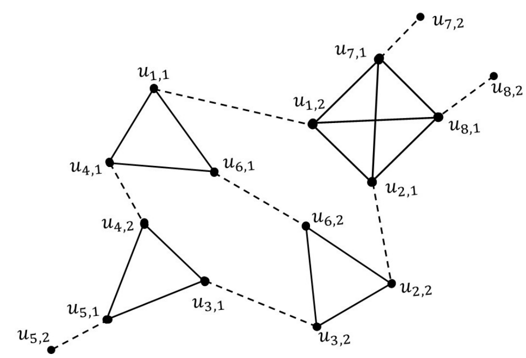

Fig. 2. Expanded line graph $L^{\cdot\cdot}(G)$.

Theorem 2.3. *For a tree $T \neq S_{1,n}$, $n \geq 2$ we have,* $\chi_{dt}(L(T)) \geq \chi_{dt}(T) = 2$.

Proof. It is easy to verify that for $n \geq 2$, $\chi_{dt}(L(S_{1,n})) = 1 \neq 2 = \chi_{dt}(S_{1,n})$. Furthermore, it is easy to verify that the line graph of any tree T of order n with say, t pendent vertices is a *cluster* of $(n-t)$ complete graphs with each in the expanded line graph, corresponding to order $deg(v)$ of the vertex it corresponds to. Upon contracting all broken edges each pair of distinct complete graphs share exactly one merged vertex. Hence, it is possible to yield a complete graph with more that two vertices, each of degree unequal to the others. Therefore, $\chi_{dt}(L(T)) \geq \chi_{dt}(T) = 2$. □

Corollary 2.5. Stars $S_{1,n}$, $n \geq 2$ are the only trees for which, $\chi_{dt}(L(T)) = 1 < \chi_{dt}(T) = 2$.

For a graph G let $\mathfrak{C} = V(G) \cup E(G)$. Then the middle graph of G denoted by, $M(G)$ is defined by, $M(G) = G(\mathfrak{C})$ subject to: if and only if v_i, v_j are adjacent edges or, v_i is a vertex, adjacent to v_j an edge in G.

Proposition 2.3. For a graph G we have that, $\chi_{dt}(M(G)) \leq \chi_{dt}(L(G)) + 1$.

Proof. The line graph $L(G)$ is the largest (maximum) induced subgraph of $M(G)$ such that $v \in V(G)$ in $M(G)$ is not in $V(L(G))$. Assign a DT-coloring in $M(G)$ to only the vertices corresponding to $L(G)$. If $v \in V(G) \subset V(M(G))$ is adjacent to u_i then $deg(u_i) \geq deg(v)+1$. Hence, a vertex $v \in V(G)$ can be assigned a color assigned to some vertex in $L(G)$ and possibly, this is possible for all $v \in V(G)$. If so then, $\chi_{dt}(M(G)) = \chi_{dt}(L(G))$. Else, all $v \in V(G)$ can be assigned the same additional color because $vw \notin E(M(G))$ for any pair $v, w \in M(G)$. Then, $\chi_{dt}(M(G)) = \chi_{dt}(L(G))+1$. Hence, the result is settled. □

For a graph G let $\mathfrak{C} = V(G) \cup E(G)$. Then the total graph of G denoted by, $T(G)$ is defined by, $T(G) = G(\mathfrak{C})$ subject to: if and only if v_i, v_j are adjacent in G.

Proposition 2.4. For a graph G we have that, $\chi_{dt}(T(G)) \leq \chi_{dt}(L(G)) + \chi_{dt}(G)$.

Proof. Consider only $v \in V(G)$ in $T(G)$. In $T(G)$ we have, $deg_{T(G)}(v) = 2deg_G(v)$, $\forall\ v \in V(G)$. Hence, to begin with a DT-coloring can be assigned to G. Either, the color set of $c(V(G))$ suffices to color the vertices of $L(G)$ in $T(G)$ or, fewer or equal than $\varepsilon(G)$ new colors are required. Therefore, $\chi_{dt}(T(G)) \leq \chi_{dt}(L(G)) + \chi_{dt}(G)$. □

3. Conclusion

This study reported on results in respect of certain unary operations and certain new graphs obtained from the subgraphs of a given graph. Since many other unary operations exist and numerous other new graphs from subgraphs of a given graph can be defined, scope for further research exists.

Of particular interest is to find an improvement on the upper bound in Proposition 2.4, if possible.

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