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On multidimensional fractional Langevin equations in terms of Caputo derivatives

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Abstract

In this paper, we consider a more general and multidimensional fractional Langevin equations with nonlinear terms that involve some unknown functions and their Caputo derivatives. Using some fixed point theorems, we obtain new results on the existence and uniqueness of solutions in addition to the existence of at least one solution. We also define and prove the generalized Ulam-Heyers stability of solutions for the considered equations. Some examples are provided to illustrate the applications of our results.

Keywords

Caputo derivative, fixed point, fractional Langevin equation, existence and uniqueness, Ulam-Hyers stability.

AMS Subject Classification

30C45, 39B72, 39B82.

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Contents

1. Introduction and Preliminaries

In the last few decades, there has been an explosion of research activities on the application of fractional differential equations to very diverse scientific fields ranging from the physics of diffusion and advection phenomena, to control systems, finance and economics. For more details, see [9, 12, 17, 18]. Furthermore, Ulam-Hyers stability is one of the important issues in the theory of differential equations and their applications. Considerable work have been done in this field of research, see, e.g., Abbas et al. [1], Chalishajar [4], Dai et al. [6], Harikrishnan et al. [10], Ibrahim et al. [11], Taïeb [24–28], Taïeb et al. [29–31] and Wang [33].

Let us now introduce some other important research papers related to the Langevin equation which has inspired our work: we know that the Langevin equation was introduced by Paul Langevin in 1908, in order to describe Brownian motion [14]. The Langevin equation was used to describe the evolution of physical phenomena in fluctuating environment [5]. The generalized Langevin equation which was concerned with describing the fractal and memory properties, was proposed by Kubo [13], in 1966.

Ever after, Langevin equations have exhausted the attention of many authors [2, 3, 7, 8, 16, 20, 21, 23, 32].

In 2008, A new type of fractional Langevin equation of two different orders is introduced by S.C. Lim et al. [15]:

$$ {}_0D_t^{\beta}\left({}_0D_t^{\alpha}+\lambda\right)u(t)=f(t,u(t)). $$

The solutions for this equation, known as the fractional Ornstein–Uhlenbeck processes, based on Weyl and Riemann–Liouville fractional derivatives are obtained.

In 2018, a class of Langevin equations is studied by R.W. Ibrahim et al. [11]:

$$ \begin{cases} {}^{\rho}D^{\alpha_1,\beta}\left({}^{\rho}D^{\alpha_2,\beta}+\lambda\right)x(t)=f(t,x(t)), \\ t\in J:=(a,b], \\ I^{1-\gamma}x(a)=x_a, \quad \gamma=(\alpha_1+\alpha_2)(1-\beta)+\beta, \end{cases} $$

where the existence, uniqueness and stability results are obtained, ${}^{\rho}D^{\alpha_1,\beta}$, ${}^{\rho}D^{\alpha_2,\beta}$ are Hilfer-Katugampola fractional differential operators of orders α_1 and α_2, $\beta,\rho>0$ and λ is any real number, $f: J\times\mathbb{R}\to\mathbb{R}$ is given continuous function.

In 2020, a coupled system of nonlinear fractional Langevin equations of α and β fractional orders, is proposed by A. Salem et al. [22]:

$$\begin{cases} {}^cD^{\beta_1}\left({}^cD^{\alpha_1}+\lambda\right)x_1(t)=f_1(t,x_1(t),x_2(t)),\\ {}^cD^{\beta_2}\left({}^cD^{\alpha_2}+\lambda\right)x_2(t)=f_2(t,x_1(t),x_2(t)),\\ t\in[0,1],\end{cases}$$

supplemented by the following:

$$\begin{cases} x_1(0)=0,\quad {}^cD^{\alpha_1}x_1(0)=\Gamma(\alpha_1+1)^{\rho_1}I^{\gamma_1}x_1(\eta_1),\\ \sum\limits_{j=1}^{m_1}a_{j_1}x_1(\zeta_{j_1})=\mu_1\,{}^{AB}I^{\gamma_2}x_1(\eta_2),\\ x_1(0)=0,\quad {}^cD^{\alpha_2}x_2(0)=\Gamma(\alpha_2+1)^{\rho_2}I^{\gamma_3}x_2(\eta_3),\\ \sum\limits_{j=1}^{m_2}a_{j_2}x_2(\zeta_{j_2})=\mu_2\,{}^{AB}I^{\gamma_4}x_2(\eta_4),\end{cases}$$

where ${}^cD$ is the Caputo fractional derivative of order $0<\alpha_i\le 1$ and $1<\beta i\le 2$, for $i=1,2$. ${}^{AB}I$ and ${}^{\rho}I^{\gamma}$ are Atangana-Baleanu, and Katugampola fractional integrals, respectively. $\rho_i>0$ and $\lambda_i,\mu_i\in\mathbb{R}$, for $i=1,2$, $\gamma_k>0$, for $k=1,2,3,4$, $a_{j_i}\in\mathbb{R}$, for $j=1,2,...,m_i$, and $i=1,2$. $0<\eta_k<\zeta_{1_i}<\zeta_{2_i}<\cdots<\zeta_{m_i}<1$, for $i=1,2$, and $k=1,...,4$,
$f_1,f_2:[0,1]\times\mathbb{R}^2\to\mathbb{R}$, are continuous functions.

In this paper, we consider a more general and multidimensional fractional Langevin equations that involve some unknown functions and their Caputo derivatives. Then, we discuss the existence, uniqueness and some types of Ulam stabilities for the proposed coupled nonlinear fractional system. So, let us consider:

$$\begin{cases} {}_0D_t^{\beta_k}\left({}_0D_t^{\alpha_k}+\lambda_k\right)x_k(t)=f_k(\Delta_x(t)),\\ k=1,...,n,\quad t\in J:=[0,1],\\ x_k(0)={}_0D_t^{\alpha_k}x_k(0)=x_k(1)+{}_0I_1^{\alpha_k}x_k(t)=0,\end{cases}\tag{1.1}$$

where

$$\Delta_x(t):=\left(t,x_1(t),...,x_n(t),{}_0D_t^{\delta_1}x_1(t),...,{}_0D_t^{\delta_n}x_n(t)\right),$$

$0<\alpha_k<1$, $1<\beta_k<2$, $0<\delta_k<\alpha_k$, $\lambda_k\in\mathbb{R}$, $f_k: J\times\mathbb{R}^{2n}\to\mathbb{R}$, $k=1,...,n$, $n\in\mathbb{N}-\{0\}$, are continous functions. The operators ${}_0D_t^{\alpha_k}$, ${}_0D_t^{\beta_k}$, ${}_0D_t^{\delta_k}$ are the derivatives in the sense of Caputo, defined by:

$$\begin{aligned}{}_0D_t^{\kappa}x(t) &= \frac{1}{\Gamma(m-\kappa)}\int_0^t(t-s)^{m-\kappa-1}x^{(m)}(s)\,ds\\ &= {}_0I_t^{m-\kappa}x^{(m)}(t),\end{aligned}$$

with $m-1<\kappa<m$, $m\in\mathbb{N}-\{0\}$. The Riemann-Liouville fractional integral ${}_0I_t^{\vartheta}$ of order $\vartheta\ge 0$ for a continuous function ψ on $[0,\infty)$ is defined by:

$${}_0I_t^{\vartheta}\psi(t)=\begin{cases}\frac{1}{\Gamma(\vartheta)}\int_0^t(t-s)^{\vartheta-1}\psi(s)\,ds,\quad \vartheta>0,\\ \psi(t),\quad \vartheta=0,\end{cases}$$

where $t\ge 0$ and $\Gamma(\vartheta):=\int_0^\infty e^{-x}x^{\vartheta-1}dx.$

We give some properties of the fractional calculus theory which can be found in [19].

(i): For $\alpha,\beta>0$; $n-1<\alpha<n$, we have:

$${}_0D_t^{\alpha}t^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}t^{\beta-\alpha-1},\quad \beta>n,$$

and ${}_0D_t^{\alpha}t^j=0$, $j=0,1,...,n-1$,

(ii):

$${}_0D_{t0}^{p}I_t^{q}f(t)={}_0I_t^{q-p}f(t),$$

where $q>p>0$ and $f\in L^1([a,b])$,

(iii): Let $n\in\mathbb{N}-\{0\}$, $n-1<\alpha<n$, and ${}_0D_t^{\alpha}u(t)=0$. Then,

$$u(t)=\sum_{j=0}^{n-1}c_jt^j,$$

and

$${}_0I_{t0}^{\alpha}D_t^{\alpha}u(t)=u(t)+\sum_{j=0}^{n-1}c_jt^j,\quad (c_j)_{j=0,1,...,n-1}\in\mathbb{R}.$$

We also need to the following fundamental Lemma to prove our existence results.

Lemma 1.1 (Shaefer Fixed Point Theorem)**.** *Let E be a Banach space. Assume that $T:E\to E$ is a completely continuous operator and the set $\Omega=\{x\in E:\ x=\lambda Tx\ ,0<\lambda<1\}$, is bounded. Then, T has a fixed point in E.*

From the following auxiliary result, we will import the integral representation of system (1.1).

Lemma 1.2. *Let given $0<\alpha_k<1$, $1<\beta_k<2$, $k=1,...,n$, $n\in\mathbb{N}-\{0\}$, $\lambda_k\in\mathbb{R}$, and a family $(G_k)_{k=1,...,n}\in C([0,1],\mathbb{R})$. Then, the following problem:*

$${}_0D_t^{\beta_k}\left({}_0D_t^{\alpha_k}+\lambda_k\right)x_k(t)=G_k(t),\quad k=1,...,n,\tag{1.2}$$

associated with the conditions:

$$x_k(0)={}_0D_t^{\alpha_k}x_k(0)=x_k(1)+\lambda_k\,{}_0I_1^{\alpha_k}x_k(t)=0,\tag{1.3}$$

has a unique solution $(x_1,...,x_n)$, where

$$\begin{aligned}x_k(t) &= \frac{1}{\Gamma(\alpha_k+\beta_k)}\int_0^t(t-s)^{\alpha_k+\beta_k-1}G_k(s)\,ds\\ &-\frac{\lambda_k}{\Gamma(\alpha_k)}\int_0^t(t-s)^{\alpha_k-1}x_k(s)\,ds\\ &-\frac{t^{\alpha_k+1}}{\Gamma(\alpha_k+\beta_k)}\int_0^1(1-s)^{\alpha_k+\beta_k-1}G_k(s)\,ds.\end{aligned}\tag{1.4}$$

Proof. The property (iii) allow us to write problem (1.2) to an equivalent integral equations:

$$\begin{aligned} x_k(t) &= \frac{1}{\Gamma(\alpha_k+\beta_k)}\int_0^t (t-s)^{\alpha_k+\beta_k-1} G_k(s)\,ds \\ &\quad -\frac{\lambda_k}{\Gamma(\alpha_k)}\int_0^t (t-s)^{\alpha_k-1} x_k(s)\,ds \\ &\quad -\frac{c_0^k t^{\alpha_k}}{\Gamma(\alpha_k+1)}-\frac{c_1^k t^{\alpha_k+1}}{\Gamma(\alpha_k+2)}-c_0'^k, \end{aligned} \tag{1.5}$$

where $c_0^k, c_1^k, c_0'^k \in \mathbb{R}$, $k=1,...,n$. Using the boundary conditions (1.3), we obtain:

$$\begin{aligned} c_0'^k &= c_0^k = 0, \\ c_1^k &= \frac{\Gamma(\alpha_k+2)}{\Gamma(\alpha_k+\beta_k)}\int_0^1 (1-s)^{\alpha_k+\beta_k-1} G_k(s)\,ds. \end{aligned} \tag{1.6}$$

Substituting Eq. (1.6) in Eq. (1.5), we receive Eq. (1.4). This completes the proof. □

Now, we introduce the Banach space:

$$B := \left\{(x_1,...,x_n) \in \mathbb{R} : x_k \in C(J,\mathbb{R}),\ {}_0D_t^{\delta_k} x_k \in C(J,\mathbb{R})\right\},$$

$k=1,...,n$, endowed with the norm:

$$\|(x_1,...,x_n)\|_B = \max_{1\le k\le n}\left(\|x_k\|_\infty, \left\|{}_0D_t^{\delta_k} x_k\right\|_\infty\right),$$

such that,

$$\|x_k\|_\infty = \max_{t\in J}|x_k(t)| \text{ and } \left\|{}_0D_t^{\delta_k} x_k\right\|_\infty = \max_{t\in J}\left|{}_0D_t^{\delta_k} x_k(t)\right|.$$

2. Main results

We begin this section by introducing the following hypotheses:

$(\mathscr{H}_1)$: There exist nonegative constants $(\eta_k)_j$, $k=1,...,n$, $j=1,...,2n$, such that,

$$|f_k(t,u_1,...,u_{2n}) - f_k(t,v_1,...,v_{2n})| \le \sum_{j=1}^{2n}(\eta_k)_j\left|u_j - v_j\right|,$$

for all $t\in J$ and all $(u_1,...,u_{2n}),(v_1,...,v_{2n}) \in \mathbb{R}^{2n}$,

$(\mathscr{H}_2)$: There exist nonnegative constants L_k, such that for all $t \in J$ and all $(u_1,...,u_{2n}) \in \mathbb{R}^{2n}$,

$$|f_k(t,u_1,...,u_{2n})| \le L_k,$$

$(\mathscr{H}_3)$: The functions $(f_k)_{k=1,...,n} : J\times\mathbb{R}^{2n}$ are continuous,

$(\mathscr{C})$: The constant

$$\Delta := \max_{1\le k\le n}\left(\Sigma_k\Theta_k + \Lambda_k, \Sigma_k\Theta_k^* + \Lambda_k^*\right),$$

satisfies $0<\Delta<1$, where

$$\begin{aligned} \Sigma_k &= \sum_{j=1}^{2n}(\eta_k)_j, \\ \Lambda_k &= \frac{|\lambda_k|}{\Gamma(\alpha_k+1)}, \\ \Lambda_k^* &= \frac{|\lambda_k|}{\Gamma(\alpha_k-\delta_k+1)}, \\ \Theta_k &= \frac{2}{\Gamma(\alpha_k+\beta_k+1)}, \\ \Theta_k^* &= \frac{1}{\Gamma(\alpha_k+\beta_k-\delta_k+1)} \\ &\quad + \frac{\Gamma(\alpha_k+2)}{\Gamma(\alpha_k-\delta_k+2)\Gamma(\alpha_k+\beta_k+1)}. \end{aligned}$$

2.1 Existence and Uniqueness of Solutions

Our first main result is based on the Banach contraction principle.

Theorem 2.1. *Assume that $(\mathscr{H}_1)$ and $(\mathscr{C})$ hold. Then, system (1.1) has a unique solution on J.*

Proof. Define the nonlinear operator $\mathscr{A} : B \to B$ by:

$$\mathscr{A}(x_1,...,x_n)(t) = (\mathscr{A}_1(x_1,...,x_n)(t), ..., \mathscr{A}_n(x_1,...,x_n)(t)),$$

with

$$\begin{aligned} &\mathscr{A}_k(x_1,...,x_n)(t) \\ &= \frac{1}{\Gamma(\alpha_k+\beta_k)}\int_0^t (t-s)^{\alpha_k+\beta_k-1} f_k(\Delta_x(t))\,ds \\ &\quad -\frac{\lambda_k}{\Gamma(\alpha_k)}\int_0^t (t-s)^{\alpha_k-1} x_k(s)\,ds \\ &\quad -\frac{t^{\alpha_k+1}}{\Gamma(\alpha_k+\beta_k)}\int_0^1 (1-s)^{\alpha_k+\beta_k-1} f_k(\Delta_x(t))\,ds, \end{aligned}$$

$$\begin{aligned} &{}_0D_t^{\delta_k}\mathscr{A}_k(x_1,...,x_n)(t) \\ &= \frac{1}{\Gamma(\alpha_k+\beta_k-\delta_k)}\int_0^t (t-s)^{\alpha_k+\beta_k-\delta_k-1} f_k(\Delta_x(t))\,ds \\ &\quad -\frac{\lambda_k}{\Gamma(\alpha_k-\delta_k)}\int_0^t (t-s)^{\alpha_k-\delta_k-1} x_k(s)\,ds \\ &\quad -\frac{\Gamma(\alpha_k+2)\,t^{\alpha_k-\delta_k+1}}{\Gamma(\alpha_k-\delta_k+2)\Gamma(\alpha_k+\beta_k)} \\ &\quad \times\int_0^1 (1-s)^{\alpha_k+\beta_k-1} f_k(\Delta_x(t))\,ds, \end{aligned}$$

for all $\quad k=1,...,n, \quad$ and all $\quad t\in J$.

We will show that the operator $\mathscr{A}$ is contractive on B.

Let $\quad (x_1,...,x_n),(y_1,...,y_n)\in B$. Then, for all $t\in J$, we have:

$$
\begin{aligned}
&\left|\mathscr{A}_k(x_1,...x_n)(t)-\mathscr{A}_k(y_1,...y_n)(t)\right|\\
\leq\;& \max_{s\in J}\left|f_k\left(\Delta_x(t)\right)-f_k\left(\Delta_y(t)\right)\right|\frac{t^{\alpha_k+\beta_k}}{\Gamma(\alpha_k+\beta_k+1)}\\
&+\frac{|\lambda_k|\,t^{\alpha_k}}{\Gamma(\alpha_k+1)}\max_{s\in J}|x_k(s)-y_k(s)|\\
&+\frac{t^{\alpha_k+1}}{\Gamma(\alpha_k+\beta_k+1)}\max_{s\in J}\left|f_k\left(\Delta_x(t)\right)-f_k\left(\Delta_y(t)\right)\right|.
\end{aligned}
$$

Using $(\mathscr{H}_1)$, we can write:

$$
\begin{aligned}
&\left\|\mathscr{A}_k(x_1,...x_n)-\mathscr{A}_k(y_1,...y_n)\right\|_\infty\\
\leq\;& \frac{2}{\Gamma(\alpha_k+\beta_k+1)}\sum_{j=1}^{2n}(\eta_k)_j\\
&\times\max\begin{pmatrix}\|x_1-y_1\|_\infty,...,\|x_n-y_n\|_\infty,\\ \left\|{}_0D_t^{\delta_1}(x_1-y_1)\right\|_\infty,...,\\ \left\|{}_0D_t^{\delta_n}(x_n-y_n)\right\|_\infty\end{pmatrix}\\
&+\frac{|\lambda_k|}{\Gamma(\alpha_k+1)}\|x_k-y_k\|_\infty.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left\|\mathscr{A}_k(x_1,...x_n)-\mathscr{A}_k(y_1,...y_n)\right\|_\infty\\
\leq\;& (\Theta_k\Sigma_k+\Lambda_k)\left\|(x_1-y_1,...,x_n-y_n)\right\|_B.
\end{aligned}
\tag{2.1}
$$

Using the same arguments, we can write:

$$
\begin{aligned}
&\left\|{}_0D_t^{\delta_k}\left(\mathscr{A}_k(x_1,...,x_n)-\mathscr{A}_k(y_1,...y_n)\right)\right\|_\infty\\
\leq\;& (\Theta_k^*\Sigma_k+\Lambda_k^*)\left\|(x_1-y_1,...,x_n-y_n)\right\|_B.
\end{aligned}
\tag{2.2}
$$

Combining inequalities (2.1) and (2.2), we get

$$
\left\|\mathscr{A}(x_1,...x_n)-\mathscr{A}(y_1,...y_n)\right\|_B\leq\Delta\left\|(x_1-y_1,...,x_n-y_n)\right\|_B. \tag{2.3}
$$

Thanks to $(\mathscr{C})$, we deduce that $\mathscr{A}$ is a contractive operator. Consequently, by the Banach fixed point Theorem $\mathscr{A}$ has a fixed point which is a solution of fractional Langevin system (1.1). This completes the proof. $\square$

Example 2.2. *Consider the following Langevin system:*

$$
\begin{cases}
{}_0D_t^{\frac{7}{4}}\left({}_0D_t^{\frac{3}{4}}-\frac{1}{3\pi}\right)x_1(t)\\
=\dfrac{\left|x_1(t)+x_2(t)+x_3(t)+{}_0D_t^{\frac{1}{2}}x_1(t)+{}_0D_t^{\frac{1}{3}}x_2(t)+{}_0D_t^{\frac{1}{4}}x_3(t)\right|}{180\pi(t^2+1)\left(1+\left|x_1(t)+x_2(t)+x_3(t)+{}_0D_t^{\frac{1}{2}}x_1(t)+{}_0D_t^{\frac{1}{3}}x_2(t)+{}_0D_t^{\frac{1}{4}}x_3(t)\right|\right)},\\
{}_0D_t^{\frac{5}{3}}\left({}_0D_t^{\frac{2}{3}}+\frac{\pi}{4e}\right)x_2(t)\\
=\dfrac{\sum_{i=1}^{3}\sin x_i(t)+\cos{}_0D_t^{\frac{1}{2}}x_1(t)+\cos{}_0D_t^{\frac{1}{3}}x_2(t)+\cos{}_0D_t^{\frac{1}{4}}x_3(t)}{32\pi(t+1)},\\
{}_0D_t^{\frac{3}{2}}\left({}_0D_t^{\frac{1}{2}}+\frac{2}{3\pi}\right)x_3(t)\\
=\frac{1}{32e^{t^2+1}}\left(\sum_{i=1}^{3}\cos x_i(t)\dfrac{\left|{}_0D_t^{\frac{1}{2}}x_1(t)+{}_0D_t^{\frac{1}{3}}x_2(t)+{}_0D_t^{\frac{1}{4}}x_3(t)\right|}{\left(1+\left|{}_0D_t^{\frac{1}{2}}x_1(t)+{}_0D_t^{\frac{1}{3}}x_2(t)+{}_0D_t^{\frac{1}{4}}x_3(t)\right|\right)}\right),\\
t\in J:=[0,1],\\
x_1(0)={}_0D_t^{\frac{3}{4}}x_1(0)=x_1(1)+{}_0I_1^{\frac{3}{4}}x_1(t)=0,\\
x_2(0)={}_0D_t^{\frac{2}{3}}x_2(0)=x_2(1)+{}_0I_1^{\frac{2}{3}}x_2(t)=0,\\
x_3(0)={}_0D_t^{\frac{1}{2}}x_3(0)=x_3(1)+{}_0I_1^{\frac{1}{2}}x_3(t)=0.
\end{cases}
\tag{2.4}
$$

For this example, we have: $\quad n=3$,

$$
\begin{aligned}
\beta_1&=\frac{7}{4},\quad \beta_2=\frac{5}{3},\quad \beta_3=\frac{3}{2},\\
\alpha_1&=\frac{3}{4},\quad \alpha_2=\frac{2}{3},\quad \alpha_3=\frac{1}{2},\\
\delta_1&=\frac{1}{2},\quad \delta_2=\frac{1}{3},\quad \delta_3=\frac{1}{4},\\
\lambda_1&=-\frac{1}{3\pi},\quad \lambda_2=\frac{\pi}{4e},\quad \lambda_3=\frac{2}{3\pi},
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
f_1(t,u_1(t),...,u_6(t))&=\frac{\left|\sum_{i=1}^{6}u_i(t)\right|}{180\pi(t^2+1)\left(1+\left|\sum_{i=1}^{6}u_i(t)\right|\right)},\\
f_2(t,u_1(t),...,u_6(t))&=\frac{\sum_{i=1}^{3}u_i(t)+\sum_{i=4}^{6}\cos u_i(t)}{32\pi(t+1)},\\
f_3(t,u_1(t),...,u_6(t))&=\left(\frac{\sum_{i=1}^{3}\cos u_i(t)\left|\sum_{i=4}^{6}u_i(t)\right|}{32e^{t^2+1}\left(1+\left|\sum_{i=4}^{6}u_i(t)\right|\right)}\right).
\end{aligned}
$$

Then for all $t \in J$ and $(u_1,...,u_6),(v_1,...,v_6) \in \mathbb{R}^6$, we have :

$$
\begin{aligned}
|f_1(t,u_1,...,u_6) - f_1(t,v_1,...,v_6)| &\leq \frac{1}{180\pi}\sum_{i=1}^{6}|u_i - v_i|,\\
|f_2(t,u_1,...,u_6) - f_2(t,v_1,...,v_6)| &\leq \frac{1}{32\pi}\sum_{i=1}^{6}|u_i - v_i|,\\
|f_3(t,u_1,...,u_6) - f_3(t,v_1,...,v_6)| &\leq \frac{1}{32e}\sum_{i=1}^{6}|u_i - v_i|.
\end{aligned}
$$

We can take

$$(\eta_1)_j = \frac{1}{180\pi},\ (\eta_2)_j = \frac{1}{32\pi},\ (\eta_3)_j = \frac{1}{32e},\ j = 1,...,6,$$

Indeed,

$$\Sigma_1 = \frac{1}{30\pi}, \quad \Sigma_2 = \frac{3}{16\pi}, \quad \Sigma_3 = \frac{3}{16e},$$

$$
\begin{aligned}
\Lambda_1 &= 0.1154, \quad \Lambda_2 = 0.32, \quad \Lambda_3 = 0.2394,\\
\Lambda_1^* &= 0.1171, \quad \Lambda_2^* = 0.3235, \quad \Lambda_3^* = 0.2341,\\
\Theta_1 &= 0.6018, \quad \Theta_2 = 0.7199, \quad \Theta_3 = 1,\\
\Theta_1^* &= 0.9272, \quad \Theta_2^* = 0.9549, \quad \Theta_3^* = 1.2083.
\end{aligned}
$$

Furthermore, we have:

$$
\begin{aligned}
\Sigma_1\Theta_1 + \Lambda_1 &= 0.1218,\\
\Sigma_2\Theta_2 + \Lambda_2 &= 0.363,\\
\Sigma_3\Theta_3 + \Lambda_3 &= 0.3084,\\
\Sigma_1\Theta_1^* + \Lambda_1^* &= 0.1269,\\
\Sigma_2\Theta_2^* + \Lambda_2^* &= 0.3805,\\
\Sigma_3\Theta_3^* + \Lambda_3^* &= 0.3175.
\end{aligned}
$$

Using Theorem 2.1, we deduce that system (2.4) has a unique solution on J.

2.2 Existence of at Least One Solution

Theorem 2.3. *Assume that $(\mathscr{H}_2)$ and $(\mathscr{H}_3)$ hold. Then, system (1.1) has at least one solution on J.*

Proof. The proof will be given in two steps:

(1) : We show that $\mathscr{A}$ is completely continuous:

We begin by proving that $\mathscr{A}$ maps bounded sets into bounded sets in B : Let us consider the set:

$$S_\rho := \{(x_1,...,x_n) \in B;\ \|(x_1,...,x_n)\|_B \leq \rho,\ \rho > 0\},$$

and $(x_1,...,x_n) \in S_\rho$. Then, for each $t \in J$, and using $(\mathscr{H}_2)$, we can obtain:

$$
\begin{aligned}
&|\mathscr{A}_k(x_1,...,x_n)(t)|\\
\leq\ &\left(\frac{t^{\alpha_k+\beta_k}}{\Gamma(\alpha_k+\beta_k+1)} + \frac{t^{\alpha_k+1}}{\Gamma(\alpha_k+\beta_k+1)}\right)\\
&\times \max_{s\in J}|f_k(\Delta_x(s))| + \frac{|\lambda_k|t^{\alpha_k}}{\Gamma(\alpha_k+1)}\max_{s\in J}|x_k(s)|.
\end{aligned}
$$

Hence,

$$\|\mathscr{A}_k(x_1,...x_n)\|_\infty \leq \Theta_k L_k + \rho\Lambda_k. \tag{2.5}$$

Similarly, it can be shown that

$$\left\|{}_0D_t^{\delta_k}\mathscr{A}_k(x_1,...x_n)\right\|_\infty \leq \Theta_k^* L_k + \rho\Lambda_k^*. \tag{2.6}$$

It follows from inequalities (2.5) and (2.6) that

$$\|\mathscr{A}(x_1,...x_n)\|_B \leq \max_{1\leq k\leq n}(\Theta_k L_k + \rho\Lambda_k, \Theta_k^* L_k + \rho\Lambda_k^*) < \infty. \tag{2.7}$$

This means that $\mathscr{A}$ maps bounded sets into bounded sets in B.

Thanks to $(\mathscr{H}_3)$, the operator $\mathscr{A}$ is continuous on B. On the other hand, for any $0 \leq t_1 < t_2 \leq 1$ and $(x_1,...,x_n) \in S_\rho$, we have:

$$
\begin{aligned}
&\|\mathscr{A}_k(x_1,...x_n)(t_2) - \mathscr{A}_k(x_1,...x_n)(t_1)\|_\infty\\
\leq\ &\frac{L_k}{\Gamma(\alpha_k+\beta_k+1)}\Bigg(2(t_2-t_1)^{\alpha_k+\beta_k}\\
&+\left(t_2^{\alpha_k+\beta_k} - t_1^{\alpha_k+\beta_k}\right) + \left(t_2^{\alpha_k+1} - t_1^{\alpha_k+1}\right)\Bigg)\\
&+\frac{|\lambda_k|\rho}{\Gamma(\alpha_k+1)}\left(2(t_2-t_1)^{\alpha_k} + \left(t_2^{\alpha_k} - t_1^{\alpha_k}\right)\right),
\end{aligned}
\tag{2.8}
$$

and

$$\begin{aligned}
&\left\| {}_0D_t^{\delta_k}\left(\mathscr{A}_k(x_1,...x_n)(t_2)-\mathscr{A}_k(x_1,...x_n)(t_1)\right)\right\|_\infty \\
\leq\ & \frac{L_k}{\Gamma(\alpha_k+\beta_k-\delta_k+1)} \\
&\times\left(2(t_2-t_1)^{\alpha_k+\beta_k-\delta_k}+\left(t_2^{\alpha_k+\beta_k-\delta_k}-t_1^{\alpha_k+\beta_k-\delta_k}\right)\right) \\
&+\frac{|\lambda_k|\rho}{\Gamma(\alpha_k-\delta_k+1)} \\
&\times\left(2(t_2-t_1)^{\alpha_k-\delta_k}+\left(t_2^{\alpha_k-\delta_k}-t_1^{\alpha_k-\delta_k}\right)\right) \\
&+\frac{\Gamma(\alpha_k+2)L_k\left(t_2^{\alpha_k-\delta_k+1}-t_1^{\alpha_k-\delta_k+1}\right)}{\Gamma(\alpha_k-\delta_k+2)\Gamma(\alpha_k+\beta_k+1)}.
\end{aligned} \tag{2.9}$$

The right-hand sides of inequalities (2.8) and (2.9) are independent of $(x_1,...,x_n)\in S_\rho$ and tend to zero as t_2 tends to t_1. Thus, $\mathscr{A}$ is equi-continuous. Finally, we can see by Arzelà-Ascoli Theorem and the above arguments that $\mathscr{A}$ is a completely continuous operator.

(2) : We consider the set

$$\Omega := \{(x_1,...,x_n)\in B;\ (x_1,...,x_n)=\mu\mathscr{A}(x_1,...,x_n),\ 0<\mu<1\},$$

and show that is bounded:

Let $(x_1,...,x_n)\in\Omega$, then,

$$(x_1,...,x_n)=\mu\mathscr{A}(x_1,...,x_n),\quad 0<\mu<1, \tag{2.10}$$

Thus, for each $t\in J$ and corresponding to inequality (2.7), we have:

$$\|(x_1,...x_n)\|_B \leq \mu\max_{1\leq k\leq n}\left(\Theta_k L_k+\rho\Lambda_k,\Theta_k^* L_k+\rho\Lambda_k^*\right)<\infty. \tag{2.11}$$

Therefore, Ω is bounded.

Consequently by the steps (1), (2) and using Lemma 1.1, we deduce that $\mathscr{A}$ has at least one fixed point which is a solution of system (1.1). Theorem 2.3 is thus proved. □

Example 2.4. *To illustrate the second main result, let us consider the Langevin system:*

$$\begin{cases}
{}_0D_t^{\frac{5}{3}}\left({}_0D_t^{\frac{2}{3}}+\frac{\sqrt{2}}{12}\right)x_1(t) \\
=\dfrac{e^t+\sin(x_1(t)+x_2(t)+x_3(t)+x_4(t))}{(t^2+2)+\cos\left({}_0D_t^{\frac{1}{3}}x_1(t)+{}_0D_t^{\frac{2}{5}}x_2(t)+{}_0D_t^{\frac{1}{4}}x_3(t)+{}_0D_t^{\frac{1}{2}}x_4(t)\right)}, \\
{}_0D_t^{\frac{8}{5}}\left({}_0D_t^{\frac{4}{5}}-\frac{\pi}{8}\right)x_2(t) \\
=\dfrac{\pi(t+1)+\arccos\left(x_1(t)+{}_0D_t^{\frac{1}{3}}x_1(t)\right)+\arcsin\left(x_2(t)+{}_0D_t^{\frac{2}{5}}x_2(t)\right)}{\pi e^t+\sin(x_3(t)+x_4(t))\cos\left({}_0D_t^{\frac{1}{4}}x_3(t)+{}_0D_t^{\frac{1}{2}}x_4(t)\right)}, \\
{}_0D_t^{\frac{4}{3}}\left({}_0D_t^{\frac{1}{2}}-\frac{e}{3}\right)x_3(t) \\
=\dfrac{e^t+\sin\left({}_0D_t^{\frac{1}{3}}x_1(t)+{}_0D_t^{\frac{2}{5}}x_2(t)+{}_0D_t^{\frac{1}{4}}x_3(t)+{}_0D_t^{\frac{1}{2}}x_4(t)\right)}{2\pi(t+1)+\arctan(x_1(t)+x_2(t)+x_3(t)+x_4(t))}, \\
{}_0D_t^{\frac{7}{4}}\left({}_0D_t^{\frac{3}{4}}+\frac{\pi}{3}\right)x_4(t) \\
=\dfrac{2(t+e)+\cos(x_1(t)+x_2(t)+x_3(t)+x_4(t))}{(t+2)^2-\cos\left({}_0D_t^{\frac{1}{3}}x_1(t)+{}_0D_t^{\frac{2}{5}}x_2(t)+{}_0D_t^{\frac{1}{4}}x_3(t)+{}_0D_t^{\frac{1}{2}}x_4(t)\right)}, \\
t\in J:=[0,1], \\
x_1(0)={}_0D_t^{\frac{2}{3}}x_1(0)=x_1(1)+{}_0I_1^{\frac{2}{3}}x_1(t)=0, \\
x_2(0)={}_0D_t^{\frac{4}{5}}x_2(0)=x_2(1)+{}_0I_1^{\frac{4}{5}}x_2(t)=0, \\
x_3(0)={}_0D_t^{\frac{1}{2}}x_3(0)=x_3(1)+{}_0I_1^{\frac{1}{2}}x_3(t)=0, \\
x_4(0)={}_0D_t^{\frac{3}{4}}x_4(0)=x_4(1)+{}_0I_1^{\frac{3}{4}}x_4(t)=0,
\end{cases} \tag{2.12}$$

We have: $n=4$,

$$\begin{aligned}
\beta_1 &= \frac{5}{3},\quad \beta_2=\frac{8}{5},\quad \beta_3=\frac{4}{3},\quad \beta_4=\frac{7}{4}, \\
\alpha_1 &= \frac{2}{3},\quad \alpha_2=\frac{4}{5},\quad \alpha_3=\frac{1}{2},\quad \alpha_4=\frac{3}{4}, \\
\delta_1 &= \frac{1}{3},\quad \delta_2=\frac{2}{5},\quad \delta_3=\frac{1}{4},\quad \delta_4=\frac{1}{2}, \\
\lambda_1 &= \frac{\sqrt{2}}{12},\quad \lambda_2=-\frac{\pi}{8},\quad \lambda_3=-\frac{e}{3},\quad \lambda_4=\frac{\pi}{3},
\end{aligned}$$

and

$$f_1(t,u_1,...,u_8) = \frac{e^t+\sin(u_1+u_2+u_3+u_4)}{(t^2+2)+\cos(u_5+u_6+u_7+u_8)},$$

$$\begin{aligned}
&f_2(t,u_1,...,u_8) \\
=\ &\frac{\pi(t+1)+\arccos(u_1+u_5)+\arcsin(u_2+u_6)}{\pi e^t+\sin(u_3+u_4)\cos(u_7+u_8)},
\end{aligned}$$

$$f_3(t,u_1,...,u_8) = \frac{e^t+\sin(u_5+u_6+u_7+u_8)}{2\pi(t+1)+\arctan(u_1+u_2+u_3+u_4)},$$

$$f_4(t,u_1,...,u_8) = \frac{2(t+e)+\cos(u_1+u_2+u_3+u_4)}{(t+2)^2-\cos(u_5+u_6+u_7+u_8)}.$$

Then, we can see that the functions f_k, $k=1,2,3,4$, are continuous and bounded on $J\times\mathbb{R}^8$. So by Theorem 2.3, system (2.12) has at least one solution on J.

2.3 Generalized Ulam-Hyers Stability

Definition 2.5. *cf. [24–28] Fractional Langevin system (1.1) is Ulam-Hyers stable if there exists a constant $\sigma_{f_k}>0$, such that for all $(\varepsilon_1,...,\varepsilon_n)>0$, and for all solution $(y_1,...,y_n)\in B$ of*

$$\begin{cases} \left|{}_0D_t^{\beta_k}\left({}_0D_t^{\alpha_k}+\lambda_k\right)y_k(t)-f_k(\Delta_y(t))\right|\le\varepsilon_k, \quad t\in J,\\ y_k(0)={}_0D_t^{\alpha_k}y_k(0)=y_k(1)+{}_0I_1^{\alpha_k}y_k(t)=0, \end{cases} \tag{2.13}$$

there exists $(x_1,...,x_n)\in B$ a solution of fractional Langevin system (1.1), with

$$\|(y_1-x_1,...,y_n-x_n)\|_B\le\sigma_{f_k}\varepsilon, \quad \varepsilon>0.$$

Definition 2.6. *cf. [24–28] Fractional Langevin system (1.1) is generalized Ulam-Hyers stable if there exists $\Upsilon_{f_k}\in C(\mathbb{R}^+,\mathbb{R}^+)$, $\Upsilon_{f_k}(0)=0$, such that for all $\varepsilon>0$ and for each solution $(y_1,...,y_n)\in B$ of (2.13), there exists a solution $(x_1,...,x_n)\in B$ of fractional Langevin system (1.1), with*

$$\|(y_1-x_1,...,y_n-x_n)\|_B\le\Upsilon_{f_k}(\varepsilon), \quad \varepsilon>0.$$

Theorem 2.7. *Let $(\mathscr{H}_1)$ and $(\mathscr{C})$ hold. Then, Langevin fractional system (1.1) is generalized Ulam-Hyers stable in B.*

Proof. Let $(y_1,...,y_n)\in B$ be a solution of inequalities (2.13). Then, by integrating inequalities (2.13), we obtain:

$$\begin{aligned} &\left|\begin{array}{l} y_k(t)-\frac{1}{\Gamma(\alpha_k+\beta_k)}\int_0^t(t-s)^{\alpha_k+\beta_k-1}f_k(\Delta_y(s))\,ds\\ +\frac{\lambda_k}{\Gamma(\alpha_k)}\int_0^t(t-s)^{\alpha_k-1}y_k(s)\,ds+\frac{t^{\alpha_k+1}}{\Gamma(\alpha_k+\beta_k)}\\ \times\int_0^1(1-s)^{\alpha_k+\beta_k-1}f_k(\Delta_y(s))\,ds \end{array}\right|\\ \le\ & {}_0I_t^{\alpha_k+\beta_k}\varepsilon_k\\ \le\ & \frac{t^{\alpha_k+\beta_k}}{\Gamma(\alpha_k+\beta_k+1)}\varepsilon_k. \end{aligned} \tag{2.14}$$

Using $(\mathscr{H}_1)$ and $(\mathscr{C})$, there exists a solution $(x_1,...,x_n)\in B$ of system (1.1) :

$$\begin{aligned} &x_k(t)\\ =\ &\frac{1}{\Gamma(\alpha_k+\beta_k)}\int_0^t(t-s)^{\alpha_k+\beta_k-1}f_k(\Delta_x(s))\,ds\\ &-\frac{\lambda_k}{\Gamma(\alpha_k)}\int_0^t(t-s)^{\alpha_k-1}x_k(s)\,ds-\frac{t^{\alpha_k+1}}{\Gamma(\alpha_k+\beta_k)}\\ &\times\int_0^1(1-s)^{\alpha_k+\beta_k-1}f_k(\Delta_x(s))\,ds, \end{aligned} \tag{2.15}$$

$k=1,...,n.$
Then, we get

$$\begin{aligned} &|y_k(t)-x_k(t)|\\ =\ &\left|\begin{array}{l} y_k(t)-\frac{1}{\Gamma(\alpha_k+\beta_k)}\int_0^t(t-s)^{\alpha_k+\beta_k-1}f_k(\Delta_y(s))\,ds\\ +\frac{\lambda_k}{\Gamma(\alpha_k)}\int_0^t(t-s)^{\alpha_k-1}y_k(s)\,ds+\frac{t^{\alpha_k+1}}{\Gamma(\alpha_k+\beta_k)}\\ \times\int_0^1(1-s)^{\alpha_k+\beta_k-1}f_k(\Delta_y(s))\,ds+\frac{1}{\Gamma(\alpha_k+\beta_k)}\\ \times\int_0^t(t-s)^{\alpha_k+\beta_k-1}\Big(f_k(\Delta_y(s))-f_k(\Delta_x(s))\Big)ds\\ -\frac{\lambda_k}{\Gamma(\alpha_k)}\int_0^t(t-s)^{\alpha_k-1}(y_k(s)-x_k(s))\,ds-\frac{t^{\alpha_k+1}}{\Gamma(\alpha_k+\beta_k)}\\ \times\int_0^1(1-s)^{\alpha_k+\beta_k-1}\Big(f_k(\Delta_y(s))-f_k(\Delta_x(s))\Big)ds \end{array}\right| \end{aligned}$$

Using inequality (2.14), we get:

$$\begin{aligned} &|y_k(t)-x_k(t)|\\ \le\ &\frac{t^{\alpha_k+\beta_k}}{\Gamma(\alpha_k+\beta_k+1)}\varepsilon_k+\left(\frac{t^{\alpha_k+\beta_k}}{\Gamma(\alpha_k+\beta_k+1)}+\frac{t^{\alpha_k+1}}{\Gamma(\alpha_k+\beta_k+1)}\right)\\ &\times\max_{s\in J}\left|f_k(\Delta_y(s))-f_k(\Delta_x(s))\right|\\ &+\frac{|\lambda_k|t^{\alpha_k}}{\Gamma(\alpha_k+1)}\max_{s\in J}|y_k(s)-x_k(s)|. \end{aligned}$$

Thanks to $(\mathscr{H}_1)$, we get

$$\begin{aligned} &\|y_k(t)-x_k(t)\|_\infty\\ \le\ &\frac{\varepsilon_k}{\Gamma(\alpha_k+\beta_k+1)}+(\Theta_k\Sigma_k+\Lambda_k)\,\|(x_1-y_1,...,x_n-y_n)\|_B. \end{aligned} \tag{2.16}$$

By differentiating inequality (2.14), we get:

$$\begin{aligned}
&\left| {}_0D_t^{\delta_k} y_k(t) - \frac{1}{\Gamma(\alpha_k+\beta_k-\delta_k)} \right. \\
&\times \int_0^t (t-s)^{\alpha_k+\beta_k-\delta_k-1} f_k(\Delta_y(s))\, ds \\
&+ \frac{\lambda_k}{\Gamma(\alpha_k-\delta_k)} \int_0^t (t-s)^{\alpha_k-\delta_k-1} y_k(s)\, ds \\
&\left. + \frac{\Gamma(\alpha_k+2) t^{\alpha_k-\delta_k+1}}{\Gamma(\alpha_k-\delta_k+2)\Gamma(\alpha_k+\beta_k)} \int_0^1 (1-s)^{\alpha_k+\beta_k-1} f_k(\Delta_y(s))\, ds \right| \\
\leq \quad & {}_0I_t^{\alpha_k+\beta_k-\delta_k} \varepsilon_k \\
\leq \quad & \frac{t^{\alpha_k+\beta_k-\delta_k}}{\Gamma(\alpha_k+\beta_k-\delta_k+1)} \varepsilon_k.
\end{aligned}$$

Similarly as before, we can show that

$$\begin{aligned}
&\|y_k(t) - x_k(t)\|_\infty \\
\leq \quad & \frac{\varepsilon_k}{\Gamma(\alpha_k+\beta_k-\delta_k+1)} \\
&+ (\Theta_k^* \Sigma_k + \Lambda_k^*) \|(x_1-y_1, \ldots, x_n-y_n)\|_B. \qquad (2.17)
\end{aligned}$$

Using inequalities (2.16) and (2.17), we get

$$\begin{aligned}
&\|(y_1-x_1, \ldots, y_n-x_n)\|_B \\
\leq \quad & \max_{1\leq k\leq n} \left(\frac{\varepsilon_k}{\Gamma(\alpha_k+\beta_k+1)}, \frac{\varepsilon_k}{\Gamma(\alpha_k+\beta_k-\delta_k+1)} \right) \\
&+ \max_{1\leq k\leq n} (\Theta_k \Sigma_k + \Lambda_k, \Theta_k^* \Sigma_k + \Lambda_k^*) \\
&\times \|(y_1-x_1, \ldots, y_n-x_n)\|_B.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|(y_1-x_1, \ldots, y_n-x_n)\|_B \\
\leq \quad & \varepsilon \mathscr{M} + \Delta \|(y_1-x_1, \ldots, y_n-x_n)\|_B,
\end{aligned}$$

where

$$\begin{aligned}
\varepsilon &= \max_{1\leq k\leq n} \varepsilon_k, \\
\mathscr{M} &= \max_{1\leq k\leq n} \left(\frac{1}{\Gamma(\alpha_k+\beta_k+1)}, \frac{1}{\Gamma(\alpha_k+\beta_k-\delta_k+1)} \right).
\end{aligned}$$

Hence,

$$\|(y_1-x_1, \ldots, y_n-x_n)\|_B \leq \frac{\varepsilon \mathscr{M}}{(1-\Delta)} := \sigma_{f_k} \varepsilon, \qquad (2.18)$$

$$\sigma_{f_k} := \frac{\mathscr{M}}{(1-\Delta)}.$$

Thanks to $(\mathscr{C})$, we get $\sigma_{f_k} > 0$. That is fractional Langevin system (1.1) is Ulam-Hyers stable. Putting $\Upsilon_{f_k}(\varepsilon) = \sigma_{f_k} \varepsilon$, we receive the generalized Ulam-Hyers stability for system (1.1). $\square$

References

[1] S. Abbas, M. Benchohra, J.R. Graef and J. Henderson, Implicit fractional differential and integral equations: existence and stability, *Walter de Gruyter GmbH* & Co KG, 2018.

[2] B. Ahmad, A. Alsaedi and S. Salem, On a nonlocal integral boundary value problem of nonlinear Langevin equation with different fractional orders, *Adv. Differ. Equ.* 2019(57)(2019).

[3] H. Baghani, Existence and uniqueness of solutions to fractional Langevin equations involving two fractional orders, *J. Fixed Point Theory Appl.* 20(2018).

[4] D. Chalishajar, A. Kumar, Existence, Uniqueness and Ulam's stability of solutions for a coupled system of fractional differential equations with integral boundary conditions, *Mathematics.* 6(96)(2018).

[5] W.T. Coffey, Y.P. Kalmykov and J.T.Waldron, The Langevin equation: with applications to stochastic problems in physics, chemistry and electrical engineering, *World Scientific Publishing Co. River Edge.* (2004).

[6] Q. Dai, R. Gao, Z. Li and C. Wang, Stability of Ulam-Hyers and Ulam-Hyers-Rassias for a class of fractional differential equations, *Adv. Differ. Equ.* 2020(103) (2020).

[7] K.S. Fa, Generalized Langevin equation with fractional derivative and long-time correlation function, *Phys. Rev. E.* 73(2006).

[8] Z.Y. Gao, X.L. Yu and J.R. Wang, On local problems for Langevin-type differential equations with two fractional-order derivatives, *Bound. Value. Probl.* 2016(2016).

[9] R. Hilfer, Applications of fractional calculus in physics, *World Scientific, River Edge, New Jersey.* (2000).

[10] S. Harikrishnan, R.W. Ibrahim and K. Kanagarajan, On the generalized Ulam-Hyers-Rassias stability for coupled fractional differential equations, 2018(2018), 1-13.

[11] R.W. Ibrahim, S. Harikrishnan and K. Kanagarajan, Existence and stability of Langevin equations with two Hilfer-Katugampola fractional derivatives, *Stud. Univ. Babeş-Bolyai Math.* 63(3)(2018), 291–302.

[12] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier B.V., Amsterdam, The Netherlands. (2006).

[13] R. Kubo, The fluctuation-dissipation theorem, *Reports on Progress in Physics*, 29(1966), 255–284.

[14] P. Langevin, Sur la théorie du mouvement brownien, *C. R. Acad. Sci. Paris.,* 146(1908), 530–533, (in French).

[15] S.C. Lim, M. Li, and L.P. Teo, Langevin equation with two fractional orders, *Physics Letters A.,* 372(42)(2008), 6309–6320.

[16] A. Lozinski, R. G. Owen and T. N. Philips, *The Langevin and Fokker-Planck Equations in Polymer Rheology, in*

Handbook of Numerical Analysis, Elsevier, Amsterdam, Netherlands. 16(2011), 211–303.

[17] F. Mainradi, P. Pironi, The fractional Langevin equation, Brownian motion revisited, *Extracta Math.,* 11(1)(1996), 140-154.

[18] F. Mainradi, *Fractional calculus, Some Basic Problems in Continuum and Statistical Mechanics, Fractals and Fractional Calculus in Continuum Mechanics*, Springer Verlag, Wien. (1997), 291-348.

[19] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York. (1993).

[20] A. Salem, F. Alzahrani and M. Alnegga, Fractional Langevin equation with nonlocal integral boundary condition, *Mathematical Problems in Engineering*, 7(5) (2019), 402–411.

[21] A. Salem, F. Alzahrani and L. Almaghamsi, Langevin equation involving one fractional order with three point boundary conditions, *J. Nonlinear Sci. Appl.,* 12(2019), 791–798.

[22] A. Salem, F. Alzahrani and M. Alnegga, Coupled system of nonlinear fractional Langevin equations with multipoint and nonlocal integral boundary conditions, *Mathematical Problems in Engineering*, 2020(2020).

[23] C. Thaiprayoon, S. K. Ntouyas and J. Tariboon, On systems of fractional Langevin equations of Riemann-Liouville type with generalization non-local fractional integral boundary conditions, *Journal of Computational Analysis and Applications*, 27(2019), 723-737.

[24] A. Taïeb, Generalized Ulam-Hyers stability of fractional system of nonlinear integro-differential equations, *Int. J. Open Problems Compt. Math.,* 12(1)(2019).

[25] A. Taïeb, Stability of singular fractional systems of nonlinear integro-differntial equations, *Lobachevskii Journal of Math.,* 40(2)(2019), 219-299.

[26] A. Taïeb, Existence of solutions and the Ulam stability for a class of singular nonlinear fractional integro-differential equations, *Commun. Optim. Theory,* 2019 (2019), 1–22.

[27] A. Taïeb, Ulam stability for a singular fractional $2D$ nonlinear system, *Konuralp Journal of Mathematics*, 7(2)(2019), 300–311.

[28] A. Taïeb, On singular systems of nonlinear equations involving $3n-$ Caputo derivatives, *Acta et Commentationes Universitatis Tartuensis de Mathematica*, 23(2)(2019), 179–192.

[29] A. Taïeb and Z. Dahmani, Fractional system of nonlinear integro-differential equations, *Journal of Fractional Calculus and Applications*, 10(1)(2019), 55–67.

[30] A. Taïeb, S. Salmi and S. Boumessaoud, Ulam-Hyers stability for high dimensional fractional systems, *Canad. J. Appl. Math.,* 2(1)(2020), 95–110.

[31] A. Taïeb and Z. Dahmani, Triangular system of higher order singular fractional differential equations,]it Kragujevac Journal of Math., 45(1)(2021), 81–101.

[32] C. Torres, Existence of solution for fractional Langevin equation: variational approach, *Electron. J. Qual. Theory Differ. Equ.,* 2014(54)(2014).

[33] J. Wang, L. Lv and Y. Zhou, Ulam stability and data dependence for fractional differential equations with Caputo derivative, *Electronic J. Quali Theory Diff. Equat.,* (63)2011), 1–10.

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