Malaya J. Mat. 2(4)(2014) 438–444

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Third Hankel determinant for a subclass of analytic univalent functions

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Abstract

This paper focuses on attaining the upper bounds on $H_3(1)$ for a class C_α^β $(0 \leq \beta < 1, \alpha \geq 0)$ in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

Keywords: Bounded turning, coefficient bounds, convex functions, Hankel determinant, starlike functions.

2010 MSC: 30C45, 30C50.

1 Introduction

Let A be the class of functions

$$f(z) = z + a_2 z^2 + \dots \tag{1.1}$$

which are analytic in $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

A function $f \in A$ is said to be of bounded turning, starlike and convex respectively if and only if for $z \in \Delta$, $Re\, f'(z) > 0$, $Re\, \frac{zf'(z)}{f(z)} > 0$ and $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$. By usual notations, we denote these classes of functions respectively by R, S^* and C. Let $n \geq 0$ and $q \geq 1$. The q^{th} Hankel determinant is defined as:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & & \cdots \\ \vdots & & \vdots & \\ a_{n+q-1} & \cdots & & a_{n+2(q-1)} \end{vmatrix}.$$

This determinant has been considered by several authors. For example, Noor in [11] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions f given by (1.1) with bounded boundary. In particular, sharp upper bounds on $H_2(2)$ were obtained by authors of articles [5, 6, 7, 13, 14] for different classes of functions.

The class C_α^β is defined as follows.

Definition 1.1. *Let f be given by (1.1). Then $f \in C_\alpha^\beta$ if and only if*

$$Re\left\{\frac{(zf'(z) + \alpha z^2 f''(z))'}{f'(z)}\right\} > \beta, \quad z \in \Delta, 0 \leq \beta < 1, 0 \leq \alpha \leq 1.$$

The choice of $\alpha = 0$, $\beta = 0$ yields $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$, $z \in \Delta$, the class of convex functions C [12].

The choice of $\alpha = 0$, yields $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta$, $z \in \Delta$, the class of convex functions of order β denoted by $C(\beta)$ [12].

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In the present investigation, our focus is on the Hankel determinant, $H_3(1)$ for the class C_α^β in Δ. By definition, $H_3(1)$ is given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

for $f \in A$, $a_1 = 1$, so that

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$$

and by triangle inequality, we have

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{1.2}$$

In this paper, we find the sharp upper bound for the functional $|a_2a_3 - a_4|$, $|a_2a_4 - a_3^2|$ and $|a_3 - a_2^2|$ respectively for the functions belonging to the class C_α^β. Our proofs are based on the techniques employed by [8, 9] which has been widely used by many authors (see for example [5, 6, 7, 14]).

2 Preliminary Results

Let P denote the class of functions

$$p(z) = 1 + c_1z + c_2z^2 + \cdots \tag{2.3}$$

which are regular in Δ and satisfy $Re\ p(z) > 0$, $z \in \Delta$. Throughout this paper, we assume that $p(z)$ is given by (2.3) and $f(z)$ is given by (1.1). To prove the main results we shall require the following lemmas.

Lemma 2.1. *[3] Let $p \in P$. Then $|c_k| \leq 2$, $k = 1, 2, \ldots$ and the inequality is sharp.*

Lemma 2.2. *[8, 9] Let $p \in P$. Then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{2.4}$$

and

$$4c_3 = c_1^3 + 2xc_1(4 - c_1^2) - x^2c_1(4 - c_1^2) + 2y(1 - |x|^2)(4 - c_1^2) \tag{2.5}$$

for some x, y such that $|x| \leq 1$ and $|y| \leq 1$.

Lemma 2.3. *[2] Let $p \in P$. Then*

$$\left| c_2 - \sigma \frac{c_1^2}{2} \right| = \begin{cases} 2(1-\sigma) & \text{if } \sigma \leq 0, \\ 2 & \text{if } 0 \leq \sigma \leq 2, \\ 2(\sigma - 1) & \text{if } \sigma \geq 2. \end{cases}$$

3 Main Results

Lemma 3.1. *Let $f \in C_\alpha^\beta$. Then, we have the best possible bound for*

$$|a_2a_3 - a_4| \leq \begin{cases} \frac{4}{9\sqrt{3}} & \alpha = 0, \beta = 0 \\ \frac{(1-\beta)}{MA_2}\sqrt{\frac{A_1}{A_2}}[B_1 + (4A_2 - A_1)(B_2 + B_3)] & 0 < \alpha \leq 1, 0 < \beta < 1, \end{cases}$$

where,
$A_1 = 4(4 + 23\alpha + 48\alpha^2 + 36\alpha^3 - \beta - 2\alpha\beta)$,
$A_2 = 3(4 + 20\alpha + 64\alpha^2 + 48\alpha^3 + 2\beta + 20\alpha\beta - 2\beta^2 - 12\alpha\beta^2)$,
$B_1 = -3\alpha + 3\beta + 22\alpha\beta - 2\beta^2 - 12\alpha\beta^2 + 16\alpha^2 + 12\alpha^3$,
$B_2 = 3 + 16\alpha + 32\alpha^2 + 24\alpha^3$,
$B_3 = 1 + 7\alpha + 16\alpha^2 + 12\alpha^3$,
$M = 48(1 + 2\alpha)^2(1 + 3\alpha)(1 + 4\alpha)$.

Proof. For $f \in C_\alpha^\beta$, there exists a $p \in P$ such that

$$f'(z) + zf''(z) + \alpha z^2 f'''(z) + 2\alpha z f''(z) = [(1-\beta)p(z) + \beta]f'(z).$$

Equating the coefficients,

$$\begin{aligned}
a_2 &= \frac{c_1(1-\beta)}{2(1+2\alpha)}, \quad a_3 = \frac{c_2(1-\beta)}{6(1+3\alpha)} + \frac{c_1^2(1-\beta)^2}{6(1+2\alpha)(1+3\alpha)},\\
a_4 &= \frac{c_3(1-\beta)}{12(1+4\alpha)} + \frac{c_1c_2(3+8\alpha)(1-\beta)^2}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{c_1^3(1-\beta)^3}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)},\\
a_5 &= \frac{1}{20(1+5\alpha)}\left\{\frac{c_1c_3(1-\beta)^2(4+4\alpha)}{3(1+2\alpha)(1+4\alpha)} + \frac{c_1^4(1-\beta)^4}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{c_2^2(1-\beta)^2}{2(1+3\alpha)}\right.\\
&\quad \left. + \frac{c_1^2c_2(1-\beta)^3(6+20\alpha)}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)} + c_4(1-\beta)\right\}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
|a_2a_3 - a_4| = \Bigg| &\frac{c_1c_2(1-\beta)^2(-1)}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{c_1^3(1-\beta)^3(1+6\alpha)}{24(1+2\alpha)^2(1+3\alpha)(1+4\alpha)}\\
&- \frac{c_3(1-\beta)}{12(1+4\alpha)}\Bigg| \qquad (3.6)
\end{aligned}$$

Suppose now that $c_1 = c$. Since $|c| = |c_1| \leq 2$,using the Lemma 2.1, we may assume without restriction $c \in [0,2]$. Substituting for c_2 and c_3, from Lemma 2.2 and applying the triangle inequality with $\rho = |x|$, we obtain

$$\begin{aligned}
|a_2a_3 - a_4| &\leq \frac{c^3(1-\beta)[-3\alpha + 3\beta + 22\alpha\beta - 2\beta^2 - 12\alpha\beta^2 + 16\alpha^2 + 12\alpha^3]}{48(1+2\alpha)^2(1+3\alpha)(1+4\alpha)}\\
&\quad + \frac{\rho c(1-\beta)(4-c^2)(3+10\alpha+12\alpha^2-\beta)}{48(1+2\alpha)(1+3\alpha)(1+4\alpha)}\\
&\quad + \frac{\rho^2(4-c^2)(1-\beta)(c-2)}{48(1+4\alpha)} + \frac{2(1-\beta)(4-c^2)}{48(1+4\alpha)}\\
&= F(\rho).
\end{aligned}$$

Differentiating $F(\rho)$, we get

$$F'(\rho) = \frac{c(1-\beta)(4-c^2)(3+10\alpha+12\alpha^2-\beta)}{48(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{\rho(4-c^2)(1-\beta)(c-2)}{24(1+4\alpha)}.$$

Note also that $F'(\rho) \geq F'(1) \geq 0$. Then there exist a $c^* \in [0,2]$ such that $F'(\rho) > 0$ for $c \in (c^*, 2]$ and $F'(\rho) \leq 0$ otherwise.
Then, for a $c \in (c^*, 2]$, $F(\rho) \leq F(1)$, that is:

$$\begin{aligned}
|a_2a_3 - a_4| &\leq \frac{c^3(1-\beta)[-3\alpha + 3\beta + 22\alpha\beta - 2\beta^2 - 12\alpha\beta^2 + 16\alpha^2 + 12\alpha^3]}{48(1+2\alpha)^2(1+3\alpha)(1+4\alpha)}\\
&\quad + \frac{c(1-\beta)(4-c^2)(3+10\alpha+12\alpha^2-\beta)}{48(1+2\alpha)(1+3\alpha)(1+4\alpha)}\\
&\quad + \frac{(4-c^2)(1-\beta)(c-2)}{48(1+4\alpha)} + \frac{2(1-\beta)(4-c^2)}{48(1+4\alpha)}\\
&= G(c).
\end{aligned}$$

If $\alpha = 0, \beta = 0$, we have $G(c) = \frac{c(4-c^2)}{12}$. By elementary calculus, we have $G'(c) = \frac{4-3c^2}{12}$, $G''(c) = -\frac{c}{2} < 0$. Since $c \in [0,2]$ by our assumption, it follows that $G(c)$ is maximum at $c = 2/\sqrt{3}$. Otherwise, again by elementary calculus $G(c)$ is maximum at $c = \sqrt{\frac{A_1}{A_2}}$ and is given by

$$G(c) \leq \frac{(1-\beta)}{MA_2}\sqrt{\frac{A_1}{A_2}}[B_1 + (4A_2 - A_1)(B_2 + B_3)]$$

Now suppose $c \in [0, c^*]$, then $F(\rho) \leq F(0)$, that is:

$$\begin{aligned} F(\rho) &\leq \frac{c^3(1-\beta)[-3\alpha+3\beta+22\alpha\beta-2\beta^2-12\alpha\beta^2+16\alpha^2+12\alpha^3]}{48(1+2\alpha)^2(1+3\alpha)(1+4\alpha)} + \frac{2(1-\beta)(4-c^2)}{48(1+4\alpha)} \\ &= G(c), \end{aligned}$$

which implies that $G(c)$ turns at $c = 0$ and $c = \frac{4(1+2\alpha)^2(1+3\alpha)}{[-3\alpha+3\beta+22\alpha\beta-2\beta^2-12\alpha\beta^2+16\alpha^2+12\alpha^3]}$
with its maximum at $c = 0$. That is $G(c) \leq \frac{(1-\beta)}{6(1+4\alpha)}$.
Thus for all admissible $c \in [0, 2]$, the maximum of the functional $|a_2a_3 - a_4|$ are given by the inequalities of the theorem.
If $p(z) \in P$, with $c_1 = 2/\sqrt{3}$, $c_2 = -2/3$ and $c_3 = -10/3\sqrt{3}$, then we obtain $p(z) = 1 + \frac{2}{\sqrt{3}}z - \frac{2}{3}z^2 - \frac{10}{3\sqrt{3}}z^3 + \cdots \in P$ which shows that the result is sharp. □

Lemma 3.2. *Let $f \in C_\alpha^\beta$. Then ,we have the best possible bound for*

$$|a_2a_4 - a_3^2| \leq \begin{cases} \frac{1}{8} & \alpha = 0, \beta = 0 \\ \frac{(1-\beta)^2}{N}[M_1V_1V_2 + (4V_2 - V_1)\{M_2V_1 + V_1P_1 + P_2\}] & 0 < \alpha \leq 1, 0 < \beta \leq 1, \end{cases}$$

where,
$M_1 = [22\alpha^3 + 31\alpha^2 + 11\alpha - 2\beta^2 - 5\beta - 3\alpha\beta - 8\alpha^2\beta]$,
$M_2 = 3 + 118\alpha^2 - 45\alpha + 44\alpha^3 - \beta - 3\alpha\beta - 8\alpha^2\beta$,
$P_1 = (1 + 27\alpha^2 - 10\alpha)(1 + 2\alpha)$,
$P_2 = (8 + 48\alpha + 64\alpha^2)(1 + 2\alpha)$,
$V_1 = 2M_1 + 8M_2 + 8P_1 - 2P_2$,
$V_2 = 4M_2 + 4P_1$,
$N = 288(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)$.

Proof. Let $f \in C_\alpha^\beta$. Then proceeding as in Lemma 3.1, we have

$$\begin{aligned} &|a_2a_4 - a_3^2| = \\ &\left| \frac{c_1c_3(1-\beta)^2}{24(1+2\alpha)(1+4\alpha)} + \frac{c_1^2c_2(3+8\alpha)(1-\beta)^3}{48(1+2\alpha)^2(1+3\alpha)(1+4\alpha)} + \frac{c_1^4(1-\beta)^4}{48(1+2\alpha)^2(1+3\alpha)(1+4\alpha)} \right. \\ &\left. - \frac{c_2^2(1-\beta)^2}{36(1+3\alpha)^2} - \frac{c_1^4(1-\beta)^4}{36(1+2\alpha)^2(1+3\alpha)^2} - \frac{2c_1^2c_2(1-\beta)^3}{36(1+2\alpha)(1+3\alpha)^2} \right|. \end{aligned} \tag{3.7}$$

Suppose now that $c_1 = c$. Since $|c| = |c_1| \leq 2$, Using Lemma 2.1, we may assume without restriction $c \in (0, 2]$. Substituting for c_2 and c_3, from Lemma 2.2 and applying triangle inequality with $\rho = |x|$, we obtain

$$\begin{aligned} |a_2a_4 - a_3^2| \leq \frac{1}{144} &\left\{ \frac{(1-\beta)^2c^4[22\alpha^3 + 23\alpha^2 + 8\alpha^2 + 11\alpha - 2\beta^2 - 5\beta - 3\alpha\beta - 8\alpha^2\beta}{2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \right. \\ &+ \frac{\rho c^2(4-c^2)(1-\beta)^2[3 + 118\alpha^2 - 45\alpha + 44\alpha^3 - \beta - 3\alpha\beta - 8\alpha^2\beta]}{2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \\ &+ \frac{\rho^2(4-c^2)(1-\beta)^2[8 + c^2 + 48\alpha + 64\alpha^2 + 27c^2\alpha^2 - 10c^2\alpha]}{2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \\ &\left. + \frac{3(4-c^2)(1-\rho)(1-\beta)^2}{(1+2\alpha)(1+4\alpha)} \right\} \\ &= F(\rho). \end{aligned}$$

Differentiating $F(\rho)$, we get,

$$\begin{aligned} F'(\rho) = \frac{1}{144} &\left\{ \frac{c^2(4-c^2)(1-\beta)^2[3 + 118\alpha^2 - 45\alpha + 44\alpha^2 - \beta - 3\alpha\beta - 8\alpha^2\beta]}{2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \right. \\ &+ \frac{2\rho(4-c^2)(1-\beta)^2[8 + c^2 + 48\alpha + 64\alpha^2 + 27c^2\alpha - 10c^2\alpha]}{2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \\ &\left. + \frac{3\rho(4-c^2)(1-\beta)^2}{(1+2\alpha)(1+4\alpha)} \right\}. \end{aligned}$$

Note also that $F'(\rho) \geq F'(1) \geq 0$. Then there exist a $c^* \in [0,2]$ such that $F'(\rho) > 0$ for $c \in (c^*, 2]$ and $F'(\rho) \leq 0$ otherwise.
Then for a $c \in (c^*, 2]$, $F(\rho) \leq F(1)$, that is:

$$
\begin{aligned}
|a_2a_4 - a_3^2| \leq \frac{1}{144}\Bigg\{ & \frac{(1-\beta)^2[22\alpha^3 + 31\alpha^2 + 11\alpha - 2\beta^2 - 5\beta - 3\alpha\beta - 8\alpha^2\beta]c^4}{2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \\
& + \frac{(1-\beta)^2[3 + 118\alpha^2 - 45\alpha + 44\alpha^3 - \beta - 3\alpha\beta - 8\alpha^2\beta]c^2(4-c^2)}{2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \\
& + \frac{(1-\beta)^2(4-c^2)(8 + c^2 + 48\alpha + 64\alpha^2 + 27c^2\alpha^2 - 10c^2\alpha)}{2(1+2\alpha)(1+3\alpha)^2(1+4\alpha)}\Bigg\} \\
= G(c). &
\end{aligned}
$$

If $\alpha = 0$, $\beta = 0$, we have $G(c) = \frac{3c^2(4-c^2)}{2} + \frac{(4-c^2)(c^2+8)}{2}$. By elementary calculus we have, $G'(c) = 8c - 8c^3$, $G''(c) = 8 - 24c^2 < 0$. Since $c \in (0,2]$, by our assumption it follows that $G(c)$ is maximum at $c = 1$. Otherwise, again by elementary calculus $G(c)$ is maximum at $c = \sqrt{\frac{V_1}{V_2}}$ and is given by

$$
G(c) \leq \frac{(1-\beta)^2}{N}[M_1V_1V_2 + (4V_2 - V_1)\{M_2V_1 + V_1P_1 + P_2\}].
$$

Now suppose $c \in [0, c^*]$, then $F(\rho) \leq F(0)$, that is:

$$
\begin{aligned}
F(\rho) \leq \frac{1}{144}\Bigg\{ & \frac{(1-\beta)^2[22\alpha^3 + 31\alpha^2 + 11\alpha - 2\beta^2 - 5\beta - 3\alpha\beta - 8\alpha^2\beta]}{2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \\
& + \frac{3(4-c^2)(1-\beta)^2}{(1+2\alpha)(1+4\alpha)}\Bigg\} \\
= G(c), &
\end{aligned}
$$

which implies that $G(c)$ turns at $c = 0$ and $c = \sqrt{\frac{(22\alpha^3+31\alpha^2+11\alpha-2\beta^2-5\beta-3\alpha\beta-8\alpha^2\beta)}{3(1+2\alpha)(1+3\alpha)^2}}$,
with its maximum at $c = 0$. That is, $G(c) \leq \frac{12(1-\beta)^2}{(1+2\alpha)(1+4\alpha)}$.
Thus for all admissible $c \in [0,2]$, the maximum of the functional $|a_2a_4 - a_3^2|$ are given by the inequalities of the theorem.
If $p(z) \in P$, with $c_1 = 1$, $c_2 = -1$, $c_3 = -2$, then $p(z) = \frac{1-z^2}{1-z+z^2} = 1 + z - z^2 - 2z^3 + \cdots \in P$ which shows that the result is sharp. $\square$

Lemma 3.3. *Let $f \in C_\alpha^\beta$. Then we have the best possible bound for*

$$
|a_3 - a_2^2| \leq \begin{cases} \frac{1}{3} & \alpha = 0, \beta = 0 \\ \frac{1-\beta}{3(1+3\alpha)} & 0 < \alpha \leq 1, 0 < \beta \leq 1. \end{cases}
$$

Proof. Let $f \in C_\alpha^\beta$. Then proceeding as in Lemma 3.1, we have

$$
|a_3 - a_2^2| = \left| \frac{c_2(1-\beta)}{6(1+3\alpha)} - \frac{c_1^2(1-\beta)^2(1+5\alpha)}{12(1+2\alpha)^2(1+3\alpha)} \right| \tag{3.8}
$$

and

$$
|a_3 - a_2^2| = \frac{(1-\beta)}{6(1+3\alpha)} \left| c_2 - \frac{c_1^2(1-\beta)(1+5\alpha)}{2(1+2\alpha)^2} \right|.
$$

Setting $\sigma = \frac{(1-\beta)(1+5\alpha)}{(1+2\alpha)^2}$, using Lemma 2.3, we have $|a_3 - a_2^2| \leq \frac{(1-\beta)}{3(1+3\alpha)}$.
If $p(z) \in P$ with $c_1 = 0$, $c_2 = 2$, then $p(z) = \frac{1+z^2}{1-z^2} = 1 + 2z^2 + 2z^4 + \cdots \in P$, which shows that the result is sharp. $\square$

Remark 3.1. *Let $f \in C_\alpha^\beta$. By Lemma 2.1, we have*

$$\begin{aligned}
|a_3| &= \left| \frac{c_2(1-\beta)}{6(1+3\alpha)} + \frac{c_1^2(1-\beta)^2}{6(1+2\alpha)(1+3\alpha)} \right|, \\
&\leq \frac{(1-\beta)(3+2\alpha-2\beta)}{3(1+2\alpha)(1+3\alpha)}, \\
|a_4| &= \left| \frac{c_3(1-\beta)}{12(1+4\alpha)} + \frac{c_1c_2(3+8\alpha)(1-\beta)^2}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{c_1^3(1-\beta)^3}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)} \right|, \\
&\leq \frac{(1-\beta)(6+6\alpha^2+2\beta^2+13\alpha-7\beta-8\alpha\beta)}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)}, \\
|a_5| &= \left| \frac{1}{20(1+5\alpha)} \left\{ \frac{c_1c_3(1-\beta)^2(4+4\alpha)}{3(1+2\alpha)(1+4\alpha)} + \frac{c_1^4(1-\beta)^4}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{c_2^2(1-\beta)^2}{2(1+3\alpha)} \right.\right. \\
&\qquad \left.\left. + \frac{c_1^2c_2(1-\beta)^3(6+20\alpha)}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)} + c_4(1-\beta) \right\} \right| \\
&\leq \frac{(1-\beta)(120+408\alpha^2+500\alpha-576\alpha\beta-432\alpha^2\beta+160\alpha\beta^2+188\beta+96\beta^2)}{120(1+2\alpha)(1+3\alpha)(1+4\alpha)(1+5\alpha)}.
\end{aligned}$$

This leads to the next theorem which gives a sharp result by using Lemmas 3.1,3.2 and 3.3 and Remark 3.1.

Theorem 3.1. *Let $f \in C_\alpha^\beta$. Then*

$$\begin{aligned}
|H_3(1)| \leq & \frac{(1-\beta)^2(3+2\alpha-2\beta)}{3(1+2\alpha)(1+3\alpha)} \\
& \left\{ \frac{(1-\beta)^2}{N} [M_1V_1V_2 + (4V_2 - V_1)\{M_2V_1 + V_1P_1 + P_2\}] \right\} \\
& + \frac{(1-\beta)(6+6\alpha^2+2\beta^3+13\alpha-7\beta-8\alpha\beta)}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)} \\
& \left\{ \frac{(1-\beta)}{MA_2} \sqrt{\frac{A_1}{A_2}} [B_1 + (4A_2 - A_1)(B_2 + B_3)] \right\} \\
& + \frac{(1-\beta)(120+408\alpha^2+500\alpha-576\alpha\beta-432\alpha^2\beta+160\alpha\beta^2+188\beta+96\beta^2)}{120(1+2\alpha)(1+3\alpha)(1+4\alpha)(1+5\alpha)} \\
& \times \frac{(1-\beta)}{3(1+3\alpha)}.
\end{aligned}$$

When $\alpha = 0$, $\beta = 0$, we have the following corollary due to [1].

Corollary 3.1. *If $\alpha = 0$, $\beta = 0$, then*

$$|H_3(1)| \leq \frac{32+33\sqrt{3}}{72\sqrt{3}} = 0.714933452973167.$$

4 Acknowledgement

The authors thank the referees for their valuable comments and suggestions to improve the presentation of the paper.

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Received: March 05, 2014; *Accepted*: July 05, 2014

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