Concept Lattice: A rough set approach

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Abstract

Concept lattice is an efficient tool for knowledge representation and knowledge discovery and is applied to many fields successfully. However, in many real life applications, the problem under investigation cannot be described by formal concepts. Such concepts are called the non-definable concepts. The hierarchical structure of formal concept (called concept lattice) represents a structural information which obtained automatically from the input data table. We deal with the problem in which how further additional information be supplied to utilize the basic object attribute data table. In this paper , we provide rough concept lattice to incorporate the rough set into the concept lattice by using equivalence relation. Some results are established to illustrate the paper.

Keywords: Rough Set, Formal Concept lattice, Equivalence Relation, Lattice.

1 Introduction

The formal concept analysis (FCA) is a mathematical framework, developed by Wille [23] and his colleagues at Darmstadt University, which is very useful for representation and analysis of data [18].The concept lattice is also called Galois lattice, was proposed by Wille in 1982 [23]. A concept lattice is an ordered hierarchy that is defined by a binary relationship between objects and attributes in a data set. As an efficient tool of data analysis and knowledge processing, the concept lattice has been applied in many fields, such as knowledge engineering, data mining, information searches, and software engineering [6]. Most of the researchers have concentrated on their attention to the concept lattice and defined on such topics as: construction of the concept lattice, pruning of the concept lattice, acquisition of rules, relationship between the concept lattice and rough set [7] and approximation. The basic formal concept analysis deals with input data in the form of a table with rows corresponding to objects and columns corresponding to attributes. The data table is formally represented by a so called formal context which is a triplet (A, B, I) where A and B are sets and I is subset of $A \times B$ (i.e, $I \subseteq A \times B$) and defined a binary relation between A and B. The elements of A are called objects while the elements of B are called attributes or simply considered as the characteristics of objects. For a object and b characteristic , $(a, b) \in I$ or aIb shall indicate the following: a object owns the b attribute. Let us assume that (A, B, I) is a formal context. The knowledge about a considered universe is the starting point. Using two operations, a lower and an upper approximations, we can describe every subset of the universe. The concepts of the lower and upper approximations in rough set theory are fundamental to the examination of granularity in knowledge.

In this paper, we discuss the rough properties of concept lattice in rough set. FCA and rough set theory are two kinds of complementary mathematical tools for data analysis and data processing ([26], [27]). Up to now, many efforts have been made to combine these two theories ([26],[27],[3],[18],[22],[8],[24],[4]), in which the

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concept lattices based on rough set theory, including the attribute oriented concept lattice [2], lattice for covering rough approximation[14], rough set approach on lattice [12] and distributive lattice [16], rough modular lattice [13], lattice for rough intervals [11] and the object oriented concept lattice ([26],[27]), are perspective concept lattices for knowledge representation and knowledge discovery. However, the concept lattices usually contain redundant attributes and objects. In this paper , we provide rough concept lattice to incorporate rough set into the concept lattice by using equivalence relation.

2 Preliminaries

In this section, we present some definitions and fundamental concept on covering lattice.

Definition 2.1. *If $X \subseteq A$, $Y \subseteq B$, then two operators α and β can be defined as $\alpha : 2^A \rightarrow 2^B, \alpha(Y) = \{a \in B : aIb, \forall b \in B\}$.*
$\beta : 2^A \rightarrow 2^B, \beta(X) = \{b \in B : aIb, \forall a \in A\}$
$\beta(X)$ will take us to the set of attributes that are common in the entire objects in X set. Similarly $\alpha(Y)$ will take us to the attribute set of A that owns the entire attributes of Y. In other word $\beta(X)$, shall give the maximum object set that it hired by the entire objects in X while $\alpha(Y)$ still give the maximum object set that it owns by the entire objects in Y. (β, α) shall form a Galois connection between 2^A and 2^B.
Definition : *Let $K = (A, B, I)$ be a formal context, $X \in P(A)$ and $Y \in P(B)$, where $P(A)$ and $P(B)$ are the power set of A and B respectively. (X, Y) is called a concept, if $\alpha(X) = Y$ and $\beta(Y) = X$ hold for X and Y, where X is called the extent of the concept and Y is called the intent of the concept. $L(K)$ denotes the set of all concepts in the formal context..*

Definition 2.2. *Let (A, B, I) be a formal context. If there exists an attribute set $D \subseteq B$ such that $Latt(A, B, I_D) \cong Latt(A, B, I)$, then D is called a consistent set of (A, B, I). And further, if $\forall d \in D$, $Latt(A, D - \{d\}, I_{D-\{d\}}) \neq Latt(A, B, I)$. Then D is called a reduct of (A, B, I). The intersection of all the reducts is called the core of (A, B, I).*
Definition : *For the formal context $K = (A, B, I)$, let $H_1 = (X_1, Y_1)$ and $H_2 = (X_2, Y_2)$ be two elements of $Latt(K)$. If there exists $H_1 \leq H_2 \Leftrightarrow Y_2 \leq Y_1$, then $\leq$ is a partial order of $Latt(K)$, which produce a lattice structure in $Latt(K)$, called concept lattice of formal context $K = (A, B, I)$, also denoted by $Latt(K)$ Table 1 is a formal context, and Figure 1 shows its Hasse diagram..*

Lemma 2.1. *Let (A, B, I) be a context. Then the following assertions hold:*

- *$X_1 \subseteq X_2$ implies $\beta(A_1) \supseteq \beta(A_2)$ for every $X_1, X_2 \subseteq A$, and $Y_1 \subseteq Y_2$ implies $\alpha(Y_1) \supseteq \alpha(Y_2)$ for every $Y_1, Y_2 \subseteq B$.*

- *$X \subseteq \alpha(\beta(X))$ and $\beta(X) = \beta(\alpha(\beta(X)))$ for all $X \subseteq A$, and $Y \subseteq \beta(\alpha(Y))$ and $\alpha(Y) = \alpha(\beta(\alpha(X)))$ for all $Y \subseteq B$.*

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3 Fundamental Theorem of FCA

The fundamental theorem of FCA states that the set of all formal concepts on a given context with the ordering $(X_1, Y_1) \leq (X_2, Y_2)$ if and only if $X_1 \subseteq X_2$ is a complete lattice called the concept lattice, in which the infima and suprema are given by $\bigwedge_{i \in J}(X_j, Y_j) = (\bigcap_{i \in J} X_j, \beta(\alpha(\bigcup_{i \in J} Y_j))) = (\bigcap_{i \in J} X_j, \beta(\bigcap_{i \in J} X_j)))$
$\bigvee_{i \in J}(X_j, Y_j) = (\alpha(\beta(\bigcup_{i \in J} X_j)), \bigcap_{i \in J} Y_j) = (\alpha(\bigcap_{i \in J} X_j)), \bigcap_{i \in J} X_j).$

Theorem 3.1. *For two elements $H_1 = (X_1, Y_1)$ and $H_2 = (X_2, Y_2)$ of concept lattice $(Latt(K), \cap, \cup)$,if $\alpha(X_1 \cap X_2) = Y_1 \cap Y_2$ and $\beta(Y_1 \cap Y_2) = X_1 \cup X_2$ then $(L(K), \cap, \cup)$ is a distributive lattice.*

Proof. Since $(X_1, Y_1) \cup (X_2, Y_2) = (X_1 \cup X_2, Y_1 \cap Y_2)$ and $(X_1, Y_1) \cap (X_2, Y_2) = (X_1 \cap X_2, Y_1 \cap Y_2)$, therefore

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\begin{aligned}
H_1 \cap (H_2 \cap H_3) &= (X_1, Y_1) \cap (X_2, Y_2) \cap (X_3, Y_3) \\
&= (X_1, Y_1)(X_2 \cup X_3, Y_2 \cup Y_3) \\
&= (X_1 \cap (X_2 \cup X_3), Y_1 \cup (Y_2 \cap Y_3)) \\
&= ((X_1 \cap X_2)(X_1 \cap X_3), (Y_1 \cup Y_2)(Y_1 \cup Y_3)).
\end{aligned}
$$

and $(H_1 \cap H_2)(H_1 \cup H_3) = (X_1 \cap X_2, Y_1 \cap Y_2)(X_1 \cap X_3, Y_1 \cap Y_3) = ((X_1 \cap X_2)(X_1 \cap X_3,), (Y_1 \cup Y_2)(Y_1 \cup Y_3))$
i.e., $H_1 \cap (H_2 \cup H_3) = (H_1 \cup H_2)(H_1 \cap H_3)$. Similarly , we can prove $H_1 \cup (H_2 \cap H_3) = (H_1 \cap H_2)(H_1 \cup H_3)$.
Thus $(Latt(K), \cap, \cup)$ is a distributive lattice. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Theorem 3.2. *Let* $Latt(K)$ *be a concept lattice in formal context* $K = (A, B, I)$. *Let* $H_1 = (X_1, Y_1)$ *and* $H_2 = (X_2, Y_2)$
be elements of $L(K)$. *The following propositions are equivalent.*
i) $H_1 \leq H_2$ *ii)* $H_1 \cap H_2 = H_1$; $H_1 \cup H_2 = H_2$.

Proof. Suppose that i) is true . Since $Y_2 \subseteq Y_1$ and $X_1 \subseteq X_2$, we have $(X_1, Y_1) \cap (X_2, Y_2) = (X_1 \cap X_2, \alpha(X_1 \cap X_2)) = (X_1, \alpha(X_1)) = (X_1, Y_1) = H_1$ and $H_1 \cup H_2 = (X_1, Y_1) \cup (X_2, Y_2) = (\beta(Y_1 \cap Y_2), Y_1 \cap Y_2)) = (\beta(Y_2), Y_2) = (X_2, Y_2) = H_2$, Hence, ii) is true. Suppose that ii) is true. Since $H_1 \cap H_2 = H_1 \Leftrightarrow (X_1, Y_1) = (X_1 \cap X_2, \alpha(X_1 \cap X_2))$, from definition 3 , it follows that $Y_1 \cup Y_2 \cap \alpha(X_1 \cap X_2) = Y_1$ and $Y_1 \cup Y_2 \subseteq Y_1$ .Thus $Y_2 \subseteq Y_1, i.e., H_1 \leq H_2$.
Hence i) is true. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Example-1: Table-1 shows a formal context (A, B, I), in which $A = \{1,2,3,4\}$ and $B = \{a,b,c,d,e\}$.
The concepts are $(\{1\}, \{a,b,d,e\}); (\{2,4\}, \{a,b,c\}); (\{1,3\}, \{d\}); (\{1,2,4\}, \{a,b\}); (U, \emptyset)$. The concept lattice
is shown in Figure-1.

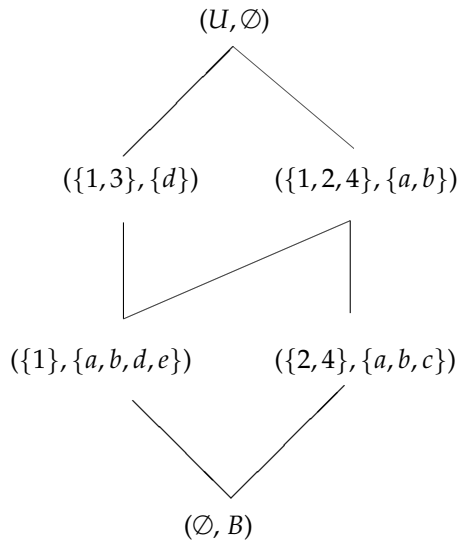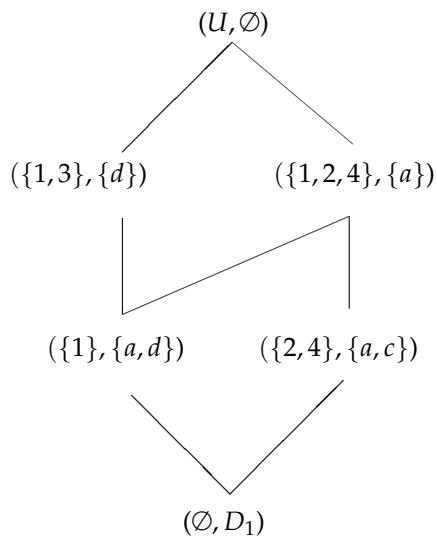|	a	b	c	d	e
1	1	1	0	1	1
2	1	1	1	0	0
3	0	0	0	1	0
4	1	1	1	0	0

Table - 1

Figure-1: $Latt(A, B, I)$ in example 1

Figure-2: $Latt(A, B, I_{D_1})$ in example 1

If we consider another example then it will able to understand how far the approximation causes for concept approximation in rough set.

Example-2: Table-2 shows a formal context (A, B, I), in which $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d, e\}$. The concepts are $(\{1\}, \{a, b, e\}); (\{4\}, \{a, c, e\}); (\{2\}, \{a, b, c\}); (\{2, 4\}, \{a, c\}); (\{1, 2\}, \{a, b\}); (\{1, 2, 4\}, \{a\}); (\{1, 3, 4\}, \{e\}); (U, \emptyset)$. The concept lattice is shown in Figure-3.

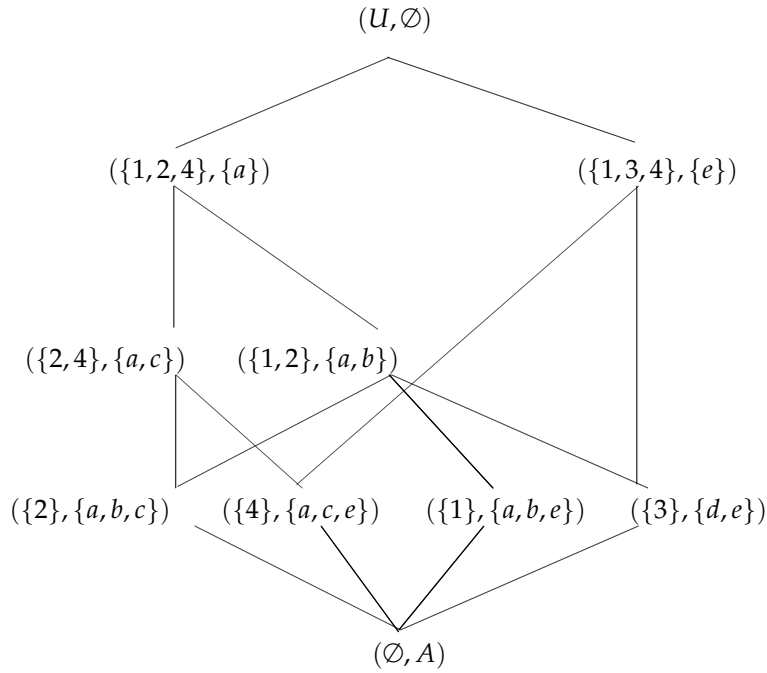|	a	b	c	d	e
1	1	1	0	0	1
2	1	1	1	0	0
3	0	0	0	1	1
4	1	0	1	0	1

Table - 2

Figure-3: Concept lattice for table-2

Let (A, B, I) be a formal context. Clearly we have the following results:

Proposition 3.1. *The core the formal context is a reduct $\Leftrightarrow$ There is only on reduct in the formal context.*

Proposition 3.2. $a \in B$ *is an unnecessary attribute $\Leftrightarrow A - \{a\}$ is a consistent set.*

Proposition 3.3. $a \in B$ *is an element of the core $\Leftrightarrow A - \{a\}$ is not a consistent set.*

If we consider another example then it will able to understand how far the approximation causes for concept approximation in rough set.

4   Rough Concept Lattice

Let (A, B, I, E) be an information system, where $A = \{a_1, a_2, \ldots, a_m\}$ is an object set, $B = \{x_1, x_2, x_3, x_4, \ldots, x_n\}$ is an attribute (property) set, E is an equivalent relation on A, $\forall X \subseteq A$, we can define the upper and lower approximations of X about E

$$X \downarrow_E \;\; = \;\; \bigcup \{M \in A/E : M \subseteq X\} = \{x \in A : E_A(x) \subseteq X\} \tag{4.1}$$

$$X \uparrow_E \;\; = \;\; \bigcup \{M \in A/E : M \cap X \neq \varnothing\} = \{x \in A : E_X(x) \cap X \neq \varnothing\}. \tag{4.2}$$

$X \downarrow_E$, $X \uparrow_E$ are called E- lower approximation and E- upper approximation of X respectively. If $X \downarrow_E = X \uparrow_E$, we say that X is definable, otherwise, X is rough. Similarly, we can define the upper and lower approximations of attributes set $Y \subseteq B$ about an equivalent relation on B.

Definition 4.3. *For all $Y \subseteq B$, we denote $E_Y = \{(a_i, a_j) \in X \times X : f_p(a_i) = f_p(a_j), p \in Y\}$, where $f_p : A \to \{0, 1\}$ is defined by $f_p(a_i) = 1$ if and only if the object a_i possesses property $p, (p \in A, a_i \in A)$, and E_Y is an equivalent relation, and E_Y can generate a partition of A, $X(Y) = \{Y(x) : x \in A\} = X/E_Y$, where $Y(x) = \{y \in A : yE_Yx\} = \{y \in A : f_p(y) = f_p(x), a_p \in Y\}$, that is, $Y(x) = \wedge\{(a_p, f_p(x)) : a_p \in Y\}, \bigcup Y(x) = \bigwedge(\wedge\{(a_p, f_p(x)) : a_p \in Y\}).*

Definition 4.4. *Let* (A, B, I, E) *is a rough formal context* $I \subseteq A \times B$ *for a set* $Y \subseteq B$ *of attributes , we define function* $\downarrow: 2^A \rightarrow 2^B, Y \downarrow = \{a \in A : (a, b) \in E, \forall b \in Y\}$ *(the set objects which have all attributes in Y). Correspondingly, for a set* $X \subseteq Y$ *of objects ,we define:* $\uparrow: 2^A \rightarrow 2^B, X \downarrow = \{b \in Y : (a, b) \in I, \forall a \in X\}$ *(the set objects which have all attributes in X).*

Example-3: The above table 1 is rough formal context (A, B, I, E) where $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d, e\}$ by rough theory $X = \{1, 2\} \subseteq A, Y = \{a, e\} \subseteq B$ then $f_a(1) = f_a(2) = f_a(4) = 1; f_a(3) = 0. f_b(1) = f_b(2) = f_b(4) = 1; f_b(3) = 0. f_c(2) = f_c(4) = 1; f_c(1) = f_c(3) = 0. f_d(1) = f_3(1) = 1; f_d(2) = f_d(4) = 0. f_e(1) = 1; f_e(2) = f_e(3) = f_e(4) = 0$, and the partition of X is : $B/X(1) = \{\{a, b, d, e\}, \{c\}\}; B/X(2) = \{\{a, b, c\}, \{d, e\}\}$. So $B/X = \{\{a, b\}, \{d, e\}, \{c\}, \varnothing\}$; and the partition of Y is $A/Y(a) = \{\{1, 2, 4\}, \{3\}\}. A/Y(e) = \{\{1\}, \{2, 3, 4\}\}$. So $A/Y = \{\{1\}, \{2, 4\}, \{3\}, \varnothing\}$. Under $B/X = E_X$ be the lower approximation of $Y = Y \downarrow_{E_X} = \varnothing$, the upper approximation of $Y = Y \uparrow_{E_X} = \{a, b, d, e\}$. Under $B/Y = E_Y$ be the lower approximation of $X = X \downarrow_{E_Y} = \{1\}$, the upper approximation of $X = Y \uparrow_{E_Y} = \{1, 2, 4\}$.

Here $f_p : A \rightarrow \{0, 1\}$, set $\{0, 1\}$ can extent to $[0, 1]$, and the condition of equivalent relation can also substitute for other relations, for example, the similar relation (that is, it satisfies reflexivity, symmetry, and transitivity). In essential, those functions are the same. The information system (A, B, I, E) which has lower and upper approximations and partition is called a rough formal context. $\forall a \in A, b \in B$, object a has attribute B then $(a, b) \in I$, aIb. For the rough formal context in Table-1, the Hasse diagram is shown in Figure 4:
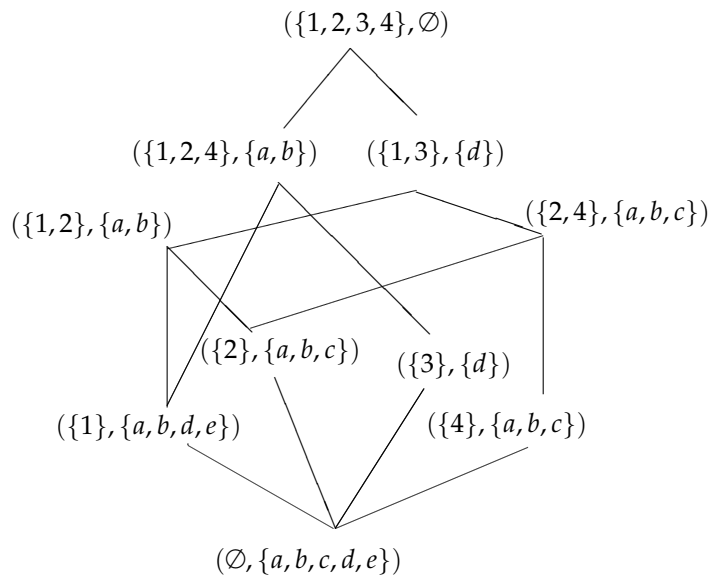

Figure-4: Rough concept lattice for the rough formal context in Table-1

Now we consider another example of a rough formal context and its corresponding concept lattice
Example-4: The following Table-3 is a rough formal context (A, B, I, E) where $A = \{1, 2, 3, 4, 5\}, B = \{h_1, h_2, ..., h_9\}$,

where 1,2,3,4,5 stands for "big", "beautiful","wooden", "cheap" and the green surroundings respectively. Also here $h_1, h_2, ..., h_9$ are nine houses.

	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9
1	1	0	1	0	0	1	0	1	0
2	1	0	1	0	0	0	1	0	1
3	1	0	0	1	0	0	1	0	1
4	0	1	1	0	0	1	0	1	0
5	0	1	0	0	1	0	1	0	0

Table -3

By rough set theory $X = \{1,2\} \subseteq A, Y = \{h_1, h_5\} \subseteq B$ then $f_{h_1}(1) = f_{h_1}(2) = f_{h_1}(3) = 1; f_{h_1}(4) = f_{h_1}(5) = 0. f_{h_2}(1) = f_{h_2}(2) = f_{h_2}(3) = 1; f_{h_2}(4) = f_{h_2}(5) = 0. f_{h_3}(1) = f_{h_3}(2) = f_{h_3}(4) = 1; f_{h_3}(3) = f_{h_3}(5) = 0....$and the partition of X are : $B/X(1) = \{\{h_1, h_3, h_6, h_8\}, \{h_2, h_4, h_5, h_7, h_9\}\}; B/X(2) = \{\{h_1, h_3, h_7\}, \{h_2, h_4, h_5, h_6, h_8\}\}$ so $B/X = \{\{h_1, h_3\}, \{h_6, h_8\}, \{h_7, h_9\}, \{h_4, h_5\}\}$; and the partition of Y are $A/Y(h_1) = \{\{1,2,3\}, \{4,5\}\}$, $A/Y(h_9) = \{\{2,3\}, \{1,4,5\}\}$. So $A/Y = \{\{1\}, \{2,3\}, \{4,5\}\}$. Under $B/X = E_X$ be the lower approximation of $Y = Y \downarrow_{E_X} = \emptyset$, the upper approximation of $Y = Y \uparrow_{E_X} = \{h_1, h_3, h_7, h_9\}$.Under $B/Y = E_Y$ be the lower approximation of $X = X \downarrow_{E_Y} = \{1\}$, the upper approximation of $X = Y \uparrow_{E_Y} = \{1,2,3\}$.

Here $I = \{0,1\}$ i.e, we only consider $\{a,b\} \in I$ or not. The following figure represents rough concept lattice, based on the information system described in Table-3.

$(\{1,2,3,4,5\}, \emptyset)$

$(\{1,2,3\}, \{h_1\})$
$(\{1,2,4\}, \{h_3\})$
$(\{2,3,5\}, \{h_7\})$
$(\{4,5\}, \{h_2\})$

$(\{1,2\}, \{h_1, h_3\})$
$(\{2,3\}, \{h_7, h_9\})$
$(\{1,4\}, \{h_3, h_6, h_8\})$

$(\{3\}, \{h_1, h_4, h_7, h_9\})$

$(\{1\}, \{h_1, h_3, h_6, h_8\})$
$(\{2\}, \{h_1, h_3, h_7, h_9\})$
$(\{4\}, \{h_2, h_3, h_6, h_8\})$
$(\{5\}, \{h_2, h_5, h_7\})$

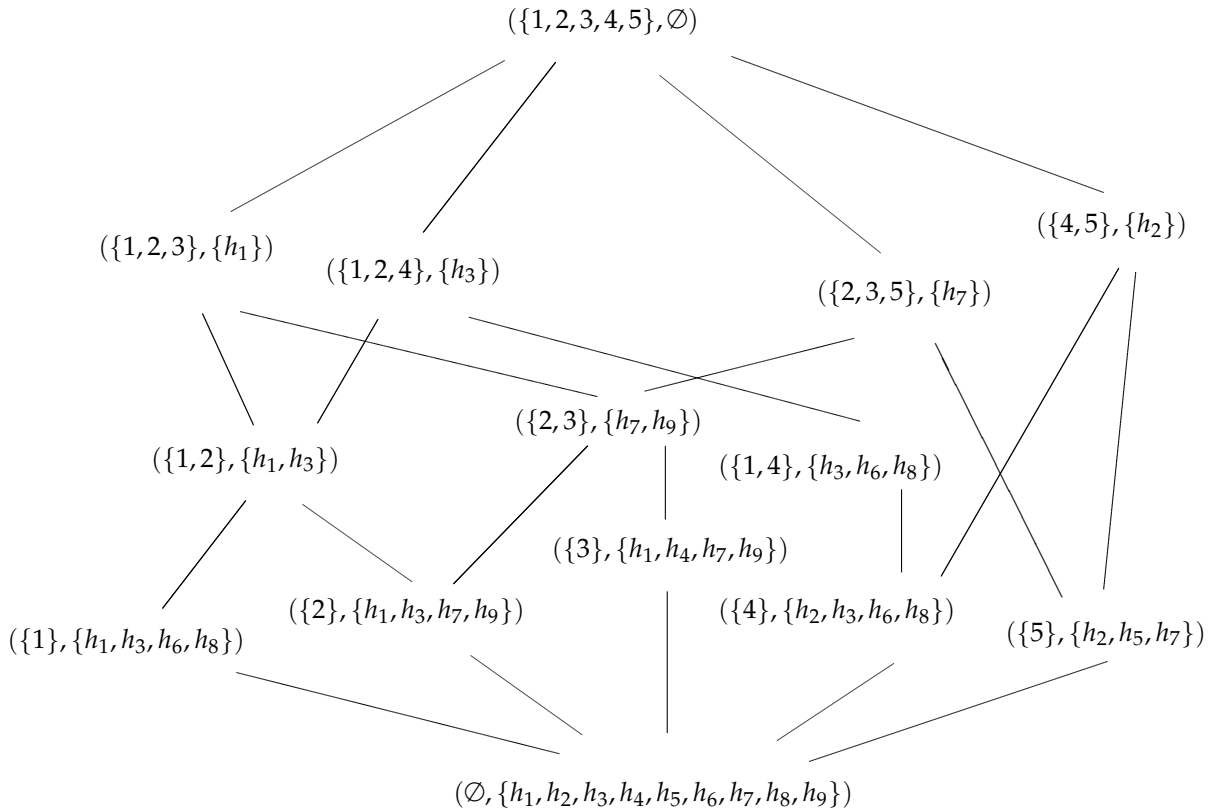$(\emptyset, \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9\})$

Figure-5: Rough concept lattice for the rough formal context in Table-3

One can find the extension of their meet by the set intersection of their extensions, and the intension of their join by the set intersection of their intensions. One cannot find directly the intension of their meet and the

extension of their join by simply applying set-theoretic operators.

5 Conclusion

This paper presents the approach to approximate concepts in the framework of the formal concept analysis. The main focus is to show how rough set techniques can be employed as an approach to the problem of knowledge extraction. The approaches show how to approximate single sets of objects, single sets of features, and non-definable concepts. we use both the set of objects and the set of features for approximating non-definable concepts, whose results in the fact that non-definable concepts with the same set of objects have different and more accurate concept approximations. Rough lattice combines the advantages of concept lattice and rough set, so it is widely used in Information Retrieval, Data Mining, Software Engineering and other fields [27].

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