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On graph differential equations and its associated matrix differential equations

J. Vasundhara Devi,[a,*] R.V.G. Ravi Kumar[b] and N. Giribabu[c]

[a,b,c] *GVP-Prof.V.Lakshmikantham Institute for Advanced Studies, Department of Mathematics, GVP College of Engineering, Visakhapatnam-530048, Andhra Pradesh, India.*

Abstract

Networks are one of the basic structures in many physical phenomena pertaining to engineering applications. As a network can be represented by a graph which is isomorphic to its adjacency matrix, the study of analysis of networks involving rate of change with respect to time reduces to the study of graph differential equations or equivalently matrix differential equations. In this paper, we develop the basic infrastructure to study the IVP of a graph differential equation and the corresponding matrix differential equation. Criteria are obtained to guarantee the existence of a solution and an iterative technique for convergence to the solution of a matrix differential equation is developed.

Keywords: Dynamic graph, adjacency matrix, graph linear space, graph differential equations, matrix differential equations, existence of a solution, monotone iterative technique.

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1 Introduction

A graph [1] represents a network of a natural or a man-made system, wherein interconnections between its constituents play an important role. Graphs have been utilized to model organizational structures in social sciences. It has been observed that the graphs which are static in nature limit the study in social phenomena where changes with time are natural. Hence, it was thought that a dynamic graph will be more appropriate in modeling such social behavior [2, 4]. The concept of a dynamic graph was introduced in [2] and a graph differential equation was utilized to describe the famous prey predator model and its stability properties were studied [2].

The importance of networks in engineering fields and the representation of a network by a graph led us to consider a graph differential equation as an important topic of study. Thus we plan to study the existence of solutions through monotone iterative technique [3] for the graph differential equation through its associated matrix differential equations.

2 Preliminaries

In this section we introduce the notions and concepts that are necessary to develop graph differential equations and and the corresponding matrix differential equations. All the basic definitions and results are taken from [2] and suitable changes are made to suit our set up. Consider a weighted directed simple graph (called digraph) $D = (V, E)$ an ordered pair, where V is a non-empty finite set of N vertices and E is the set of all directed edges. To each directed edge (v_i, v_j) we assign a nonzero weight $e_{ij} \in \mathbb{R}$ if $(v_i, v_j) \in E$ while $e_{ij} = 0$ if $(v_i, v_j) \notin E$. Corresponding to a digraph D we associate an adjacency matrix $E = (e_{ij})$. This association is

*Corresponding author.

E-mail address: jvdevi@gmail.com (J. Vasundhara Devi).

an isomorphism.

Graph linear space. Let $v_1, v_2, \dots v_N$ be N vertices, N fixed and $\mathcal{D}_N$ be the set of all weighted directed simple graphs (called digraphs), $D = (V, E)$. Then $(\mathcal{D}_N, +, .)$ is a linear space over the field of real numbers with the following definition of the addition and scalar multiplication.
Let D_1, D_2 be two digraphs $D_1 = (V, E_1)$ and $D_2 = (V, E_2)$.
Then the sum $D_1 + D_2$ is defined as

$$D_1 + D_2 = (V, E_1 + E_2)$$

where $E_1 + E_2$ is the set of all edges $(v_i, v_j) \in E_1 \cup E_2$ where the weight of (v_i, v_j) is defined as the sum of the weights of the edges (v_i, v_j) in the respective digraphs D_1 and D_2.
Let $D = (V, E)$ be a graph then by $\alpha D = (V, \alpha E)$ where αE is the set of all edges (v_i, v_j) whose weight is α times the weight of (v_i, v_j). Observe that if $\alpha = 0$ then $\alpha D = 0 \in \mathcal{D}_N$ is the graph consisting of N isolated vertices. Hence the set of edges is empty. With the fore mentioned operations, $(\mathcal{D}_N, +, .)$ is a linear space. This space is isomorphic to the linear space $\mathcal{M}_N$ of all $N \times N$ adjacency matrices with entries of the principal diagonal being zero, defined over the field of real numbers, with the usual definition of matrix addition and scalar multiplication.
Let γ be a matrix norm defined as

$$\gamma : \mathcal{M}_N \to \mathbb{R}_+ \quad \text{satisfying}$$

(i) $\gamma(m) > 0 \;\; \forall \;\; m \in \mathcal{M}_N, m \neq 0$
(ii) $\gamma(\alpha\, m) = |\alpha|\, \gamma(m), \;\forall\;\; m \in \mathcal{M}_N, \alpha \in \mathbb{R}$
(iii) $\gamma(m_1 + m_2) \leq \gamma(m_1) + \gamma(m_2), \;\forall\;\; m_1, m_2 \in \mathcal{M}_N$.

Once a matrix norm is chosen we can define an associated matrix norm on $\mathcal{D}_N$ and induced metric η is given by

$$\eta(m_1, m_2) = \gamma\,(m_1 - m_2), \;\; \forall\, m_1, m_2 \in \mathcal{M}_N.$$

In order to study graph functions that vary over time, we use an axiomatic definition of the abstract linear space $\mathcal{D}_N$ into itself.
Consider the space $\mathcal{D}_N$ and a family of mappings $\Phi : \mathbb{R}_+ \times \mathcal{D}_N \to \mathcal{D}_N$, where to any graph $D \in \mathcal{D}_N$ and any parameter (time) $t \in \mathbb{R}_+$ assigns a graph $\Phi\,(t, D) \in \mathcal{D}_N$.

Dynamic graph. A dynamic graph $\widehat{D} = \Phi_D(t)$ is a one parameter mapping $\Phi_D : \mathbb{R}_+ \to \mathcal{D}_N$ with $\Phi_D(t) = \Phi\,(t, D) \in \mathcal{D}_N$ satisfying the following axioms.
(i). $\Phi\,(t_0, D_0) = D_0$
(ii). Φ is continuous
(iii). $\Phi(t_2, \Phi\,(t_1, D)) = \Phi(t_1 + t_2, D), \;\forall\;\; t_1, t_2 \in \mathbb{R}_+, \forall\, D \in \mathcal{D}_N$.

The first axiom establishes $D(t_0) = D_0$ as the initial graph. The second axiom requires continuity of mapping $\Phi\,(t, D)$ with respect to t and D which includes continuity with respect to t_0 and D_0. The third axiom establishes that dynamic graph D as a one parameter graph $\Phi\,(t, D)$ of transformations of the space $\mathcal{D}_N$ into itself. Corresponding to a dynamic graph the dynamic adjacency matrix is defined as follows.

Definition 2.1. *A dynamic adjacency matrix $\widehat{E}$ is a one-parameter mapping $\psi : \mathbb{R}_+ \times \mathcal{E}_N \to \mathcal{E}_N$ of the space $\mathcal{E}_N$ into itself satisfying the following axioms.*

(i) $\psi(t_0, E_0) = E_0$.

(ii) $\psi(t, E)$ *is continuous.*

(iii) $\psi(t_2, \psi(t_1, E)) = \psi(t_1 + t_2, E), \;\forall\; t_1, t_2 \in \mathbb{R}_+$ *and* $\forall\; E \in \mathcal{E}_N$.

Examples.

A dynamic graph can be defined by the corresponding adjacency matrix and a few examples are given below.
(1) Let $\psi(t, E) = E, \;\forall\;\; t \in \mathbb{R}_+, \;\forall\; E \;\in \mathcal{D}_N$

Then $\psi(t_0, E_0) = E_0$ and $\psi(t, E) = E$ is continuous $\forall\; t$ and $\forall\; E$ and

$$\psi(t_2, \psi(t_1, E)) \;\; = \;\; \psi(t_2, E) = E$$

$$= \quad \psi(t_1+t_2,E)$$

Therefore, the dynamic graph $\widehat{D}=D$ for all $t\in\mathbb{R}_+$.

(2) Let $\psi(t,E)=E_0\ \ \forall\ \ t\in\mathbb{R}_+,\ \forall\ E\ \in\mathcal{E}_N$. Then the dynamic graph $\widehat{D}=E_0$ for all $t\in\mathbb{R}_+$.

(3) Let $\psi(t,E)=t\,E_0+E,\ \forall\ t\in\mathbb{R}_+,\ \forall\ E\ \in\ \mathcal{E}_N$ and E_0 be any initial adjacency matrix.
Then $\psi(0,E_0)=E_0$
and $\psi(t_n,E)\to\psi(t_0,E)$ whenever $t_n\to t_0$
and $\psi(t,E_n)\to\psi(t,E_0)$ whenever $E_n\to E_0$

$$\begin{aligned}\text{Further}\quad \psi(t_2,\psi(t_1,E)) &= \psi(t_2,t_1E_0+E)\\ &= t_2\,E_0+(t_1E_0+E)\\ &= (t_1+t_2)\,E_0+E\\ &= \psi(t_1+t_2,E).\end{aligned}$$

Therefore, the dynamic graph $\widehat{D}=tE_0+D$ for all $t\in\mathbb{R}_+$.

Motion of the graph. The mapping $\Phi(t,D)=\widehat{D}$ is called the motion of the graph. The mapping $\psi(t,E)$ is called as the motion of the adjacency matrix $\widehat{E}$. A graph D^e satisfying $\Phi(t,D^e)=D^e$ is called as the equilibrium graph.

In order to define the time evolution of a graph one needs the concept of a derivative in the abstract space, we can use the theory of abstract differential equations. Introducing the concept of Frechet derivative, if it exists on the notion of a generalized derivative we consider the time-evolution of a dynamic graph abstractly by the equation $\Delta D=\mathcal{G}(t,D)$ where ΔD represents the tendency of the graph to change in time t.

In order to introduce the corresponding concept in the adjacency matrices we need the following notions.

(1) The adjacency matrix $\widehat{E}=E(t)$ is said to be continuous if the entry $e_{ij}(t)$ is continuous for all $i,j=1,2,\dots N$.

(2) The continuous adjacency matrix $\widehat{E}=E(t)$ is said to be differentiable if each continuous entry $e_{ij}(t)$ is differentiable for all $i,j=1,2,\dots N$, and is denoted by $E'=(e'_{ij})_{N\times N}$. With the above definitions in place we can express the corresponding changes in an adjacency matrix that evolved in time 't' for a dynamic graph by the equation
$$\frac{dE}{dt}=F(t,E).$$
With the concept of rate of change of a graph with respect to time t, one can consider the differentiable equation in the abstract space $\mathcal{D}_N$. Using the theory of differential equations in abstract spaces one can study the graph differential equations.

An alternative approach that is more useful for practical purposes would be to consider the corresponding adjacency matrix differential equation or simply the matrix differential equation.

3 Linear Matrix differential equations

In this section, we study a graph differential equation that can be expressed as a linear matrix differential equation. Now consider a matrix differential equation (MDE) given by
$$E'=F(t,E).$$
where $F(t,E)$ is a $N\times N$ matrix in which each entry $f_{ij}(t)$ is a function of t,e_{ij} where $i,j=1,2,\dots,N$ and satisfies certain smoothness conditions.

In order to analyze the graph differential equation through the Matrix differential equation (MDE) we first consider those equations that can be transformed to a linear system.

Consider the IVP of a MDE, corresponding to some graph differential equation, given by

$$\left.\begin{aligned}E' &= F(t,E)\\ E(t_0) &= E_0=(k_{ij})_{N\times N}\end{aligned}\right\}\tag{3.1}$$

where $F : I \times \mathcal{E}_N \to \mathcal{E}_N$ is continuous, $I = [t_0, T]$. This means that $F(t, E) = (f_{ij}(t, e_{11}, e_{12}, \ldots, e_{1N}, e_{21}, e_{22}, \ldots, e_{2N}, \ldots, e_{N1}, e_{N2}, \ldots, e_{NN}))_{N\times N}$ and f_{ij} is a continuous, real valued function. Suppose that $f_{ij}(t)$ are linear combinations of the functions $e_{ij}(t)$. Then the system (3.1) can be written as a linear system

$$\left.\begin{array}{rcl} X' & = & AX \\ X(t_0) & = & X_0 \end{array}\right\} \tag{3.2}$$

where X is the vector given by

$$X^T = [e_{11}, e_{12}, \ldots, e_{1N}, e_{21}, e_{22}, \ldots, e_{2N}, \ldots, e_{N1}, e_{N2}, \ldots, e_{NN}],$$

A is $N^2 \times N^2$ coefficient matrix and

$$X_0^T = [k_{11}, k_{12}, \ldots, k_{1N}, k_{21}, k_{22}, \ldots, k_{2N}, \ldots, k_{N1}, k_{N2}, \ldots, k_{NN}].$$

As the qualitative theory of the system (3.2) is well established, using it one can easily analyze the linear system (3.2) and the corresponding graph differential equation.
Next suppose that MDE(3.2) along with its initial condition is of the form

$$\left.\begin{array}{rcl} E' & = & AE \\ E(t_0) & = & E_0 = (k_{ij})_{N\times N} \end{array}\right\} \tag{3.3}$$

where A is the coefficient matrix of order $N \times N$. The system (3.2) can be considered as N subsystems given by

$$X_j' = AX_j, \;\; X_j(t_0) = k_j, \; j = 1, 2, \ldots, N \tag{3.4}$$

where $X_j = \begin{pmatrix} e_{1j} \\ e_{2j} \\ \vdots \\ e_{Nj} \end{pmatrix}$ and $K_j = \begin{pmatrix} k_{1j} \\ k_{2j} \\ \vdots \\ k_{Nj} \end{pmatrix}$.

The N subsystems given by (3.4) can be completely understood through the theory of ordinary differential systems and the corresponding graph differential equation can be analyzed.

4 Nonlinear matrix differential equation

We proceed to introduce an initial value problem of the nonlinear matrix differential equation in this section. Further we prove some basic inequality theorems.
Consider the Matrix differential equation (MDE) given by,

$$\left.\begin{array}{rcl} E' & = & F(t, E) \\ E(t_0) & = & E_0, \end{array}\right\} \tag{4.1}$$

where $E' = (e'_{ij})_{N\times N}$ and $F(t, E)$ is the matrix given by $F(t, E) = (f_{ij}(t, e_{rs}))$ $i, j = 1, 2, \ldots, N$ and $r, s = 1, 2, \ldots, N$ and f_{ij} are real valued functions which are nonlinear in terms of the entries e_{rs}. In order to study the MDE (4.1) we need to devlop new notions that would help us to develop basic Matrix differential inequality results. We begin with a Partial Order $\leq$.

Definition 4.1. *Consider two matrices A and B of order N. We say that $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for all $i, j = 1, 2, \ldots, N$.*

Definition 4.2. *A matrix function $E : I \to \mathcal{E}_N$ defined by $E(t) = (e_{ij}(t))$ is said to be continuous if and only if $e_{ij} : I \to \mathbb{R}$ is continuous for all $i, j = 1, 2, \ldots, N$.*

Definition 4.3. *A matrix function $E : I \to \mathcal{E}_N$ is said to be continuous and differentiable if and only if $e_{ij} : I \to \mathbb{R}$ is continuous and differentiable for all $i, j = 1, 2, \ldots, N$.*

Definition 4.4. *By a solution of the IVP(4.1) we mean a matrix function $E : I \to \mathcal{E}_N$ which is continuous, differentiable and satisfies the equation(4.1) along with the initial condition.*

In order to state the basic differential inequality theorem we introduce the following notions.

Definition 4.5. *By a lower solution of the MDE(4.1) we mean a continuous differentiable matrix function $V(t)$ satisfying the inequalities*

$$V' \leq F(t,V), \quad V(t_0) \leq E_0 \tag{4.2}$$

Definition 4.6. *By a upper solution of the MDE(4.1) we mean a continuous differentiable matrix function $W(t)$ satisfying the inequalities*

$$W' \geq F(t,W), \quad W(t_0) \geq E_0 \tag{4.3}$$

Definition 4.7. *A function $F(t,U) \in C[I \times \mathcal{E}_N, \mathcal{E}_N]$ is said to be quasi monotone nondecreasing in U for each t, if and only if $V \leq W$ and $v_{mn} = w_{mn}$ for some m,n implies $f_{ij}(t,V(t)) \leq f_{ij}(t,W(t))$ for all $i,j = 1,2,\ldots,N$.*

The basic matrix differential inequlity results is given below.

Theorem 4.1. *Assume that*

$$V' \leq F(t,V) \tag{4.4}$$

$$W' \geq F(t,W) \tag{4.5}$$

where $V,W \in C^1[I,\mathcal{E}_N]$ and $F \in C[I \times \mathcal{E}_N, \mathcal{E}_N]$ and $F(t,U)$ be quasi monotone nondecreasing in U for each t. Further assume that $V_0 < W_0$ where $V(t_0) = V_0 \in \mathcal{E}_N$ and $W(t_0) = W_0 \in \mathcal{E}_N$. Then $V(t) < W(t),\ t \in I$, where $I = [t_0, T]$ provided one of the inequalities in (4.4) and (4.5) is strict.

Proof. Assume that $V' \leq F(t,V),\ W' > F(t,W)$. Suppose that the conclusion does not hold. Then there exists an element $t_1 \in I$ such that $V(t) < W(t)$ for $t_0 < t < t_1$ and there exists a pair of indices k and l such that $v_{kl}(t_1) = w_{kl}(t_1)$. Now since $F(t,U)$ is quasi monotone nondecreasing in U, this implies that

$$f_{ij}(t,V(t)) \leq f_{ij}(t,W(t)),\ t \in I \tag{4.6}$$

for $i,j = 1,2,\ldots,N$. Further $v_{kl}(t) < w_{kl}(t), t_0 < t < t_1$ and $v_{kl}(t_1) = w_{kl}(t_1)$ implies for small $h < 0,\ \ v_{kl}(t_1 + h) - v_{kl}(t_1) < w_{kl}(t_1 + h) - w_{kl}(t_1)$, which further implies that

$$\frac{v_{kl}(t_1+h) - v_{kl}(t_1)}{h} > \frac{w_{kl}(t_1+h) - w_{kl}(t_1)}{h}$$

taking limit as $h \to 0$, we get

$$v'_{kl}\,(t_1) \geq w'_{kl}\,(t_1) \tag{4.7}$$

Using the inequalities (4.4), (4.5) and (4.7), yield
$f_{kl}\,(t_1, V(t_1)) \geq v'_{kl}\,(t_1) \geq w'_{kl}\,(t_1) > f_{kl}\,(t_1, W(t_1)) = f_{kl}\,(t_1, V(t_1))$, which is a contradiction. Hence the conclusion holds and the proof is complete. □

Next we state and prove a theorem involving non strict inequalities in this set up.

Theorem 4.2. *Suppose (4.4) and (4.5) holds and that $F(t,U)$ is quasi monotone nondecreasing in U for each t. Further, suppose that F satisfies,*

$$F(t,W) - F(t,V) \leq L\,(W - V) \text{ for } W \geq V, \text{ where } L > 0 \text{ is a } N \times N \text{ matrix.}$$

Then $V_0 \leq W_0$ implies that $V(t) \leq W(t), t \in I$.

Proof. Let us define

$$\begin{aligned} W_\epsilon(t) &= W(t) + \epsilon\, e^{2Lt}, \text{ where } \epsilon > 0 \text{ is sufficiently small.} \\ \text{Then } W'_\epsilon(t) &= W'(t) + 2L\,\epsilon\, e^{2\,Lt} \end{aligned}$$

$$\begin{aligned} &\geq F(t,W(t)) + 2L\,\epsilon\, e^{2Lt} \\ &\geq F(t,W(t)) - F(t,W_\epsilon(t)) + F(t,W_\epsilon(t)) + 2L\,\epsilon\, e^{2Lt} \end{aligned}$$

$$\begin{aligned}
&\geq -L(W_\epsilon(t) - W(t)) + F(t, W_\epsilon(t)) + 2L\,\epsilon\, e^{2Lt} \\
&= F(t, W_\epsilon(t)) + L\,\epsilon\, e^{2Lt} \\
&> F(t, W_\epsilon(t)).
\end{aligned}$$

$$\begin{aligned}
\text{Further, } W_\epsilon(t_0) &= W(t_0) + \epsilon\, e^{2Lt_0} \\
&> W_0 \\
&\geq V_0
\end{aligned}$$

Hence we are in a position to apply the result for strict differential inequalities which yields $V(t) < W_\epsilon(t)$, $t \in I$ which implies as $\epsilon \to 0$,

$V(t) \leq W(t)$ and the proof is complete. □

The study of existence of a solution in a sector is essential to develop the monotone iterative technique. The following theorem deals with the existence of a solution in a sector.

Theorem 4.3. *Let $V, W \in C^1[I, \mathcal{E}_N]$ be lower and upper solutions of the Matrix differential equation*

$$\left.\begin{aligned} E' &= F(t, E) \\ E(t_0) &= E_0 \end{aligned}\right\} \tag{4.8}$$

such that $V(t) \leq W(t)$ on I and $F \in C[\Omega, \mathcal{E}_N]$, where
$\Omega = \{(t, E) : V(t) \leq E \leq W(t), t \in I\}$. *Then there exists a solution $E(t)$ of (4.8) such that*
$V(t) \leq E(t) \leq W(t)$ *on I.*

Proof. Let $P : I \times \mathcal{E}_N \to \mathcal{E}_N$ be defined by $P(t, E) = (p_{ij}(t))_{N\times N}$ where
$p_{ij}(t) = Max\{v_{ij}(t), Min\{e_{ij}, w_{ij}(t)\}\}$

Then $F(t, P) = (f_{ij}(t, P(t, E))$ defines a continuous extension of F to $I \times \mathcal{E}_N$ and is also bounded since F is bounded on Ω, which implies that E' is bounded on Ω. Hence the system
$E' = F(t, P(t, E))$, $E(t_0) = E_0$ has a solution $E(t)$ on I.

For $\epsilon > 0$, consider
$w_{\epsilon_{ij}}(t) = w_{ij}(t) + \epsilon(1 + t)$ and $v_{\epsilon_{ij}}(t) = v_{ij}(t) - \epsilon(1 + t)$ for $i, j = 1, 2, \ldots, N$.
We claim that $V_\epsilon(t) < E(t) < W_\epsilon(t)$. Since $v_{\epsilon_{ij}}(0) < e_{ij}(0) < w_{\epsilon_{ij}}(0)$ for any i and j we have $V_\epsilon(0) < E(0) < W_\epsilon(0)$. Suppose that there exists an element $t_1 \in (t_0, T]$ and a pair of indices k and l such that $v_{\epsilon_{kl}}(t) < e_{kl}(t) < w_{\epsilon_{kl}}(t)$ on $[t_0, t_1)$ and $e_{kl}(t_1) = w_{\epsilon_{kl}}(t_1)$.
Then $e_{kl}(t_1) > w_{kl}(t_1)$ and hence $p_{kl}(t_1) = w_{kl}(t_1)$.
Also we have $V(t_1) \leq P(t_1, E(t_1)) \leq W(t_1)$.
Since F is quasi monotone nondecreasing, we have

$$F(t_1, P(t_1, E(t_1))) \leq F(t_1, W(t_1))$$

$$\begin{aligned}
\text{Then } \ w'_{kl}(t_1) &\geq f_{kl}(t_1, W(t_1)) \\
&\geq f_{kl}(t_1, P(t_1, E(t_1)) \\
&= e'_{kl}(t_1)
\end{aligned}$$

Since $w'_{\epsilon_{kl}}(t_1) > w'_{kl}(t_1)$, we have $w'_{\epsilon_{kl}}(t_1) > e'_{kl}(t_1)$, which is a contradiction to the fact that $e_{kl}(t) < w_{\epsilon_{kl}}(t)$ for $t \in [t_0, t_1)$ and $e_{kl}(t_1) = w_{\epsilon_{kl}}(t_1)$.
Therefore $V_\epsilon(t) < E(t) < W_\epsilon(t)$ on I.
Now as $\epsilon \to 0$, we obtain that $V(t) \leq E(t) \leq W(t)$ and the proof is complete. □

5 Monotone iterative technique

In this section we shall construct monotone sequences that converges to the solutions of

$$\left.\begin{aligned} E' &= F(t, E) \\ E(t_0) &= E_0 \end{aligned}\right\} \tag{5.1}$$

Theorem 5.1. *Assume that $V_0, W_0 \in C^1[I, \mathcal{E}_N]$, $I = [t_0, T]$ are lower and upper solutions of the IVP (5.1) such that $V_0 \leq W_0$ on I. Let $F \in C[I \times \mathcal{E}_N, \mathcal{E}_N]$. Suppose further that $F(t, X) - F(t, Y) \geq -M(X - Y)$, for $V_0 \leq Y \leq X \leq W_0$, $M \in \mathbb{R}^{N \times N}, M \geq 0$. Then there exists monotone sequences $\{V_n\}, \{W_n\}$ such that $\{V_n\}$ converges to ρ and $\{W_n\}$ converges to R as $n \to \infty$ uniformly and monotonically on I and that ρ and R are the minimal and maximal solutions of IVP (5.1) respectively.*

Proof. For any $Y \in C^1[I, \mathcal{E}_N]$ such that $V_0 \leq Y \leq W_0$, we consider the linear Matrix differential equation

$$X' = F(t, Y) - M(X - Y),\ X(t_0) = X_0. \tag{5.2}$$

Then there exists a unique solution of (5.2) given by

$$X(t) = e^{M(t-t_0)} X_0 + \int_{t_0}^{t} e^{M(t-s)} [F(s, Y(s)) + MY(s)] ds$$

Define a sequence $\{V_n\}$ by

$$V_n' = F(t, V_{n-1}) - M(V_n - V_{n-1}), V_n(t_0) = X_0, \quad n = 1, 2, \ldots, \tag{5.3}$$

Let V_1 be the solution of (5.3) for $n = 1$.

$$\begin{aligned} \text{Consider} \quad P &= V_0 - V_1 \\ \text{Then} \quad P' &= V_0' - V_1' \\ &\leq F(t, V_0) - F(t, V_0) + M(V_1 - V_0), \\ &\leq -MP. \end{aligned}$$

and $P(t_0) \leq 0$ which implies that $P \leq 0$ on I, and thus $V_0 \leq V_1$ on I.
Similarly, we consider a sequence $\{W_n\}$ by

$$W_n' = F(t, W_{n-1}) - M(W_n - W_{n-1}), W_n(t_0) = X_0 \tag{5.4}$$

Let W_1 be the solution of (5.4) for $n = 1$.

$$\begin{aligned} \text{Consider} \quad Q &= W_1 - W_0 \\ \text{Then} \quad Q' &= W_1' - W_0' \\ &\leq F(t, W_0) - M(W_1 - W_0) - F(t, W_0) \\ &= -M\,Q \end{aligned}$$

and $Q(t_0) \leq 0$ which implies that $Q(t) \leq 0$. Hence $W_1 \leq W_0$ on I.
Now we proceed to show that $V_1 \leq W_1$ on I.

$$\begin{aligned} \text{Set} \quad R &= V_1 - W_1 \\ \text{Then} \quad R' &= V_1' - W_1' \\ &= F(t, V_0) - M(V_1 - V_0) - F(t, W_0) + M(W_1 - W_0) \\ &\leq M(W_0 - V_0) - M(V_1 - V_0 - W_1 + W_0) \\ &= -MR \end{aligned}$$

and $R(t_0) = 0$, which implies that $R \leq 0$ on I and thus $V_1 \leq W_1$ on I.
Hence we have shown that $V_0 \leq V_1 \leq W_1 \leq W_0$ on I.
Now suppose that for some $n = k$, the result $V_{k-1} \leq V_k \leq W_k \leq W_{k-1}$ holds on I. We claim that $V_k \leq V_{k+1} \leq W_{k+1} \leq W_k$ on I. To prove this we first set $n = k$ in (5.3) and (5.4). Then clearly there exists unique solutions $V_{k+1}(t)$ and $W_{k+1}(t)$ satisfying (5.3) and (5.4) respectively on I.

$$\begin{aligned} \text{Consider} \quad S &= V_k - V_{k+1} \\ \text{Then} \quad S' &= V_k' - V_{k+1}' \\ &= F(t, V_{k-1}) - M(V_k - V_{k-1}) - F(t, V_k) + M(V_{k+1} - V_k), \end{aligned}$$

$$
\begin{aligned}
&\leq \; M(V_k - V_{k-1}) + M\left(V_{k+1} - V_k - V_k + V_{k-1}\right), \\
&\leq \; -MS.
\end{aligned}
$$

and $S(t_0) = 0$ which implies that $S \leq 0$ on I and thus $V_k \leq V_{k+1}$ on I.
Similarly we can show that $W_{k+1} \leq W_k$ on I.

$$
\begin{aligned}
\text{Set} \quad T &= V_{k+1} - W_{k+1} \\
\text{Then} \quad T' &= V'_{k+1} - W'_{k+1} \\
&= F(t, V_k) - M(V_{k+1} - V_k) - F(t, W_k) + M\left(W_{k+1} - W_k\right), \\
&\leq M(W_k - V_k) + M\left(W_{k+1} - W_k - V_{k+1} + V_k\right), \\
&\leq -MT.
\end{aligned}
$$

and $T\left(t_0\right) = 0$, which implies that $T \leq 0$ on I and thus $V_{k+1} \leq W_{k+1}$ on I.
We have shown that $V_k \leq V_{k+1} \leq W_{k+1} \leq W_k$ on I.
Therefore we have

$$
V_0 \leq V_1 \leq \cdots \leq V_n \leq W_n \leq \cdots \leq W_1 \leq W_0 \;\text{ on }\; [t_0, T]. \tag{5.5}
$$

The sequences $\{V_n\}, \{W_n\}$ are uniformly bounded on $[t_0, T]$ and by (5.3) and (5.4) it follows that $\{|V_n'|\}, \{|W_n'|\}$ are also uniformly bounded. As a result, the sequences $\{V_n\}$ and $\{W_n\}$ are equicontinuous on $[t_0, T]$ and consequently by Ascoli-Arzela's Theorem there exists subsequences $\{V_{n_k}\}$, $\{W_{n_k}\}$ that converge uniformly on $[t_0, T]$. In view of (5.5) it also follows that the entire sequences $\{V_n\}, \{W_n\}$ converge uniformly and monotonically to ρ and R respectively as $n \to \infty$. By considering the integral equations corresponding to the IVP of MDE (5.3) and (5.4) respectively, we can show that ρ and R are solutions of IVP(5.1). The proof uses the concepts of uniform convergence and uniform continuity and is well established.
To prove that ρ, R are respectively the minimal and maximal solutions of (5.1) we have to show that if X is any solution of (5.1) such that $V_0 \leq X \leq W_0$ on I, then $V_0 \leq \rho \leq X \leq R \leq W_0$ on I. To do this, suppose that for some $n, V_n \leq X \leq W_n$ on I and set $\phi = X - V_{n+1}$ so that

$$
\begin{aligned}
&\phi' = F(t, X) - F(t, V_n) + M(V_{n+1} - V_n) \\
&\geq -M(X - V_n) + M(V_{n+1} - V_n) = -M\phi;
\end{aligned}
$$

and $\phi(t_0) = 0$.
Hence, it follows that $V_{n+1} \leq X$ on I. Similarly $X \leq W_{n+1}$ on I.
Hence $\qquad V_{n+1} \leq X \leq W_{n+1}$ on I.
Since $V_0 \leq X \leq W_0$ on I, this proves by induction that $V_n \leq X \leq W_n$ on I for all n. Taking the limit as $n \to \infty$, we conclude that $\rho \leq X \leq R$ on I and the proof is complete. □

Corollary 5.1. *If in addition to the assumption Theorem 5.1, if F satisfies the following condition*

$$
F(t, X) - F(t, Y) \leq M(X - Y), X \geq Y
$$

then the solution is unique.

Proof. We have $\rho \leq R$ on I.

$$
\begin{aligned}
\text{Consider} \quad \phi(t) &= R(t) - \rho(t) \\
\text{Then} \quad \phi' &= R'(t) - \rho'(t) \\
&= F(t, R) - F(t, \rho) \\
&\leq M(R - \rho) \\
&\leq M\phi.
\end{aligned}
$$

and $\phi(t_0) = 0$ which implies that $\phi(t) \leq 0$ on I and thus $R(t) \leq \rho(t)$ on I. Hence $\rho(t) = X(t) = R(t)$ on I, and the proof is complete. □

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References

[1] Narsingh Deo, *Graph Theory with Application to Engineering and Computer Science*

[2] D.D.Siljak, *Dynamic graphs*, Nnlinear Analysis: Hybrid Systems 2(2008) 544-567.

[3] G.S.Ladde, V.Lakshmikantham and A.S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations, Pitman Publishing INC., 1985.*

[4] A.R. Radcliffe-Brown, On social structure, Journal of the Royal Anthropological Institute of Great Britain and Ireland 70(1940)1-12.

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