Malaya J. Mat. 2(4)(2014) 373–391

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Analytical solutions of incomplete elliptic integrals motivated by the work of Ramanujan

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Abstract

In this paper, we obtain exact solutions of some unsolved incomplete elliptic integrals of first, second and third kinds, given in Entry 7 of Chapter XVII of second notebook of Srinivasa Ramanujan. Furthermore, we generalize these elliptic integrals in the forms of multiple series identities involving bounded multiple sequences.

Keywords: Gaussian Hypergeometric Function; Incomplete Elliptic Integrals; Multiple Series Identities; Srivastava-Daoust double Hypergeometric Function ; Kampé de Fériet double Hypergeometric Function; Srivastava's Triple Hypergeometric Function.

2010 MSC: 33B10, 33C05,, 33C65.

1 Introduction and Preliminaries

Some Interesting Series Identities

We recall the following identities which are potentially useful in the series rearrangement techniques.

$$\sum_{m=0}^{\infty}\sum_{r=0}^{m-1}\Theta(m,\ r)=\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Theta(m+r+1,\ r) \tag{1.1}$$

$$\sum_{m=0}^{\infty}\sum_{r=0}^{2m-1}\Theta(m,\ r)=\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Theta(m+r+1,\ r)+\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Theta(m+r+1,\ 2r+m+1) \tag{1.2}$$

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$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{m+n-1}\Psi(m,\,n,\,r)=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(0,\,n+r+1,\,r)+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(m+r+1,\,n,\,r)+$$
$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(m+1,\,n+r+1,\,r+m+1) \tag{1.3}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{m+n}\Psi(m,\,n,\,r)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(m+r,\,n,\,r)+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(m,\,n+r+1,\,r+m+1) \tag{1.4}$$

where $\{\Theta(m,r)\}_{m,r=0}^{\infty}$ and $\{\Psi(m,n,r)\}_{m,n,r=0}^{\infty}$ are suitably bounded double and triple sequences of essentially arbitrary(real or complex) parameters respectively.

Legendre's Normal Forms of Incomplete Elliptic Integrals

Following integrals arise in the solutions of certain classes of vibration problems and problems involving computations of the radiation field off axis from a uniform circular disc radiating according to an arbitrary angular distribution law.

Following elliptic integrals (R.H.S.) have been represented in different notations (L.H.S.) by researchers

First Kind :
$$F(x,\,\phi)=\int_0^{\phi}\frac{\mathrm{d}\theta}{\sqrt{(1-x^2\sin^2\theta)}} \tag{1.5}$$

Second Kind :
$$E(x,\,\phi)=\int_0^{\phi}\sqrt{(1-x^2\sin^2\theta)}\,\mathrm{d}\theta \tag{1.6}$$

Third Kind :
$$\prod(a,\,x,\,\phi)=\int_0^{\phi}\frac{\mathrm{d}\theta}{(1-a\sin^2\theta)\sqrt{(1-x^2\sin^2\theta)}} \tag{1.7}$$

where $\quad 0\le\ x\le\ 1,\ 0\le\ \phi\le\ \frac{\pi}{2},\ -\infty<a<\infty,\ a\neq 1$

The integrands of elliptic integrals are periodic functions with a period π. Here x, ϕ and a are called modulus, amplitude and characteristic parameter respectively. In case $x=\sin\delta$, δ is called modular angle.

Some Useful Indefinite Integrals

When $m=0,1,2,3,\cdots,$ then

$$\int\sin^{2m}(c\theta)\mathrm{d}\theta=\left\{\frac{-(\frac{1}{2})_m\sin(c\theta)\cos(c\theta)}{(1)_m\ c}\sum_{r=0}^{m-1}\frac{(1)_r\sin^{2r}(c\theta)}{(\frac{3}{2})_r}\right\}+\left\{\frac{\theta\,(\frac{1}{2})_m}{(1)_m}\right\}+Constant \tag{1.8}$$

$$\int\cos^{2m}(c\theta)\mathrm{d}\theta=\left\{\frac{(\frac{1}{2})_m\sin(c\theta)\cos(c\theta)}{(1)_m\ c}\sum_{r=0}^{m-1}\frac{(1)_r\cos^{2r}(c\theta)}{(\frac{3}{2})_r}\right\}+\left\{\frac{\theta\,(\frac{1}{2})_m}{(1)_m}\right\}+Constant \tag{1.9}$$

$$\int\sin^{2m+1}(c\theta)\mathrm{d}\theta=\frac{-(1)_m\cos(c\theta)}{(\frac{3}{2})_m\ c}\sum_{r=0}^{m}\frac{(\frac{1}{2})_r\sin^{2r}(c\theta)}{(1)_r}+Constant \tag{1.10}$$

$$\int\cos^{2m+1}(c\theta)\mathrm{d}\theta=\frac{(1)_m\sin(c\theta)}{(\frac{3}{2})_m\ c}\sum_{r=0}^{m}\frac{(\frac{1}{2})_r\cos^{2r}(c\theta)}{(1)_r}+Constant \tag{1.11}$$

Above formulas (1.8)-(1.11) can be verified for $m=0,1,2,3,\cdots$ and it is the convention that the empty sum $\sum_{r=0}^{-1}F(r)$ is treated as zero.

2 Seventh entry of Chapter Seventeenth of Second Notebook of Ramanujan[9,pp.104-107,pp.112-113]

Ramanujan's notebooks have been divided into several chapters and contains large numbers of important and useful results on elliptic integrals. Many results on elliptic integrals have been proved by B. C. Berndt [8, pp.104-113] and R. Y. Denis *et al.*[15].

We have also verified the following entries of Ramanujan by using the methods of B. C. Berndt and R. Y. Denis *et al.*

Entry 7(i): If $\sin\alpha = \sqrt{x}\sin\beta$ or $\frac{\tan\alpha}{\tan\beta} = \frac{\sqrt{x}\cos\beta}{\sqrt{(1-x\sin^2\beta)}}$, then

$$\int_0^{\alpha} \frac{d\theta}{\sqrt{(x-\sin^2\theta)}} = \int_0^{\beta} \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} \tag{2.1}$$

In a paper of Denis *et al.* [15, p.113(1)], a misprint condition is written in Entry 7(i).

Entry 7(ii): If $\tan\alpha = \sqrt{(1-x)}\tan\beta$ or $\sin\alpha = \frac{\sqrt{(1-x)}\sin\beta}{\sqrt{(1-x\sin^2\beta)}}$, then

$$\int_0^{\alpha} \frac{d\theta}{\sqrt{(1-x\cos^2\theta)}} = \int_0^{\beta} \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} \tag{2.2}$$

Entry 7(iii): If $\tan\alpha = \sqrt{(1-b)}\tan\beta$ or $\sin\beta = \frac{\sin\alpha}{\sqrt{(1-b\cos^2\alpha)}}$, then

$$\int_0^{\alpha} \frac{d\theta}{\sqrt{\{1-(\frac{a-b}{1-b})\sin^2\theta\}}} = \sqrt{(1-b)}\int_0^{\beta} \frac{d\theta}{\sqrt{(1-a\sin^2\theta)(1-b\sin^2\theta)}} \tag{2.3}$$

Entry 7(iv): If $\tan\alpha = \sqrt{(1+x)}\tan\beta$, then

$$\int_0^{\alpha} \frac{d\theta}{\sqrt{(1+x\cos 2\theta)}} = \int_0^{\beta} \frac{d\theta}{\sqrt{(1-x^2\sin^4\theta)}} \tag{2.4}$$

which is the correct form of a misprint result of Denis *et al.* [15, p.115(4)].

Entry 7(v): Degenerate form of addition theorem

If $\cot\alpha\ \cot\beta = \sqrt{(1-x)}$, then

$$\int_0^{\alpha} \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} + \int_0^{\beta} \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} = \frac{\pi}{2}\,{}_2F_1\left[\begin{array}{lll} \frac{1}{2},\frac{1}{2} & ; & \\ & & x \\ 1 & ; & \end{array}\right] \tag{2.5}$$

Entry 7(vi): Classical duplication formula (Special Case of the converse of Entry 7(viii)a)

If $\cot\alpha\ \tan(\frac{\beta}{2}) = \sqrt{(1-x\sin^2\alpha)}$, then

$$2\int_0^{\alpha} \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} = \int_0^{\beta} \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} \tag{2.6}$$

Entry 7(vii): Jacobi's imaginary transformation (For incomplete elliptic integral of first kind having imaginary amplitude)

If $\alpha = \ln\{\tan(\frac{\pi}{4} + \frac{\beta}{2})\}$ or $e^{\alpha} = \frac{\cos\beta}{1-\sin\beta} = \frac{1+\sin\beta}{\cos\beta}$ or $\sinh(\alpha) = \tan\beta$, then

$$\int_0^{i\alpha} \frac{d\theta}{\sqrt{(1 - x\sin^2\theta)}} = i\int_0^{\beta} \frac{d\theta}{\sqrt{(1-(1-x)\sin^2\theta)}} \tag{2.7}$$

which is the correct form of a misprint result of Denis *et al.* [15, p.116(7)].

Entry 7(viii): Addition theorem

If

$$\int_0^{\alpha} \frac{d\theta}{\sqrt{(1 - x\sin^2\theta)}} + \int_0^{\beta} \frac{d\theta}{\sqrt{(1 - x\sin^2\theta)}} = \int_0^{\gamma} \frac{d\theta}{\sqrt{(1 - x\sin^2\theta)}} \tag{2.8}$$

then four different sets of hypothesis (implications) are given by

$$(a)\ \tan\left(\frac{\gamma}{2}\right) = \frac{\sin\alpha\sqrt{(1 - x\sin^2\beta)} + \sin\beta\sqrt{(1 - x\sin^2\alpha)}}{\cos\alpha + \cos\beta}$$

which is the correct form of a misprint condition of Denis *et al.* [15, p.117(i)].

$$(b)\ \gamma = \tan^{-1}\{\tan\alpha\sqrt{(1 - x\sin^2\beta)}\} + \tan^{-1}\{\tan\beta\sqrt{(1 - x\sin^2\alpha)}\}$$

(c) Legendre's canonical form of the addition theorem

$$\cot\alpha\ \cot\beta = \frac{\cos\gamma}{\sin\alpha\sin\beta} + \sqrt{(1 - x\sin^2\gamma)}$$

$$(d)\ \frac{\sqrt{x}}{2} = \frac{\sqrt{\{\sin(s)\ \sin(s-\alpha)\ \sin(s-\beta)\ \sin(s-\gamma)\}}}{\sin\alpha\ \sin\beta\ \sin\gamma}\ \ \text{;where}\ \ s = \frac{\alpha+\beta+\gamma}{2}$$

The four different sets of hypothesis (implications) $(a) - (d)$ in Entry 7(viii) are both necessary and sufficient.

Entry 7(xii): Gauss' transformation

If $(1 + x\sin^2\alpha)\sin\beta = (1 + x)\ \sin\alpha$, then

$$(1 + x)\int_0^{\alpha} \frac{d\theta}{\sqrt{(1 - x^2\sin^2\theta)}} = \int_0^{\beta} \frac{d\theta}{\sqrt{\{1 - \frac{4x}{(1+x)^2}\sin^2\theta\}}} \tag{2.9}$$

which is the correct form of misprint result of Denis *et al.* [15, p.118(9)].

Entry 7(xiii): Landen's transformation (i.e. The first geometric representation)

If $x\sin\alpha = \sin(2\beta - \alpha)$, then

$$(1+x)\int_0^{\alpha}\frac{\mathrm{d}\theta}{\sqrt{(1-x^2\sin^2\theta)}}=2\int_0^{\beta}\frac{\mathrm{d}\theta}{\sqrt{\{1-\frac{4x}{(1+x)^2}\sin^2\theta\}}} \tag{2.10}$$

Entries 7(xii) and 7(xiii) are very similar in appearance.

3 A General Family of Multiple-Series Identities

Theorem 3.1. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\sin^{2m}(c\theta)\mathrm{d}\theta\right)\frac{y^m}{m!}$$
$$=-\frac{y\sin(c\gamma)\cos(c\gamma)}{2c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{3}{2})_{m+r}(1)_r\,y^m(y\sin^2(c\gamma))^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_r}+\gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{2})_m y^m}{(m!)^2} \tag{3.1}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.2. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\cos^{2m}(c\theta)\mathrm{d}\theta\right)\frac{y^m}{m!}$$
$$=\frac{y\sin(c\gamma)\cos(c\gamma)}{2c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{3}{2})_{m+r}(1)_r\,y^m(y\cos^2(c\gamma))^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_r}+\gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{2})_m y^m}{(m!)^2} \tag{3.2}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.3. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\sin^{4m}(c\theta)\mathrm{d}\theta\right)\frac{y^m}{m!}=\gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{4})_m(\frac{3}{4})_m y^m}{(\frac{1}{2})_m(m!)^2}-$$
$$-\frac{3y\sin(c\gamma)\cos(c\gamma)}{8c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{5}{4})_{m+r}(\frac{7}{4})_{m+r}(1)_r\,y^m(y\sin^2(c\gamma))^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_{m+r}(\frac{3}{2})_r}-$$
$$-\frac{y\sin^3(c\gamma)\cos(c\gamma)}{4c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{5}{4})_{m+r}(\frac{7}{4})_{m+r}(2)_{m+2r}\,(y\sin^2(c\gamma))^m(y\sin^4(c\gamma))^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_{m+r}(\frac{5}{2})_{m+2r}} \tag{3.3}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.4. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\cos^{4m}(c\theta)\mathrm{d}\theta\right)\frac{y^m}{m!}=\gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{4})_m(\frac{3}{4})_m y^m}{(\frac{1}{2})_m(m!)^2}+$$
$$+\frac{3y\sin(c\gamma)\cos(c\gamma)}{8c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{5}{4})_{m+r}(\frac{7}{4})_{m+r}(1)_r\,y^m(y\cos^2(c\gamma))^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_{m+r}(\frac{3}{2})_r}+$$
$$+\frac{y\sin(c\gamma)\cos^3(c\gamma)}{4c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{5}{4})_{m+r}(\frac{7}{4})_{m+r}(2)_{m+2r}\,(y\cos^2(c\gamma))^m(y\cos^4(c\gamma))^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_{m+r}(\frac{5}{2})_{m+2r}} \tag{3.4}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.5. *Let $\{\Lambda_{m,n}\}_{m,n=0}^{\infty}$ be a suitably bounded double sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda_{m,n}\left(\int_0^{\gamma}\sin^{2m+2n}(c\theta)\mathrm{d}\theta\right)\frac{y^m z^n}{m!\ n!}$$

$$= \gamma\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda_{m,n}\frac{(\frac{1}{2})_{m+n}y^m z^n}{(m+n)!\ m!\ n!} - \frac{z\sin(c\gamma)\cos(c\gamma)}{2c}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{0,n+r+1}\frac{(\frac{3}{2})_{n+r}(1)_r\ z^n(z\sin^2(c\gamma))^r}{(2)_{n+r}(2)_{n+r}(\frac{3}{2})_r} -$$
$$- \frac{y\sin(c\gamma)\cos(c\gamma)}{2c}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m+r+1,n}\frac{(\frac{3}{2})_{m+n+r}(1)_r\ y^m z^n(y\sin^2(c\gamma))^r}{(2)_{m+n+r}(2)_{m+r}(\frac{3}{2})_r(1)_n} -$$
$$- \frac{yz\sin^3(c\gamma)\cos(c\gamma)}{4c}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m+1,n+r+1}\frac{(\frac{5}{2})_{m+n+r}(2)_{m+r}\ (y\sin^2(c\gamma))^m z^n(z\sin^2(c\gamma))^r}{(3)_{m+n+r}(\frac{5}{2})_{m+r}(2)_{n+r}(2)_m} \tag{3.5}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.6. *Let $\{\Lambda_{m,n}\}_{m,n=0}^{\infty}$ be a suitably bounded double sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda_{m,n}\left(\int_0^{\gamma}\cos^{2m+2n}(c\theta)\mathrm{d}\theta\right)\frac{y^m z^n}{m!\ n!}$$

$$= \gamma\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda_{m,n}\frac{(\frac{1}{2})_{m+n}y^m z^n}{(m+n)!\ m!\ n!} + \frac{z\sin(c\gamma)\cos(c\gamma)}{2c}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{0,n+r+1}\frac{(\frac{3}{2})_{n+r}(1)_r\ z^n(z\cos^2(c\gamma))^r}{(2)_{n+r}(2)_{n+r}(\frac{3}{2})_r} +$$
$$+ \frac{y\sin(c\gamma)\cos(c\gamma)}{2c}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m+r+1,n}\frac{(\frac{3}{2})_{m+n+r}(1)_r\ y^m z^n(y\cos^2(c\gamma))^r}{(2)_{m+n+r}(2)_{m+r}(\frac{3}{2})_r(1)_n} +$$
$$+ \frac{yz\sin(c\gamma)\cos^3(c\gamma)}{4c}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m+1,n+r+1}\frac{(\frac{5}{2})_{m+n+r}(2)_{m+r}\ (y\cos^2(c\gamma))^m z^n(z\cos^2(c\gamma))^r}{(3)_{m+n+r}(\frac{5}{2})_{m+r}(2)_{n+r}(2)_m} \tag{3.6}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.7. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\sin^m(c\theta)\mathrm{d}\theta\right)\frac{y^m}{m!}$$

$$= \gamma\sum_{m=0}^{\infty}\Omega_{2m}\frac{y^{2m}}{4^m\ (m!)^2} - \frac{y^2\sin(c\gamma)\cos(c\gamma)}{4c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{2m+2r+2}\frac{(1)_r\ y^{2m}(y\sin(c\gamma))^{2r}}{4^{m+r}(2)_{m+r}(2)_{m+r}(\frac{3}{2})_r} +$$
$$+\frac{y}{c}\sum_{m=0}^{\infty}\Omega_{2m+1}\frac{y^{2m}}{4^m(\frac{3}{2})_m(\frac{3}{2})_m} - \frac{y\cos(c\gamma)}{c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{2m+2r+1}\frac{(\frac{1}{2})_r\ y^{2m}(y\sin(c\gamma))^{2r}}{4^{m+r}(\frac{3}{2})_{m+r}(\frac{3}{2})_{m+r}(1)_r} \tag{3.7}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.8. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\cos^m(c\theta)\mathrm{d}\theta\right)\frac{y^m}{m!}$$

$$= \gamma\sum_{m=0}^{\infty}\Omega_{2m}\frac{y^{2m}}{4^m\ (m!)^2} + \frac{y^2\sin(c\gamma)\cos(c\gamma)}{4c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{2m+2r+2}\frac{(1)_r\ y^{2m}(y\cos(c\gamma))^{2r}}{4^{m+r}(2)_{m+r}(2)_{m+r}(\frac{3}{2})_r} +$$
$$+\frac{y\sin(c\gamma)}{c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{2m+2r+1}\frac{(\frac{1}{2})_r\ y^{2m}(y\cos(c\gamma))^{2r}}{4^{m+r}(\frac{3}{2})_{m+r}(\frac{3}{2})_{m+r}(1)_r} \tag{3.8}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.9. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\sin^{2m+1}(c\theta)\mathrm{d}\theta\right)\frac{y^m}{m!}=\frac{1}{c}\sum_{m=0}^{\infty}\Omega_m\frac{y^m}{(\frac{3}{2})_m}-\frac{\cos(c\gamma)}{c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r}\frac{(\frac{1}{2})_r\,y^m(y\sin^2(c\gamma))^r}{(\frac{3}{2})_{m+r}(1)_r} \tag{3.9}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.10. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\cos^{2m+1}(c\theta)\mathrm{d}\theta\right)\frac{y^m}{m!}=\frac{\sin(c\gamma)}{c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r}\frac{(\frac{1}{2})_r\,y^m(y\cos^2(c\gamma))^r}{(\frac{3}{2})_{m+r}(1)_r} \tag{3.10}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.11. *Let $\{\Lambda_{m,n}\}_{m,n=0}^{\infty}$ be a suitably bounded double sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda_{m,n}\left(\int_0^{\gamma}\sin^{2m+2n+1}(c\theta)\mathrm{d}\theta\right)\frac{y^mz^n}{m!\,n!}$$
$$=-\frac{\cos(c\gamma)}{c}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m+r,n}\frac{(1)_{m+n+r}(\frac{1}{2})_r\,y^m\,z^n(y\sin^2(c\gamma))^r}{(\frac{3}{2})_{m+n+r}(1)_{m+r}(1)_n\,(1)_r}-$$
$$-\frac{z\cos(c\gamma)\sin^2(c\gamma)}{3c}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m,n+r+1}\frac{(2)_{m+n+r}(\frac{3}{2})_{m+r}\,(y\sin^2(c\gamma))^m\,z^n\,(z\sin^2(c\gamma))^r}{(\frac{5}{2})_{m+n+r}(2)_{n+r}(2)_{m+r}\,m!}+$$
$$+\frac{1}{c}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda_{m,n}\frac{(1)_{m+n}\,y^m\,z^n}{(\frac{3}{2})_{m+n}\,m!\,n!} \tag{3.11}$$

provided that each of the series involved is absolutely convergent.

Theorem 3.12. *Let $\{\Lambda_{m,n}\}_{m,n=0}^{\infty}$ be a suitably bounded double sequence of arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda_{m,n}\left(\int_0^{\gamma}\cos^{2m+2n+1}(c\theta)\mathrm{d}\theta\right)\frac{y^mz^n}{m!\,n!}$$
$$=\frac{\sin(c\gamma)}{c}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m+r,n}\frac{(1)_{m+n+r}(\frac{1}{2})_r\,y^m\,z^n(y\cos^2(c\gamma))^r}{(\frac{3}{2})_{m+n+r}(1)_{m+r}(1)_n\,(1)_r}+$$
$$+\frac{z\sin(c\gamma)\cos^2(c\gamma)}{3c}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Lambda_{m,n+r+1}\frac{(2)_{m+n+r}(\frac{3}{2})_{m+r}(y\cos^2(c\gamma))^m\,z^n\,(z\cos^2(c\gamma))^r}{(\frac{5}{2})_{m+n+r}(2)_{m+r}(2)_{n+r}\,m!} \tag{3.12}$$

provided that each of the series involved is absolutely convergent.

Proof of (3.1) : $\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\sin^{2m}(c\theta)\mathrm{d}\theta\right)\frac{y^m}{m!}$

$$=-\sum_{m=0}^{\infty}\sum_{r=0}^{m-1}\Omega_m\frac{\sin(c\gamma)\cos(c\gamma)(\frac{1}{2})_m(1)_r\,y^m(\sin^2(c\gamma))^r}{c(1)_m(1)_m(\frac{3}{2})_r}+\gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{2})_my^m}{(m!)^2}$$
$$=-\sum_{m=0}^{\infty}\sum_{r=0}^{m}\Omega_{m+1}\frac{\sin(c\gamma)\cos(c\gamma)(\frac{1}{2})_{m+1}(1)_r\,y^{m+1}(\sin^2(c\gamma))^r}{c(1)_{m+1}(1)_{m+1}(\frac{3}{2})_r}+\gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{2})_my^m}{(m!)^2}$$
$$=-\frac{y\sin(c\gamma)\cos(c\gamma)}{2c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{3}{2})_{m+r}(1)_r\,y^m(y\sin^2(c\gamma))^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_r}+\gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{2})_my^m}{(m!)^2}$$

Similarly we can derive (3.2) to (3.12) by means of series identities (1.1) to (1.4).

4 Hypergeometric Generalizations of Incomplete Elliptic Integrals of Ramanujan and their solutions

Putting $c = 1$ in theorems (3.1) to (3.6), (3.9) to (3.12) and setting

$$\Omega_m = \frac{(a_1)_m(a_2)_m(a_3)_m \cdots (a_A)_m}{(b_1)_m(b_2)_m(b_3)_m \cdots (b_B)_m} = \frac{((a_A))_m}{((b_B))_m},\ \Lambda_{m,n} = \frac{((a_A))_{m+n}((d_D))_m((g_G))_n}{((b_B))_{m+n}((e_E))_m((h_H))_n},$$

in theorems (3.1) to (3.12), using some algebraic properties of Pochhammer symbol and interpreting the multiple power series in hypergeometric forms of Gauss [34, p. 42(1)], Kampé de Fériet [34, p.63(16); see also 33], Srivastava [34, p.69(39,40)] and Srivastava-Daoust [33, p.37(21, 22, 23); 34, pp.64-65(18, 19, 20)], we get the analytical solutions of generalized incomplete elliptic integrals.

$$\int_0^\gamma {}_AF_B\left[\begin{array}{ll}(a_j)_{j=1}^A ; & \\ & y\sin^2\theta \\ (b_j)_{j=1}^B ; & \end{array}\right] \mathrm{d}\theta = \gamma\ {}_{A+1}F_{B+1}\left[\begin{array}{ll}\frac{1}{2},(a_j)_{j=1}^A ; & \\ & y \\ 1,(b_j)_{j=1}^B ; & \end{array}\right] - $$
$$- \frac{y\sin\gamma\ \cos\gamma \prod_{i=1}^{A}(a_i)}{2\prod_{i=1}^{B}(b_i)} F^{A+1:1;2}_{B+2:0;1}\left[\begin{array}{ll}\frac{3}{2},(1+a_j)_{j=1}^A :1\ ;1,1\ ; & \\ & y, y\sin^2\gamma \\ 2,2,(1+b_j)_{j=1}^B : -;\frac{3}{2}\ ; & \end{array}\right] \tag{4.1}$$

$$\int_0^\gamma {}_AF_B\left[\begin{array}{ll}(a_j)_{j=1}^A ; & \\ & y\cos^2\theta \\ (b_j)_{j=1}^B ; & \end{array}\right] \mathrm{d}\theta = \gamma\ {}_{A+1}F_{B+1}\left[\begin{array}{ll}\frac{1}{2},(a_j)_{j=1}^A ; & \\ & y \\ 1,(b_j)_{j=1}^B ; & \end{array}\right] + $$
$$+ \frac{y\sin\gamma\ \cos\gamma \prod_{i=1}^{A}(a_i)}{2\prod_{i=1}^{B}(b_i)} F^{A+1:1;2}_{B+2:0;1}\left[\begin{array}{ll}\frac{3}{2},(1+a_j)_{j=1}^A :1\ ;1\ ,\ 1\ ; & \\ & y, y\cos^2\gamma \\ 2,2,(1+b_j)_{j=1}^B : -;\ \frac{3}{2}\ ; & \end{array}\right] \tag{4.2}$$

$$\int_0^\gamma {}_AF_B\left[\begin{array}{ll}(a_j)_{j=1}^A ; & \\ & y\sin^4\theta \\ (b_j)_{j=1}^B ; & \end{array}\right] \mathrm{d}\theta = \gamma\ {}_{A+2}F_{B+2}\left[\begin{array}{ll}\frac{1}{4},\frac{3}{4},(a_j)_{j=1}^A ; & \\ & y \\ 1,\frac{1}{2},(b_j)_{j=1}^B ; & \end{array}\right] - $$
$$- \frac{3y\ \sin\gamma\cos\gamma \prod_{i=1}^{A}(a_i)}{8\prod_{i=1}^{B}(b_i)} F^{A+2:1;2}_{B+3:0;1}\left[\begin{array}{ll}\frac{5}{4},\frac{7}{4},(1+a_j)_{j=1}^A\ :\ 1;\ 1,1\ ; & \\ & y,\ y\sin^2\gamma \\ 2,2,\frac{3}{2},(1+b_j)_{j=1}^B : -;\ \frac{3}{2}\ ; & \end{array}\right] - $$
$$- \frac{y\sin^3\gamma\ \cos\gamma \prod_{i=1}^{A}(a_i)}{4\prod_{i=1}^{B}(b_i)} F^{A+3:1;1}_{B+4:0;0}\left(\begin{array}{l}[(1+a_j):1,1]_{j=1}^A,\ [\frac{5}{4}:1,1],\ [\frac{7}{4}:1,1],\ [2:1,2]\quad : \\ \\ [(1+b_j):1,1]_{j=1}^B, [2:1,1], [2:1,1], [\frac{3}{2}:1,1], [\frac{5}{2}:1,2]\ : \end{array}\right.$$

$$\left.\begin{matrix}[1{:}1]\,;[1{:}1] & ; & \\ & & y\sin^2\gamma,\ y\sin^4\gamma \\ \text{—}\,;\text{—} & ; & \end{matrix}\right) \tag{4.3}$$

$$\int_0^\gamma {}_AF_B\left[\begin{matrix}(a_j)_{j=1}^A & ; & \\ & & y\cos^4\theta \\ (b_j)_{j=1}^B & ; & \end{matrix}\right]\mathrm{d}\theta=\gamma\ {}_{A+2}F_{B+2}\left[\begin{matrix}\frac{1}{4},\frac{3}{4},(a_j)_{j=1}^A & ; & \\ & & y \\ 1,\frac{1}{2},(b_j)_{j=1}^B & ; & \end{matrix}\right]+$$

$$+\frac{3y\ \sin\gamma\cos\gamma\prod_{i=1}^{A}(a_i)}{8\prod_{i=1}^{B}(b_i)}F^{A+2:1;2}_{B+3:0;1}\left[\begin{matrix}\frac{5}{4},\frac{7}{4},(1+a_j)_{j=1}^A\ :\ 1;1,1 & ; & \\ & & y,\ y\cos^2\gamma \\ 2,2,\frac{3}{2},(1+b_j)_{j=1}^B:\text{—};\frac{3}{2} & ; & \end{matrix}\right]+$$

$$+\frac{y\sin\gamma\ \cos^3\gamma\prod_{i=1}^{A}(a_i)}{4\prod_{i=1}^{B}(b_i)}F^{A+3:1;1}_{B+4:0;0}\left(\begin{matrix}[(1+a_j){:}1,1]_{j=1}^A,\ \ [\frac{5}{4}{:}1,1],\ \ [\frac{7}{4}{:}1,1],\ \ [2{:}1,2] & : \\ \\ [(1+b_j)\ {:}1,1]_{j=1}^B,\ [2{:}1,1],\ [2{:}1,1],\ [\frac{3}{2}{:}1,1],\ [\frac{5}{2}{:}1,2] & :\end{matrix}\right.$$

$$\left.\begin{matrix}[1{:}1]\,;[1{:}1] & ; & \\ & & y\cos^2\gamma,\ y\cos^4\gamma \\ \text{—}\,;\text{—} & ; & \end{matrix}\right) \tag{4.4}$$

$$\int_0^\gamma F^{A:D;G}_{B:E;H}\left[\begin{matrix}(a_j)_{j=1}^A:(d_j)_{j=1}^D;(g_j)_{j=1}^G & ; & \\ & & y\sin^2\theta,\ z\sin^2\theta \\ (b_j)_{j=1}^B:(e_j)_{j=1}^E;(h_j)_{j=1}^H & ; & \end{matrix}\right]\mathrm{d}\theta$$

$$=\gamma\ F^{A+1:D;G}_{B+1:E;H}\left[\begin{matrix}(a_j)_{j=1}^A,\ \frac{1}{2}{:}\,(d_j)_{j=1}^D;(g_j)_{j=1}^G & ; & \\ & & y,\ z \\ (b_j)_{j=1}^B,\ 1{:}\,(e_j)_{j=1}^E;(h_j)_{j=1}^H & ; & \end{matrix}\right]-\frac{z\ \sin\gamma\ \cos\gamma\prod_{i=1}^{A}(a_i)\ \prod_{i=1}^{G}(g_i)}{2\prod_{i=1}^{B}(b_i)\ \prod_{i=1}^{H}(h_i)}\times$$

$$\times\ F^{A+G+1:1;2}_{B+H+2:0;1}\left[\begin{matrix}\frac{3}{2},\ (1+a_j)_{j=1}^A,\ (1+g_j)_{j=1}^G\ \ :\ 1\ ;1,1 & ; & \\ & & z,\ z\ \sin^2\gamma \\ 2,\ 2,\ (1+b_j)_{j=1}^B,\ (1+h_j)_{j=1}^H:\text{—};\ \frac{3}{2} & ; & \end{matrix}\right]-$$

$$-\frac{y\ \sin\gamma\ \cos\gamma\prod_{i=1}^{A}(a_i)\ \prod_{i=1}^{D}(d_i)}{2\prod_{i=1}^{B}(b_i)\ \prod_{i=1}^{E}(e_i)}\times$$

$$\times\ F^{(3)}\left[\begin{matrix}\frac{3}{2},(1+a_j)_{j=1}^A{::}\ \text{—};\text{—};(1+d_j)_{j=1}^D\ \ {:}1\ ;(g_j)_{j=1}^G;1,1 & ; & \\ & & y,\ z,\ y\sin^2\gamma \\ 2,\ (1+b_j)_{j=1}^B{::}\ \text{—};\text{—};(1+e_j)_{j=1}^E,2{:}\text{—};(h_j)_{j=1}^H;\frac{3}{2} & ; & \end{matrix}\right]-$$

$$
-\frac{y\, z\, \sin^3\gamma\, \cos\gamma \prod\limits_{i=1}^{A}(a_i) \prod\limits_{i=1}^{A}(1+a_i) \prod\limits_{i=1}^{D}(d_i) \prod\limits_{i=1}^{G}(g_i)}{4\prod\limits_{i=1}^{B}(b_i) \prod\limits_{i=1}^{B}(1+b_i) \prod\limits_{i=1}^{E}(e_i) \prod\limits_{i=1}^{H}(h_i)} \times
$$

$$
\times F^{(3)}\left[\begin{array}{lr} \frac{5}{2},(2+a_j)_{j=1}^{A}::-;(1+g_j)_{j=1}^{G}\;\; ;2:1,(1+d_j)_{j=1}^{D};1\,;1 \quad ; & \\ & y\sin^2\gamma,\, z,\, z\sin^2\gamma \\ 3,\,(2+b_j)_{j=1}^{B}::-;(1+h_j)_{j=1}^{H},2;\frac{5}{2}:2,(1+e_j)_{j=1}^{E};-;- \quad ; & \end{array}\right] \tag{4.5}
$$

$$
\int_0^{\gamma} F^{A:D;G}_{B:E;H}\left[\begin{array}{lr} (a_j)_{j=1}^{A}:(d_j)_{j=1}^{D};(g_j)_{j=1}^{G} \quad ; & \\ & y\cos^2\theta,\, z\cos^2\theta \\ (b_j)_{j=1}^{B}:(e_j)_{j=1}^{E};(h_j)_{j=1}^{H} \quad ; & \end{array}\right] \mathrm{d}\theta
$$

$$
= \gamma\, F^{A+1:D;G}_{B+1:E;H}\left[\begin{array}{lr} (a_j)_{j=1}^{A},\,\frac{1}{2}:(d_j)_{j=1}^{D};(g_j)_{j=1}^{G} \quad ; & \\ & y,\, z \\ (b_j)_{j=1}^{B},\,1:(e_j)_{j=1}^{E};(h_j)_{j=1}^{H} \quad ; & \end{array}\right] + \frac{z\,\sin\gamma\,\cos\gamma \prod\limits_{i=1}^{A}(a_i) \prod\limits_{i=1}^{G}(g_i)}{2\prod\limits_{i=1}^{B}(b_i) \prod\limits_{i=1}^{H}(h_i)} \times
$$

$$
\times F^{A+G+1:1;2}_{B+H+2:0;1}\left[\begin{array}{lr} \frac{3}{2},\,(1+a_j)_{j=1}^{A},\,(1+g_j)_{j=1}^{G} \;:1\,;1,1 \quad ; & \\ & z,\, z\,\cos^2\gamma \\ 2,\,2,\,(1+b_j)_{j=1}^{B},\,(1+h_j)_{j=1}^{H}:-;\,\frac{3}{2} \quad ; & \end{array}\right] +
$$

$$
+\frac{y\,\sin\gamma\,\cos\gamma \prod\limits_{i=1}^{A}(a_i) \prod\limits_{i=1}^{D}(d_i)}{2\prod\limits_{i=1}^{B}(b_i) \prod\limits_{i=1}^{E}(e_i)} \times
$$

$$
\times F^{(3)}\left[\begin{array}{lr} \frac{3}{2},(1+a_j)_{j=1}^{A}::-;-;(1+d_j)_{j=1}^{D} \;\;:1\,;(g_j)_{j=1}^{G};\;1,1 \quad ; & \\ & y,\, z,\, y\cos^2\gamma \\ 2,\,(1+b_j)_{j=1}^{B}::-;-;(1+e_j)_{j=1}^{E},2:-;(h_j)_{j=1}^{H};\;\frac{3}{2} \quad ; & \end{array}\right] +
$$

$$
+\frac{y\, z\,\sin\gamma\,\cos^3\gamma \prod\limits_{i=1}^{A}(a_i) \prod\limits_{i=1}^{A}(1+a_i) \prod\limits_{i=1}^{D}(d_i) \prod\limits_{i=1}^{G}(g_i)}{4\prod\limits_{i=1}^{B}(b_i) \prod\limits_{i=1}^{B}(1+b_i) \prod\limits_{i=1}^{E}(e_i) \prod\limits_{i=1}^{H}(h_i)} F^{(3)}\left[\begin{array}{l} \frac{5}{2},(2+a_j)_{j=1}^{A}::-;(1+g_j)_{j=1}^{G} \quad ;2: \\ \\ 3,\,(2+b_j)_{j=1}^{B}::-;(1+h_j)_{j=1}^{H},2\;;\frac{5}{2}: \end{array}\right.
$$

$$
\left.\begin{array}{lr} 1,(1+d_j)_{j=1}^{D};\;1\;;\;1 \quad ; & \\ & y\cos^2\gamma,\, z,\, z\cos^2\gamma \\ 2,(1+e_j)_{j=1}^{E};-;- \quad ; & \end{array}\right] \tag{4.6}
$$

$$
\int_0^{\gamma} {}_AF_B\left[\begin{array}{lr} (a_j)_{j=1}^{A} \quad ; & \\ & y\sin(c\theta) \\ (b_j)_{j=1}^{B} \quad ; & \end{array}\right] \mathrm{d}\theta
$$

$$= \gamma\; {}_{2A}F_{2B+1}\left[\begin{array}{lr} (\frac{a_j}{2})_{j=1}^{A},\, (\frac{1+a_j}{2})_{j=1}^{A} \;; & \\ & \frac{y^2}{4^{(1+B-A)}} \\ 1,\, (\frac{b_j}{2})_{j=1}^{B},\, (\frac{1+b_j}{2})_{j=1}^{B} \;; & \end{array}\right] - \frac{y\cos(c\gamma)\prod_{i=1}^{A}(a_i)}{c\prod_{i=1}^{B}(b_i)}\times$$

$$\times F^{2A\;:1;1}_{2B+2:0;0}\left[\begin{array}{lr} (\frac{1+a_j}{2})_{j=1}^{A},\, (\frac{2+a_j}{2})_{j=1}^{A} : 1\,;\, \frac{1}{2} \;; & \\ & \frac{y^2}{4^{(1+B-A)}},\, \frac{y^2\sin^2(c\gamma)}{4^{(1+B-A)}} \\ \frac{3}{2},\, \frac{3}{2},\, (\frac{1+b_j}{2})_{j=1}^{B},\, (\frac{2+b_j}{2})_{j=1}^{B} : \text{—}\,;\, \text{—} \;; & \end{array}\right] -$$

$$- \frac{y^2\sin(c\gamma)\cos(c\gamma)\prod_{i=1}^{A}(a_i)\prod_{i=1}^{A}(1+a_i)}{4c\prod_{i=1}^{B}(b_i)\prod_{i=1}^{B}(1+b_i)}\times$$

$$\times F^{2A\;:1;2}_{2B+2:0;1}\left[\begin{array}{lr} (\frac{2+a_j}{2})_{j=1}^{A},\, (\frac{3+a_j}{2})_{j=1}^{A} : 1\,;\, 1, 1 \;; & \\ & \frac{y^2}{4^{(1+B-A)}},\, \frac{y^2\sin^2(c\gamma)}{4^{(1+B-A)}} \\ 2,\, 2,\, (\frac{2+b_j}{2})_{j=1}^{B},\, (\frac{3+b_j}{2})_{j=1}^{B} : \text{—}\,;\, \frac{3}{2} \;; & \end{array}\right] +$$

$$+ \frac{y\prod_{i=1}^{A}(a_i)}{c\prod_{i=1}^{B}(b_i)}\; {}_{2A+1}F_{2B+2}\left[\begin{array}{lr} 1, (\frac{1+a_j}{2})_{j=1}^{A},\, (\frac{2+a_j}{2})_{j=1}^{A} \;; & \\ & \frac{y^2}{4^{(1+B-A)}} \\ \frac{3}{2},\, \frac{3}{2},\, (\frac{1+b_j}{2})_{j=1}^{B},\, (\frac{2+b_j}{2})_{j=1}^{B} \;; & \end{array}\right] \tag{4.7}$$

$$\int_0^{\gamma} {}_AF_B\left[\begin{array}{lr} (a_j)_{j=1}^{A} \;; & \\ & y\cos(c\theta) \\ (b_j)_{j=1}^{B} \;; & \end{array}\right] \mathrm{d}\theta$$

$$= \gamma\; {}_{2A}F_{2B+1}\left[\begin{array}{lr} (\frac{a_j}{2})_{j=1}^{A},\, (\frac{1+a_j}{2})_{j=1}^{A} \;; & \\ & \frac{y^2}{4^{(1+B-A)}} \\ 1,\, (\frac{b_j}{2})_{j=1}^{B},\, (\frac{1+b_j}{2})_{j=1}^{B} \;; & \end{array}\right] + \frac{y\sin(c\gamma)\prod_{i=1}^{A}(a_i)}{c\prod_{i=1}^{B}(b_i)}\times$$

$$\times F^{2A\;:1;1}_{2B+2:0;0}\left[\begin{array}{lr} (\frac{1+a_j}{2})_{j=1}^{A},\, (\frac{2+a_j}{2})_{j=1}^{A} : 1\,;\, \frac{1}{2} \;; & \\ & \frac{y^2}{4^{(1+B-A)}},\, \frac{y^2\cos^2(c\gamma)}{4^{(1+B-A)}} \\ \frac{3}{2},\, \frac{3}{2},\, (\frac{1+b_j}{2})_{j=1}^{B},\, (\frac{2+b_j}{2})_{j=1}^{B} : \text{—}\,;\, \text{—} \;; & \end{array}\right] +$$

$$+ \frac{y^2\sin(c\gamma)\cos(c\gamma)\prod_{i=1}^{A}(a_i)\prod_{i=1}^{A}(1+a_i)}{4c\prod_{i=1}^{B}(b_i)\prod_{i=1}^{B}(1+b_i)}\times$$

$$\times F^{2A\;:1;2}_{2B+2:0;1}\left[\begin{array}{lr} (\frac{2+a_j}{2})_{j=1}^{A},\, (\frac{3+a_j}{2})_{j=1}^{A} : 1\,;\, 1, 1 \;; & \\ & \frac{y^2}{4^{(1+B-A)}},\, \frac{y^2\cos^2(c\gamma)}{4^{(1+B-A)}} \\ 2,\, 2,\, (\frac{2+b_j}{2})_{j=1}^{B},\, (\frac{3+b_j}{2})_{j=1}^{B} : \text{—}\,;\, \frac{3}{2} \;; & \end{array}\right] \tag{4.8}$$

$$\int_0^{\gamma} \sin\theta \; {}_AF_B\left[\begin{array}{ll} (a_j)_{j=1}^{A} ; & \\ & y\sin^2\theta \\ (b_j)_{j=1}^{B} ; & \end{array}\right] d\theta = {}_{A+1}F_{B+1}\left[\begin{array}{ll} 1, (a_j)_{j=1}^{A} ; & \\ & y \\ \frac{3}{2}, (b_j)_{j=1}^{B} ; & \end{array}\right] -$$

$$- \cos\gamma \; F^{A\;:1;1}_{B+1:0;0}\left[\begin{array}{ll} (a_j)_{j=1}^{A} :1 ; \frac{1}{2} ; & \\ & y,\; y\sin^2\gamma \\ \frac{3}{2},\; (b_j)_{j=1}^{B} : -;- ; & \end{array}\right] \tag{4.9}$$

$$\int_0^{\gamma} \cos\theta \; {}_AF_B\left[\begin{array}{ll} (a_j)_{j=1}^{A} ; & \\ & y\cos^2\theta \\ (b_j)_{j=1}^{B} ; & \end{array}\right] d\theta = \sin\gamma \; F^{A\;:1;1}_{B+1:0;0}\left[\begin{array}{ll} (a_j)_{j=1}^{A} :1 ; \frac{1}{2} ; & \\ & y,\; y\cos^2\gamma \\ \frac{3}{2},\; (b_j)_{j=1}^{B} : -;- ; & \end{array}\right] \tag{4.10}$$

$$\int_0^{\gamma} \sin\theta \; F^{A:D;G}_{B:E;H}\left[\begin{array}{ll} (a_j)_{j=1}^{A} : (d_j)_{j=1}^{D} ; (g_j)_{j=1}^{G} ; & \\ & y\sin^2\theta,\; z\sin^2\theta \\ (b_j)_{j=1}^{B} : (e_j)_{j=1}^{E} ; (h_j)_{j=1}^{H} ; & \end{array}\right] d\theta$$

$$= F^{A+1:D;G}_{B+1:E;H}\left[\begin{array}{ll} 1,\; (a_j)_{j=1}^{A} : (d_j)_{j=1}^{D} ; (g_j)_{j=1}^{G} ; & \\ & y,\; z \\ \frac{3}{2},\; (b_j)_{j=1}^{B} : (e_j)_{j=1}^{E} ; (h_j)_{j=1}^{H} ; & \end{array}\right] -$$

$$- \cos\gamma \; F^{(3)}\left[\begin{array}{ll} 1,\; (a_j)_{j=1}^{A} :: -;-;(d_j)_{j=1}^{D} :1 ;(g_j)_{j=1}^{G} ; \frac{1}{2} ; & \\ & y,\; z,\; y\sin^2\gamma \\ \frac{3}{2},\; (b_j)_{j=1}^{B} :: -;-;1,\; (e_j)_{j=1}^{E} : -;(h_j)_{j=1}^{H} ; - ; & \end{array}\right] -$$

$$- \frac{z\sin^2\gamma\cos\gamma \prod_{i=1}^{A}(a_i) \prod_{i=1}^{G}(g_i)}{3\prod_{i=1}^{B}(b_i) \prod_{i=1}^{H}(h_i)} \times$$

$$\times F^{(3)}\left[\begin{array}{ll} 2,\; (1+a_j)_{j=1}^{A} :: -;(1+g_j)_{j=1}^{G} ; \frac{3}{2} : (d_j)_{j=1}^{D} ; 1 ; 1 ; & \\ & y\sin^2\gamma,\; z,\; z\sin^2\gamma \\ \frac{5}{2},\; (1+b_j)_{j=1}^{B} :: -;2,\; (1+h_j)_{j=1}^{H} ; 2 : (e_j)_{j=1}^{E} ; - ; - ; & \end{array}\right] \tag{4.11}$$

$$\int_0^{\gamma} \cos\theta \; F^{A:D;G}_{B:E;H}\left[\begin{array}{ll} (a_j)_{j=1}^{A} : (d_j)_{j=1}^{D} ; (g_j)_{j=1}^{G} ; & \\ & y\cos^2\theta,\; z\cos^2\theta \\ (b_j)_{j=1}^{B} : (e_j)_{j=1}^{E} ; (h_j)_{j=1}^{H} ; & \end{array}\right] d\theta$$

$$= \sin\gamma \; F^{(3)}\left[\begin{array}{ll} 1,\; (a_j)_{j=1}^{A} :: -;-;(d_j)_{j=1}^{D} :1 ;(g_j)_{j=1}^{G} ; \frac{1}{2} ; & \\ & y,\; z,\; y\cos^2\gamma \\ \frac{3}{2},\; (b_j)_{j=1}^{B} :: -;-;1,\; (e_j)_{j=1}^{E} : -;(h_j)_{j=1}^{H} ; - ; & \end{array}\right] +$$

$$+ \frac{z\sin\gamma\cos^2\gamma \prod_{i=1}^{A}(a_i) \prod_{i=1}^{G}(g_i)}{3\prod_{i=1}^{B}(b_i) \prod_{i=1}^{H}(h_i)} \times$$

$$\times F^{(3)}\left[\begin{array}{l} 2,\,(1+a_j)_{j=1}^{A} :: -\,;\;\; (1+g_j)_{j=1}^{G}\;\; ;\, \frac{3}{2} : (d_j)_{j=1}^{D}\,;\;\; 1\,;\; 1\;\;\; ; \\ \hspace{20em} y\cos^2\gamma,\, z,\, z\cos^2\gamma \\ \frac{5}{2},\,(1+b_j)_{j=1}^{B} :: -\,; 2,\,(1+h_j)_{j=1}^{H}\,; 2 : (e_j)_{j=1}^{E}\;\, ;\, -\,; -\;\;\; ; \end{array}\right] \tag{4.12}$$

provided that each of the series as well as associated integrals involved are convergent.

5 Solutions of Ramanujan's incomplete elliptic integrals

Setting $A = 1, B = 0$ and $a_1 = \frac{1}{2}$ in (4.1) and (4.2) respectively, we get

$$F(\sqrt{y},\,\gamma) = \int_0^{\gamma} \frac{d\theta}{\sqrt{(1-y\sin^2\theta)}} = \gamma\, {}_2F_1\left[\begin{array}{l} \frac{1}{2},\frac{1}{2}\;\; ; \\ \hspace{4em} y \\ 1 \;\;\;\;\; ; \end{array}\right] - \frac{y\sin\gamma\,\cos\gamma}{4}\times$$

$$\times F^{2:1;2}_{2:0;1}\left[\begin{array}{l} \frac{3}{2},\frac{3}{2}: 1\,; 1,1 \;\; ; \\ \hspace{8em} y, y\sin^2\gamma \\ 2,2 : -;\; \frac{3}{2} \;\;\;\; ; \end{array}\right] \quad ;|y|<1 \tag{5.1}$$

which is the exact solution of incomplete elliptic integral of first kind.

$$\int_0^{\gamma} \frac{d\theta}{\sqrt{(1-y\cos^2\theta)}} = \gamma\, {}_2F_1\left[\begin{array}{l} \frac{1}{2},\frac{1}{2}\;\; ; \\ \hspace{4em} y \\ 1 \;\;\;\;\; ; \end{array}\right] + \frac{y\sin\gamma\,\cos\gamma}{4}\times$$

$$\times F^{2:1;2}_{2:0;1}\left[\begin{array}{l} \frac{3}{2},\frac{3}{2}: 1; 1,1 \;\; ; \\ \hspace{8em} y, y\cos^2\gamma \\ 2,2 : -;\; \frac{3}{2} \;\;\;\; ; \end{array}\right] \quad ;|y|<1 \tag{5.2}$$

Putting $\gamma = \frac{\pi}{2}$ in (5.1) and (5.2), we get

$$\mathbf{K}(\sqrt{y}) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(1-y\sin^2\theta)}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(1-y\cos^2\theta)}} = \frac{\pi}{2}\, {}_2F_1\left[\begin{array}{l} \frac{1}{2},\frac{1}{2}\;\; ; \\ \hspace{4em} y \\ 1 \;\;\;\;\; ; \end{array}\right] \quad ;|y|<1 \tag{5.3}$$

which is well-known complete elliptic integral of first kind.

Putting $A = 1, B = 0$ and $a_1 = -\frac{1}{2}$ in (4.1)and (4.2)respectively, we get

$$E(\sqrt{y},\,\gamma) = \int_0^{\gamma} \sqrt{(1-y\sin^2\theta)}\,d\theta = \gamma_2F_1\left[\begin{array}{l} \frac{1}{2},-\frac{1}{2}\;\; ; \\ \hspace{4em} y \\ 1 \;\;\;\;\; ; \end{array}\right] + \frac{y\sin\gamma\cos\gamma}{4}\times$$

$$\times F^{2:1;2}_{2:0;1}\left[\begin{array}{ll} \frac{1}{2},\frac{3}{2}:1;1,1 & ; \\ & y, y\sin^2\gamma \\ 2,2:\text{—};\ \frac{3}{2} & ; \end{array}\right] \quad ;|y|<1 \tag{5.4}$$

which is the exact solution of incomplete elliptic integral of second kind.

$$\int_0^{\gamma}\sqrt{(1-y\cos^2\theta)}\,\mathrm{d}\theta = \gamma\,{}_2F_1\left[\begin{array}{ll} \frac{1}{2},-\frac{1}{2} & ; \\ & y \\ 1 & ; \end{array}\right] - \frac{y\sin\gamma\cos\gamma}{4}\times$$

$$\times F^{2:1;2}_{2:0;1}\left[\begin{array}{ll} \frac{1}{2},\frac{3}{2}:1;1,1 & ; \\ & y, y\cos^2\gamma \\ 2,2:\text{—};\ \frac{3}{2} & ; \end{array}\right] \quad ;|y|<1 \tag{5.5}$$

Setting $\gamma=\frac{\pi}{2}$ in (5.4) and (5.5), we get

$$\mathbf{E}(\sqrt{y}) = \int_0^{\frac{\pi}{2}}\sqrt{(1-y\sin^2\theta)}\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}}\sqrt{(1-y\cos^2\theta)}\,\mathrm{d}\theta = \frac{\pi}{2}\,{}_2F_1\left[\begin{array}{ll} -\frac{1}{2},\frac{1}{2} & ; \\ & y \\ 1 & ; \end{array}\right] \quad ;|y|<1 \tag{5.6}$$

which is well-known complete elliptic integral of second kind.

Putting $A=1, B=0,\ a_1=\frac{1}{2},\ \gamma=\beta$ and $y=x^2$ in (4.3), we get

$$\int_0^{\beta}\frac{\mathrm{d}\theta}{\sqrt{(1-x^2\sin^4\theta)}} = \beta\,{}_2F_1\left[\begin{array}{ll} \frac{1}{4},\frac{3}{4} & ; \\ & x^2 \\ 1 & ; \end{array}\right] - \frac{3x^2\sin\beta\cos\beta}{16}\times$$

$$\times F^{2:1;2}_{2:0;1}\left[\begin{array}{ll} \frac{5}{4},\frac{7}{4}:1;1,1 & ; \\ & x^2, x^2\sin^2\beta \\ 2,2:\text{—};\ \frac{3}{2} & ; \end{array}\right] - \frac{x^2\sin^3\beta\cos\beta}{8}\times$$

$$\times F^{3:1;1}_{3:0;0}\left(\begin{array}{ll} [\frac{5}{4}:1,1],\ [\frac{7}{4}:1,1],\ [2:1,2]:\ [1:1];\ [1:1] & ; \\ & x^2\sin^2\beta,\ x^2\sin^4\beta \\ [2:1,1],\ [2:1,1],\ [\frac{5}{2}:1,2]:\ \text{—};\ \text{—} & ; \end{array}\right) \quad ;|x|<1 \tag{5.7}$$

which is the exact solution of unsolved-incomplete elliptic integral given in Ramanujan's Entry 7(iv).

Setting $A=B=E=H=0,\ D=G=1,\ d_1=g_1=\frac{1}{2},\ \gamma=\beta,\ y=a,\ z=b$ in (4.5), we get

$$\int_0^{\beta}\frac{\mathrm{d}\theta}{\sqrt{(1-a\sin^2\theta)(1-b\sin^2\theta)}} = \beta\,\mathrm{F}^{1:1;1}_{1:0;0}\left[\begin{array}{ll} \frac{1}{2}:\ \frac{1}{2};\ \frac{1}{2} & ; \\ & a,\ b \\ 1:\text{—};\text{—} & ; \end{array}\right] -$$

$$-\frac{b\sin\beta\cos\beta}{4}\,F^{2:1;2}_{2:0;1}\left[\begin{array}{ll} \frac{3}{2},\ \frac{3}{2}:1;1,1 & ; \\ & b,\ b\sin^2\beta \\ 2,\ 2:\text{—};\ \frac{3}{2} & ; \end{array}\right] -$$

$$-\frac{a\sin\beta\cos\beta}{4}F^{(3)}\left[\begin{array}{ll}\frac{3}{2}::-;-;\frac{3}{2}:1;\frac{1}{2};1,1 ; & \\ & a,\, b,\, a\sin^2\beta \\ 2::-;-;2:-;-;\ \frac{3}{2} ; & \end{array}\right]-$$

$$-\frac{a\,b\sin^3\beta\cos\beta}{16}F^{(3)}\left[\begin{array}{ll}\frac{5}{2}::-;\frac{3}{2};2:1,\frac{3}{2};1;1 ; & \\ & a\sin^2\beta,\, b,\, b\sin^2\beta \\ 3::-;2\ ;\frac{5}{2}:2;-;- ; & \end{array}\right] \quad ;\max\{|a|,|b|\}<1 \tag{5.8}$$

which is the exact solution of unsolved-incomplete elliptic integral given in Ramanujan's Entry 7(iii). Here $F^{1:1;1}_{1:0;0}[\,\cdot\,]$ is Appell's function of first kind, in the notation of Srivastava and Panda.

Setting $A=B=E=H=0,\ D=G=1,\ d_1=1,\ g_1=\frac{1}{2},\ \gamma=\phi,\ y=a,\ z=b^2$ in (4.5), we get

$$\prod(a,\,b,\,\phi)=\int_0^{\phi}\frac{d\theta}{(1-a\sin^2\theta)\sqrt{(1-b^2\sin^2\theta)}}=\phi\,F^{1:1;1}_{1:0;0}\left[\begin{array}{ll}\frac{1}{2}:1;\ \frac{1}{2} ; & \\ & a,\, b^2 \\ 1:-;- ; & \end{array}\right]-$$

$$-\frac{b^2\sin\phi\cos\phi}{4}F^{2:1;2}_{2:0;1}\left[\begin{array}{ll}\frac{3}{2},\frac{3}{2}:1;\ 1,1 ; & \\ & b^2,\, b^2\sin^2\phi \\ 2,\,2:-;\frac{3}{2} ; & \end{array}\right]-$$

$$-\frac{a\sin\phi\cos\phi}{2}F^{(3)}\left[\begin{array}{ll}\frac{3}{2}::-;-;-:1\ ;\ \frac{1}{2}\ ;1,1 ; & \\ & a,\, b^2,\, a\sin^2\phi \\ 2::-;-;-:-;-\ ;\frac{3}{2} ; & \end{array}\right]-$$

$$-\frac{a\,b^2\sin^3\phi\cos\phi}{8}F^{(3)}\left[\begin{array}{ll}\frac{5}{2}::-;\frac{3}{2};2:1\ ;1\ ;1 ; & \\ & a\sin^2\phi,\, b^2,\, b^2\sin^2\phi \\ 3::-;2;\frac{5}{2}:-;-;- ; & \end{array}\right] \tag{5.9}$$

where $\max\{|a|,|b|\}<1$

which is the exact solution of incomplete elliptic integral of third kind.

In (5.9), putting $\phi=\frac{\pi}{2}$ we get the exact solution of complete elliptic integral of third kind.

$$\prod\left(a,\,b,\,\frac{\pi}{2}\right)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1-a\sin^2\theta)\sqrt{(1-b^2\sin^2\theta)}}=\frac{\pi}{2}F_1\left[\frac{1}{2};1,\frac{1}{2};1;a,b^2\right] \quad ;\max\{|a|,|b|\}<1 \tag{5.10}$$

where $F_1\left[\frac{1}{2};1,\frac{1}{2};1;a,b^2\right]$ is Appell's function of first kind[34,p.53(1.6.4)].

These solutions are not found in Ramanujan's notebooks[29-31], Five notebooks of B. C. Berndt [6-10], Three volumes of R. P. Agarwal [2-4] and other literature [5; 17; 18; 20; 22(pp.100-106); 24; 25; 26; 28; 32; 35] on special functions.

Setting $A=1,\ B=0,\ a_1=\frac{1}{2},\ \gamma=\alpha,\ y=-x,\ c=2$ in (4.8), we get

$$\int_0^{\alpha}\frac{d\theta}{\sqrt{(1+x\cos\ 2\theta)}}=\alpha\,{}_2F_1\left[\begin{array}{lr}\frac{1}{4},\frac{3}{4}\ ; & \\ & x^2\\ 1,\ ; & \end{array}\right]-\frac{x\sin(2\alpha)}{4}\times$$

$$\times\,F^{2:1;1}_{2:0;0}\left[\begin{array}{lr}\frac{3}{4},\frac{5}{4}:1;\frac{1}{2}\ ; & \\ & x^2,\ x^2\cos^2(2\alpha)\\ \frac{3}{2},\frac{3}{2}:-;-\ ; & \end{array}\right]+$$

$$+\frac{3x^2\sin(4\alpha)}{64}F^{2:1;2}_{2:0;1}\left[\begin{array}{lr}\frac{5}{4},\frac{7}{4}:1;1,1\ ; & \\ & x^2,\ x^2\cos^2(2\alpha)\\ 2,2:-;\frac{3}{2}\ ; & \end{array}\right];|x|<1 \tag{5.11}$$

Further on using (5.2) in above integral, we get an elegant formula in the following form:

$$\sqrt{(1-x)}\int_0^{\alpha}\frac{d\theta}{\sqrt{(1+x\cos\ 2\theta)}}=\int_0^{\alpha}\frac{d\theta}{\sqrt{\{1-(\frac{2x}{x-1})\cos^2\theta)\}}}$$

$$=\alpha\,{}_2F_1\left[\begin{array}{lr}\frac{1}{2},\frac{1}{2}\ ; & \\ & \frac{2x}{x-1}\\ 1\ ; & \end{array}\right]+\frac{x\sin 2\alpha}{4(x-1)}F^{2:1;2}_{2:0;1}\left[\begin{array}{lr}\frac{3}{2},\frac{3}{2}:1;1,1\ ; & \\ & \frac{2x}{x-1},\ \frac{2x\cos^2\alpha}{x-1}\\ 2,2:-;\frac{3}{2}\ ; & \end{array}\right];\left|\frac{2x}{x-1}\right|<1 \tag{5.12}$$

which is the exact solution of Ramanujan's unsolved-incomplete elliptic integral given in Entry 7(iv).

6 Special cases

In (4.1), set $A=2,B=1,a_1=b,a_2=1-b,b_1=\frac{1}{2}$ and $\gamma=\frac{\pi}{2}$, we obtain

$$\int_0^{\frac{\pi}{2}}\frac{\cos\{(2b-1)\sin^{-1}(\sqrt{y}\sin\theta)\}}{\sqrt{(1-y\sin^2\theta)}}d\theta=\frac{\pi}{2}\,{}_2F_1\left[\begin{array}{lr}b,1-b\ ; & \\ & y\\ 1\ ; & \end{array}\right] \tag{6.1}$$

which is the known result of Ramanujan [8, p.88(Entry 1)].

In (4.1), put $A=2,B=1,a_1=\frac{1}{3},a_2=\frac{2}{3},b_1=\frac{1}{2}$ and $\gamma=\frac{\pi}{2}$, we obtain

$$\int_0^{\frac{\pi}{2}}{}_2F_1\left[\begin{array}{lr}\frac{1}{3},\frac{2}{3}\ ; & \\ & y\sin^2\theta\\ \frac{1}{2}\ ; & \end{array}\right]d\theta=\frac{\pi}{2}\,{}_2F_1\left[\begin{array}{lr}\frac{1}{3},\frac{2}{3}\ ; & \\ & y\\ 1\ ; & \end{array}\right] \tag{6.2}$$

which is a known result of B.C.Berndt[10, p.133].

In (5.7), set $\beta = \frac{\pi}{2}$ and $x^2 = y$, we have

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(1 - y\sin^4\theta)}} = \frac{\pi}{2}\, {}_2F_1\left[\begin{array}{ccc} \frac{1}{4}, \frac{3}{4} & ; & \\ & & y \\ 1 & ; & \end{array}\right] \tag{6.3}$$

which is another known result of B.C.Berndt[8, p.110].

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Received: March 14, 2014; *Accepted*: August 23, 2014

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