Malaya J. Mat. 3(1)(2015) 30–44

Malaya Journal of Matematik

www.malayajournal.org

MJM an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Some Results for a Four-Point Boundary Value Problems for a Coupled System Involving Caputo Derivatives

M. Houas[a,*] , M. Benbachir[a] and Z. Dahmani[b]

[a]Faculty of Sciences and Technology Khemis Miliana University, Ain Defla, Algeria.

[b]LPAM, Faculty SEI, UMAB Mostaganem, Algeria.

Abstract

Motivated by the problem (1.1) in [5], in this paper, we prove the existence and uniqueness of solutions for the following system of fractional differential equations with four point boundary conditions:

$$\begin{cases} D^{\alpha}x(t)+f\left(t,y(t),D^{\delta}y(t)\right)=0, t\in J,\\ D^{\beta}y(t)+g\left(t,x(t),D^{\sigma}x(t)\right)=0, t\in J,\\ x(0)=y(0)=0, x(1)-\lambda_1 x(\eta)=0, y(1)-\lambda_1 y(\eta)=0,\\ x''(0)=y''(0)=0, x''(1)-\lambda_2 x''(\xi)=0, y''(1)-\lambda_2 y''(\xi)=0, \end{cases}$$

where $3<\alpha,\beta\leq 4, \alpha-2<\sigma\leq\alpha-1, \beta-2<\delta\leq\beta-1, 0<\xi,\eta<1$, and $D^{\alpha}, D^{\beta}, D^{\delta}$ and D^{σ}, are the Caputo fractional derivatives, $J=[0,1]$, λ_1,λ_2 are real constants with $\lambda_1\eta\neq 1, \lambda_2\xi\neq 1$ and f,g continuous functions on $[0,1]\times\mathbb{R}^2$.

Keywords: Caputo derivative; Boundary Value Problem; fixed point theorem.

*2010 MSC:*26A33, 34B25, 34B15.

1 Introduction

Differential equations of fractional order have been shown to be very useful in the study of models of many phenomena in various fields of science and engineering, such as electrochemistry, physics, chemistry, viscoelasticity, control, image and signal processing, biophysics. For more details, we refer the reader to [4, 7, 10, 12, 13, 15, 17, 18] and references therein. There has been a significant progress in the investigation of these equations in recent years, see [6, 8, 9, 15, 16, 27]. More recently, some basic theory for the initial boundary value problems of fractional differential equations has been discussed in [1, 14, 15]. Recently, existence

*Corresponding author.

E-mail address: houasmed@yahoo.fr(M. Houas), mbenbachir2001@gmail.com.in (M. Benbachir) ,zzdahmani@yahoo.fr(Z. Dahmani)

and uniqueness of solutions to boundary value problems for fractional differential equations had attracted the attention of many authors, see for example, [4, 6, 8, 9, 15, 16, 19, 27] and the references therein. The study of coupled system of fractional order is also important as such systems occur in various problems of applied science [3, 11, 20, 21, 24, 26]. In the last decade, many authors have established the existence and uniqueness for solutions of some systems of nonlinear fractional differential equations, one can see [20, 23, 24, 25] and references cited therein. For example in [2, 5, 21, 26] the authors established sufficient conditions for the existence of solutions for a two-point and three-point boundary value problem for a coupled system of fractional differential equations.

In [2, 5, 21, 22, 26], the existence and uniqueness of solutions was investigated for a nonlinear coupled system for fractional differential equations with two-point and three-point boundary conditions by using Schauder's fixed point theorem.

Motivated by the problem (1.1) in [5], this paper deals with the existence of solution for the following fractional differential problem:

$$\begin{cases} D^{\alpha}x(t) + f\left(t, y(t), D^{\delta}y(t)\right) = 0, t \in J, \\ D^{\beta}y(t) + g(t, x(t), D^{\sigma}x(t)) = 0, t \in J, \\ x(0) = y(0) = 0, x(1) - \lambda_1 x(\eta) = 0, y(1) - \lambda_1 y(\eta) = 0, \\ x''(0) = y''(0) = 0, x''(1) - \lambda_2 x''(\xi) = 0, y''(1) - \lambda_2 y''(\xi) = 0, \end{cases} \tag{1.1}$$

where $3 < \alpha, \beta \leq 4, \alpha - 2 < \sigma \leq \alpha - 1, \beta - 2 < \delta \leq \beta - 1, 0 < \xi, \eta < 1$, and $D^{\alpha}, D^{\beta}, D^{\delta}$ and D^{σ}, are the Caputo fractional derivatives, $J = [0,1], \lambda_1, \lambda_2$ are real constants with $\lambda_1\eta \neq 1, \lambda_2\xi \neq 1$ and f, g are continuous functions on $[0,1] \times \mathbb{R}^2$.

The rest of this paper is organized as follows. In section 2, we present some preliminaries and lemmas. Section 3 is devoted to existence of solution of problem (1.1). In section 4 examples are treated illustrating our results.

2 Preliminaries

The following notations, definitions, and preliminary facts will be used throughout this paper.

Definition 2.1. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[0, \infty[$ is defined as:*

$$J^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \alpha > 0, \tag{2.2}$$

$$J^0 f(t) = f(t),$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} du$.

Definition 2.2. *The fractional derivative of $f \in C^n([0, \infty[)$ in the Caputo's sense is defined as:*

$$D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, n-1 < \alpha, n \in N^*. \tag{2.3}$$

For more details about fractional calculus, we refer the reader to [15, 18].

We will consider the following spaces:

$$X = \{x : x \in C([0,1]), D^{\sigma}x \in C([0,1])\},$$

and

$$Y = \{y : y \in C([0,1]), D^{\delta}y \in C([0,1])\},$$

endowed with the norms:

$$\|x\|_X = \|x\| + \|D^{\sigma}x\|, \ \|x\| = \sup_{t\in J}|x(t)|, \|D^{\sigma}x\| = \sup_{t\in J}|D^{\sigma}x(t)|,$$

and

$$\|y\|_Y = \|y\| + \left\|D^{\delta}y\right\|, \ \|y\| = \sup_{t\in J}|y(t)|, \left\|D^{\delta}y\right\| = \sup_{t\in J}\left|D^{\delta}y(t)\right|.$$

We know that $(X, \|\ .\ \|_X)$ and $(Y, \|\ .\ \|_Y)$, is a Banach space. The product space $\left(X \times Y, \|(x,y)\|_{X\times Y}\right)$ is also a Banach space, with norm $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$.

We recall the following important lemmas [13]:

Lemma 2.1. *For $\alpha > 0$, the general solution of the fractional differential equation $D^{\alpha}x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1}, \tag{2.4}$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, .., n-1, n = [\alpha] + 1$.

Lemma 2.2. *Let $\alpha > 0$. Then*

$$J^{\alpha}D^{\alpha}x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1}, \tag{2.5}$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1, n = [\alpha] + 1$.

We prove the following result:

Lemma 2.3. *Let $g \in C([0,1])$, the solution of the equation*

$$D^{\alpha}x(t) + g(t) = 0, t \in J, \ 3 < \alpha \leq 4, \tag{2.6}$$

subject to the conditions

$$x(0) = 0, \ x(1) - \lambda_1 x(\eta) = 0, \tag{2.7}$$
$$x''(0) = 0, \ x''(1) - \lambda_2 x''(\xi) = 0,$$

is given by:

$$\begin{aligned} x(t) = &-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)\,ds \\ &+ \frac{\lambda_1 t}{(\lambda_1\eta - 1)\Gamma(\alpha)}\int_0^{\eta} (\eta - s)^{\alpha-1} g(s)\,ds \\ &- \frac{t}{(\lambda_1\eta - 1)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} g(s)\,ds \\ &+ \frac{\left(\lambda_2 - \lambda_2\lambda_1\eta^3\right)t + (\lambda_2\lambda_1\eta - \lambda_2)t^3}{6(\lambda_1\eta - 1)(\lambda_2\xi - 1)\Gamma(\alpha-2)}\int_0^{\xi} (\xi - s)^{\alpha-3} g(s)\,ds \\ &- \frac{\left(1 - \lambda_1\eta^3\right)t + (\lambda_1\eta - 1)t^3}{6(\lambda_1\eta - 1)(\lambda_2\xi - 1)\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3} g(s)\,ds. \end{aligned} \tag{2.8}$$

Proof. We use the same technics as in [5]. For $c_i \in \mathbb{R}, i = 0, 1, 2, 3$, and by Lemmas 2.1, 2.2, we have

$$x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds - c_0 - c_1 t - c_2 t^2 - c_3 t^3 \tag{2.9}$$

Using (2.7), we get $c_0 = c_2 = 0$, and

$$\begin{aligned} c_1 = &-\frac{\lambda_1}{(\lambda_1\eta - 1)\Gamma(\alpha)} \int_0^\eta (\eta - s)^{\alpha-1} g(s)\, ds \\ &+ \frac{1}{(\lambda_1\eta - 1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} g(s)\, ds \\ &- \frac{\lambda_2 (1-\lambda_1\eta^3)}{6(\lambda_1\eta - 1)(\lambda_2\xi - 1)\Gamma(\alpha-2)} \int_0^\xi (\xi - s)^{\alpha-3} g(s)\, ds \\ &+ \frac{(1-\lambda_1\eta)}{6(\lambda_1\eta - 1)(\lambda_2\xi - 1)\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} g(s)\, ds \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} c_3 = &-\frac{\lambda_2}{6(\lambda_2\xi - 1)\Gamma(\alpha-2)} \int_0^\xi (\xi - s)^{\alpha-3} g(s)\, ds \\ &+ \frac{1}{6(\lambda_2\xi - 1)\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} g(s)\, ds \end{aligned} \tag{2.11}$$

Substituting the value of c_1 and c_3 in (2.9), we obtain the desired quantity in Lemma. □

3 Main Results

Let us set:

$$\begin{aligned} M_1 = &\frac{|\lambda_1\eta-1|+|\lambda_1|\eta^\alpha+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)} \\ &+ \frac{\left(|\lambda_2-\lambda_2\lambda_1\eta^3|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+|1-\lambda_1\eta^3|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}, \\ M_2 = &\frac{1}{\Gamma(\alpha-\sigma+1)} + \frac{|\lambda_1|\eta^\alpha+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)} \\ &+ \frac{|\lambda_2-\lambda_2\lambda_1\eta^3|\xi^{\alpha-2}+|1-\lambda_1\eta^3|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)} + \frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}, \\ M_3 = &\frac{|\lambda_1\eta-1|+|\lambda_1|\eta^\beta+1}{|\lambda_1\eta-1|\Gamma(\beta+1)} + \frac{\left(|\lambda_2-\lambda_2\lambda_1\eta^3|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\beta-2}+|1-\lambda_1\eta^3|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}, \\ M_4 = &\frac{1}{\Gamma(\beta-\delta+1)} + \frac{|\lambda_1|\eta^\beta+1}{|\lambda_1\eta-1|\Gamma(\beta+1)\Gamma(2-\delta)} \\ &+ \frac{|\lambda_2-\lambda_2\lambda_1\eta^3|\xi^{\beta-2}+|1-\lambda_1\eta^3|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)} + \frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}, \\ L_1 = &\frac{\left(|\lambda_2-\lambda_2\lambda_1\eta^3|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+|1-\lambda_1\eta^3|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}, \\ L_2 = &\frac{|\lambda_2-\lambda_2\lambda_1\eta^3|\xi^{\alpha-2}+|1-\lambda_1\eta^3|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)} + \frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}, \\ L_3 = &\frac{\left(|\lambda_2-\lambda_2\lambda_1\eta^3|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\beta-2}+|1-\lambda_1\eta^3|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}, \\ L_4 = &\frac{|\lambda_2-\lambda_2\lambda_1\eta^3|\xi^{\beta-2}+|1-\lambda_1\eta^3|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)} + \frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}. \end{aligned} \tag{3.12}$$

Let us also consider the following hypotheses:

$(H1)$: There exist two constants k_1 and k_2 such that for all $t \in [0,1]$ and $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, we have

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq k_1 (|x_1 - x_2| + |y_1 - y_2|), \tag{3.13}$$
$$|g(t,x_1,y_1) - g(t,x_2,y_2)| \leq k_2 (|x_1 - x_2| + |y_1 - y_2|).$$

$(H2))$: The functions $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous

$(H3)$: There exists positive constants N_1 and N_2 such that

$$|f(t,x,y)| \leq N_1, \ |g(t,x,y)| \leq N_2 \text{ for each } t \in J \text{ and all } x,y \in \mathbb{R}.$$

We prove the following theorem:

Theorem 3.1. *Assume that* $(H1)$ *holds.*

If

$$k_1 (M_1 + M_2) + k_2 (M_3 + M_4) < 1, \tag{3.14}$$

then the problem (1.1) *has a unique solution.*

Proof. The proof is similarly to that of Theorem 3.1 in [5] by taking $k_1 = \omega_1 + \omega_2$ and $k_2 = \varpi_1 + \varpi_2$.

Now, we prove the following result:

Theorem 3.2. *Assume that the hypotheses* $(H1) - (H2)$ *and* $(H3)$ *are satisfied, such that*

$$k_1\theta_1 + k_2\theta_2 < 1, \tag{3.15}$$

where

$$\theta_1 = \frac{|\lambda_1\eta-1|+|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\sigma+1)} + \frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)},$$
$$\theta_2 = \frac{|\lambda_1\eta-1|+|\lambda_1|\eta^{\beta}+1}{|\lambda_1\eta-1|\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta-\delta+1)} + \frac{|\lambda_1|\eta^{\beta}+1}{|\lambda_1\eta-1|\Gamma(\beta+1)\Gamma(2-\delta)}.$$

If there exists $\mu \in \mathbb{R}$ *such that*

$$N_1 (M_1 + M_2) + N_2 (M_3 + M_4) \leq \mu, \tag{3.16}$$

then, the problem (1.1) *has at least a solution.*

Proof. We shall use Krasnselskii's fixed point theorem to prove that ϕ has at least a fixed point on $X \times Y$.

Suppose that (3.16) holds and let us take

$$\phi(x,y)(t) := T(x,y)(t) + R(x,y)(t), \tag{3.17}$$

where

$$T(x,y)(t) := (T_1 y(t), T_2 x(t)), \tag{3.18}$$

$$\begin{aligned} T_1 y(t) = & -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s), D^{\delta} y(s)\right) ds \\ & + \frac{\lambda_1 t}{(\lambda_1\eta-1)\Gamma(\alpha)} \int_0^{\eta} (\eta-s)^{\alpha-1} f\left(s, y(s), D^{\delta} y(s)\right) ds \\ & - \frac{t}{(\lambda_1\eta-1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f\left(s, y(s), D^{\delta} y(s)\right) ds, \end{aligned} \tag{3.19}$$

$$T_2x\left(t\right)=-\tfrac{1}{\Gamma(\beta)}\int_0^t\left(t-s\right)^{\beta-1}g\left(s,x\left(s\right),D^{\sigma}x\left(s\right)\right)ds \quad (3.20)$$
$$+\tfrac{\lambda_1 t}{(\lambda_1\eta-1)\Gamma(\beta)}\int_0^{\eta}\left(\eta-s\right)^{\alpha-1}g\left(s,x\left(s\right),D^{\sigma}x\left(s\right)\right)ds$$
$$-\tfrac{t}{(\lambda_1\eta-1)\Gamma(\beta)}\int_0^{1}\left(1-s\right)^{\alpha-1}g\left(s,x\left(s\right),D^{\sigma}x\left(s\right)\right)ds,$$

and

$$R\left(x,y\right)\left(t\right):=\left(R_1y\left(t\right),R_2x\left(t\right)\right), \quad (3.21)$$

where,

$$R_1y\left(t\right)=\tfrac{\left(\lambda_2-\lambda_2\lambda_1\eta^3\right)t+\left(\lambda_2\lambda_1\eta-\lambda_2\right)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)}\int_0^{\xi}\left(\xi-s\right)^{\alpha-3}f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)ds$$
$$-\tfrac{\left(1-\lambda_1\eta^3\right)t+\left(\lambda_1\eta-1\right)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)}\int_0^{1}\left(1-s\right)^{\alpha-3}f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)ds, \quad (3.22)$$

$$R_2x\left(t\right)=\tfrac{\left(\lambda_2-\lambda_2\lambda_1\eta^3\right)t+\left(\lambda_2\lambda_1\eta-\lambda_2\right)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\beta-2)}\int_0^{\xi}\left(\xi-s\right)^{\alpha-3}g\left(s,x\left(s\right),D^{\sigma}x\left(s\right)\right)ds$$
$$-\tfrac{\left(1-\lambda_1\eta^3\right)t+\left(\lambda_1\eta-1\right)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\beta-2)}\int_0^{1}\left(1-s\right)^{\alpha-3}g\left(s,x\left(s\right),D^{\sigma}x\left(s\right)\right)ds. \quad (3.23)$$

The proof will be given in several steps.

Step1: We shall prove that for any $(x,y),(x_1,y_1)\in B_{\mu}$, then $T(x,y)+R(x_1,y_1)\in B_{\mu}$, Such that $B_{\mu}=\{(x,y)\in X\times Y;\|(x,y)\|_{X\times Y}\leq\mu\}$.

For any $(x,y),(x_1,y_1)\in B_{\mu}$ and for each $t\in J$ we have:

$$\left|T_1y\left(t\right)+R_1y_1\left(t\right)\right|=\left|-\tfrac{1}{\Gamma(\alpha)}\int_0^t\left(t-s\right)^{\alpha-1}f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)ds\right.$$
$$+\tfrac{\lambda_1 t}{(\lambda_1\eta-1)\Gamma(\alpha)}\int_0^{\eta}\left(\eta-s\right)^{\alpha-1}f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)ds$$
$$-\tfrac{t}{(\lambda_1\eta-1)\Gamma(\alpha)}\int_0^{1}\left(1-s\right)^{\alpha-1}f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)ds$$
$$+\tfrac{\left(\lambda_2-\lambda_2\lambda_1\eta^3\right)t+\left(\lambda_2\lambda_1\eta-\lambda_2\right)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)}\int_0^{\xi}\left(\xi-s\right)^{\alpha-3}f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)ds$$
$$\left.-\tfrac{\left(1-\lambda_1\eta^3\right)t+\left(\lambda_1\eta-1\right)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\beta-2)}\int_0^{1}\left(1-s\right)^{\alpha-3}f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)ds\right|$$

then,

$$\left|T_1y\left(t\right)+R_1y_1\left(t\right)\right|\leq\tfrac{1}{\Gamma(\alpha)}\int_0^t\left(t-s\right)^{\alpha-1}\left|f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)\right|ds$$
$$+\tfrac{|\lambda_1|}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{\eta}\left(\eta-s\right)^{\alpha-1}\left|f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)\right|ds$$
$$+\tfrac{1}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{1}\left(1-s\right)^{\alpha-1}\left|f\left(s,y\left(s\right),D^{\delta}y\left(s\right)\right)\right|ds$$
$$+\tfrac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+\left|\lambda_2\lambda_1\eta-\lambda_2\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{\xi}\left(\xi-s\right)^{\alpha-3}\left|f\left(s,y_1\left(s\right),D^{\delta}y_1\left(s\right)\right)\right|ds$$
$$+\tfrac{\left|1-\lambda_1\eta^3\right|+\left|\lambda_1\eta-1\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{1}\left(1-s\right)^{\alpha-3}\left|f\left(s,y_1\left(s\right),D^{\delta}y_1\left(s\right)\right)\right|ds.$$

Using $(H3)$, we obtain

$$\begin{aligned}|T_1 y(t)+R_1 y_1(t)| &\leq N_1\left[\tfrac{|\lambda_1\eta-1|+|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)}\right]\\ &+N_1\left[\tfrac{\left[\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|\right]}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right].\end{aligned}$$

Consequently,

$$|T_1 y(t)+R_1 y_1(t)| \leq N_1 M_1.$$

Thus,

$$\|T_1(y)+R_1(y_1)\| \leq N_1 M_1, \tag{3.24}$$

and

$$\begin{aligned}|D^{\sigma}T_1 y(t)+D^{\sigma}R_1 y_1(t)| &\leq \tfrac{1}{\Gamma(\alpha-\sigma)}\int_0^t (t-s)^{\alpha-\sigma-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds\\ &+\tfrac{|\lambda_1|}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds\\ &+\tfrac{1}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^1(1-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds\\ &+\left[\begin{array}{c}\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)}\\ +\frac{|\lambda_2\lambda_1\eta-\lambda_2|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\end{array}\right]\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,y_1(s),D^{\delta}y_1(s)\right)\right|ds\\ &+\left[\begin{array}{c}\frac{\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)}\\ +\frac{|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\end{array}\right]\int_0^1(1-s)^{\alpha-3}\left|f\left(s,y_1(s),D^{\delta}y_1(s)\right)\right|ds.\end{aligned}$$

By $(H3)$, we have

$$\begin{aligned}|D^{\sigma}T_1 y(t)+D^{\sigma}R_1 y_1(t)| &\leq N_1\left[\tfrac{1}{\Gamma(\alpha-\sigma+1)}+\tfrac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\right]\\ &+N_1\left[\tfrac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}+\tfrac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\right].\end{aligned}$$

Consequently, we obtain

$$|D^{\sigma}T_1 y(t)+D^{\sigma}R_1 y_1(t)| \leq N_1 M_2.$$

Hence,

$$\|D^{\sigma}T_1(y)+D^{\sigma}R_1(y_1)\| \leq N_1 M_2. \tag{3.25}$$

Combining (3.24) and (3.25) , yields

$$\|T_1(y)+R_1(y_1)\|_X \leq N_1(M_1+M_2). \tag{3.26}$$

Analogously, we have

$$\|T_2(x)+R_2(x_1)\|_Y \leq N_2(M_3+M_4). \tag{3.27}$$

Hence, it follows from (3.26) and (3.27) that

$$\|T(x,y)+R(x_1,y_1)\|_{X\times Y} \leq N_1(M_1+M_2)+N_2(M_3+M_4)<\mu. \tag{3.28}$$

Step2: We shall prove that R is continuous and compact.

[1*] : The continuity of f and g implies that the operator R is continuous.

[2*] : Now, we prove that R maps bounded sets into bounded sets of $X \times Y$.

For $(x, y) \in B_{\mu}$ and for each $t \in J$, we have:

$$|R_1 y(t)| \leq \frac{|\lambda_2 - \lambda_2\lambda_1\eta^3| t + |\lambda_2\lambda_1\eta - \lambda_2| t^3}{6|\lambda_1\eta - 1||\lambda_2\xi - 1|\Gamma(\alpha-2)} \int_0^{\xi} (\xi - s)^{\alpha-3} \left| f\left(s, y(s), D^{\delta} y(s)\right)\right| ds$$
$$+ \frac{|1 - \lambda_1\eta^3| t + |\lambda_1\eta - 1| t^3}{6|\lambda_1\eta - 1||\lambda_2\xi - 1|\Gamma(\alpha-2)} \int_0^{1} (1 - s)^{\alpha-3} \left| f\left(s, y(s), D^{\delta} y(s)\right)\right| ds.$$

Using $(H3)$, we obtain

$$|R_1 y(t)| \leq \frac{N_1\left[\left(|\lambda_2 - \lambda_2\lambda_1\eta^3| + |\lambda_2\lambda_1\eta - \lambda_2|\right)\xi^{\alpha-2} + |1 - \lambda_1\eta^3| + |\lambda_1\eta - 1|\right]}{6|\lambda_1\eta - 1||\lambda_2\xi - 1|\Gamma(\alpha-1)}$$
$$\leq N_1\left(\frac{\left(|\lambda_2 - \lambda_2\lambda_1\eta^3| + |\lambda_2\lambda_1\eta - \lambda_2|\right)\xi^{\alpha-2} + |1 - \lambda_1\eta^3| + |\lambda_1\eta - 1|}{6|\lambda_1\eta - 1||\lambda_2\xi - 1|\Gamma(\alpha-1)}\right).$$

Thus,

$$|R_1 y(t)| \leq N_1 L_1, t \in J,$$

Therefore,

$$\|R_1(y)\| \leq N_1 L_1. \tag{3.29}$$

On the other hand,

$$|D^{\sigma} R_1 y(t)| \leq \frac{1}{\Gamma(\alpha-2)} \left(\begin{array}{c} \frac{|\lambda_2 - \lambda_2\lambda_1\eta^3| t^{1-\sigma}}{6|\lambda_1\eta - 1||\lambda_2\xi - 1|\Gamma(2-\sigma)} + \\ \frac{|\lambda_2\lambda_1\eta - \lambda_2| t^{3-\sigma}}{|\lambda_1\eta - 1||\lambda_2\xi - 1|\Gamma(4-\sigma)} \end{array}\right) \int_0^{\xi} (\xi - s)^{\alpha-3} \left| f\left(s, y(s), D^{\delta} y(s)\right)\right| ds$$
$$+ \frac{1}{\Gamma(\alpha-2)} \left(\begin{array}{c} \frac{|1 - \lambda_1\eta^3| t^{1-\sigma}}{6\Gamma(2-\sigma)|\lambda_1\eta - 1||\lambda_2\xi - 1|} + \\ \frac{|\lambda_1\eta - 1| t^{3-\sigma}}{|\lambda_1\eta - 1||\lambda_2\xi - 1|\Gamma(4-\sigma)} \end{array}\right) \int_0^{1} (1 - s)^{\alpha-3} \left| f\left(s, y(s), D^{\delta} y(s)\right)\right| ds.$$

By $(H3)$, we have

$$|D^{\sigma} \phi_1 y(t)| \leq N_1 \left[\frac{|\lambda_2 - \lambda_2\lambda_1\eta^3| \xi^{\alpha-2} + |1 - \lambda_1\eta^3|}{6|\lambda_1\eta - 1||\lambda_2\xi - 1|\Gamma(\alpha-1)\Gamma(2-\sigma)} + \frac{|\lambda_2\lambda_1\eta - \lambda_2| \xi^{\alpha-2} + |\lambda_1\eta - 1|}{|\lambda_1\eta - 1||\lambda_2\xi - 1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\right]$$
$$\leq N_1 \left[\frac{|\lambda_2 - \lambda_2\lambda_1\eta^3| \xi^{\alpha-2} + |1 - \lambda_1\eta^3|}{6|\lambda_1\eta - 1||\lambda_2\xi - 1|\Gamma(\alpha-1)\Gamma(2-\sigma)} + \frac{|\lambda_2\lambda_1\eta - \lambda_2| \xi^{\alpha-2} + |\lambda_1\eta - 1|}{|\lambda_1\eta - 1||\lambda_2\xi - 1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\right].$$

Consequently, we obtain

$$|D^{\sigma} R_1 y(t)| \leq N_1 L_2, t \in J.$$

Therefore,

$$\|D^{\sigma} R_1(y)\| \leq N_1 L_2. \tag{3.30}$$

Hence, from (3.29) and (3.30), we have

$$\|R_1(y)\|_X \leq N_1(L_1 + L_2). \tag{3.31}$$

Similarly, it can be shown that

$$\|R_2(x)\|_Y \leq N_2(L_3 + L_4). \tag{3.32}$$

It follows from (3.31) and (3.32) that

$$\|R(x, y)\|_{X \times Y} \leq N_1(L_1 + L_2) + N_2(L_3 + L_4). \tag{3.33}$$

Consequently,

$$\|R(x,y)\|_{X\times Y}<\infty.$$

[3*] : We show that R is equi-continuous:

Let $t_1, t_2 \in J$, such that $t_1 < t_2$ and $(x,y) \in B_\mu$. Then, we have:

$$\begin{aligned}|R_1y(t_2)-R_1y(t_1)| &\leq \frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|(t_2-t_1)+|\lambda_2\lambda_1\eta-\lambda_2|\left(t_2^3-t_1^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^\xi(\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds\\&+\frac{\left|1-\lambda_1\eta^3\right|(t_1-t_2)+|\lambda_1\eta-1|\left(t_1^3-t_2^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^1(1-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds.\end{aligned}$$

Using $(H3)$, we obtain

$$\begin{aligned}|R_1y(t_2)-R_1y(t_1)| &\leq \frac{N_1\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}(t_2-t_1)\\&+\frac{N_1\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}(t_1-t_2)\\&+\frac{N_1|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_2^3-t_1^3\right)+\frac{N_1|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_1^3-t_2^3\right),\end{aligned}\tag{3.34}$$

and

$$\begin{aligned}|D^\sigma R_1y(t_2)-D^\sigma R_1y(t_1)| &\leq \left[\begin{array}{c}\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\left(t_2^{1-\sigma}-t_1^{1-\sigma}\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)}\\+\frac{|\lambda_2\lambda_1\eta-\lambda_2|\left(t_2^{3-\sigma}-t_1^{3-\sigma}\right)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\end{array}\right]\int_0^\xi(\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds\\&+\left[\begin{array}{c}\frac{\left|1-\lambda_1\eta^3\right|\left(t_1^{1-\sigma}-t_2^{1-\sigma}\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)}\\+\frac{|\lambda_1\eta-1|\left(t_1^{3-\sigma}-t_2^{3-\sigma}\right)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\end{array}\right]\int_0^1(1-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds.\end{aligned}$$

By $(H3)$, we have:

$$\begin{aligned}|D^\sigma R_1y(t_2)-D^\sigma R_1y(t_1)| &\leq \frac{N_1\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\left(t_2^{1-\sigma}-t_1^{1-\sigma}\right)\\&+\frac{N_1\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\left(t_1^{1-\sigma}-t_2^{1-\sigma}\right)\\&+\frac{N_1|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_2^{3-\sigma}-t_1^{3-\sigma}\right)\\&+\frac{N_1|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_1^{3-\sigma}-t_2^{3-\sigma}\right).\end{aligned}\tag{3.35}$$

Hence, by (3.34) and (3.35), we obtain

$$\begin{aligned}\|R_1y(t_2)-R_1y(t_1)\|_X &\leq \frac{N_1\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}(t_2-t_1)+\frac{N_1\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}(t_1-t_2)\\&+\frac{N_1|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_2^3-t_1^3\right)+\frac{N_1|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_1^3-t_2^3\right)\\&+\frac{N_1\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\left(t_2^{1-\sigma}-t_1^{1-\sigma}\right)\\&+\frac{N_1\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\left(t_1^{1-\sigma}-t_2^{1-\sigma}\right)\\&+\frac{N_1|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_2^{3-\sigma}-t_1^{3-\sigma}\right)\\&+\frac{N_1|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_1^{3-\sigma}-t_2^{3-\sigma}\right).\end{aligned}\tag{3.36}$$

Analogously, we can obtain

$$\begin{aligned}\left\|R_2x(t_2)-R_2x(t_1)\right\|_Y\leq{}&\tfrac{N_2\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\beta-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}(t_2-t_1)+\tfrac{N_2\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}(t_1-t_2)\\&+\tfrac{N_2|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}\left(t_2^3-t_1^3\right)+\tfrac{N_2|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}\left(t_1^3-t_2^3\right)\\&+\tfrac{N_2\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\beta-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)}\left(t_2^{1-\delta}-t_1^{1-\delta}\right)\\&+\tfrac{N_2\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)}\left(t_1^{1-\delta}-t_2^{1-\delta}\right)\\&+\tfrac{N_2|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}\left(t_2^{3-\delta}-t_1^{3-\delta}\right)\\&+\tfrac{N_2|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}\left(t_1^{3-\delta}-t_2^{3-\delta}\right).\end{aligned}\tag{3.37}$$

Thanks to (3.36) and (3.37), we can state that $\|\phi(x,y)(t_2)-\phi(x,y)(t_1)\|\to 0$ as $t_1\to t_2$. Then, as a consequence of steps $([1^*],[2^*],[3^*])$, we can conclude that R is continuous and compact.

Step3: Now, we prove that T is contractive.

Let $(x,y),(x_1,y_1)\in X\times Y$. Then, for each $t\in J$, we have

$$\begin{aligned}|T_1y(t)-T_1y_1(t)|\leq{}&\tfrac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left|\begin{array}{c}f\left(s,y(s),D^\delta y(s)\right)\\-f\left(s,y_1(s),D^\delta y_1(s)\right)\end{array}\right|ds\\&+\tfrac{\lambda_1t}{(\lambda_1\eta-1)\Gamma(\alpha)}\int_0^\eta(\eta-s)^{\alpha-1}\left|\begin{array}{c}f\left(s,y(s),D^\delta y(s)\right)\\-f\left(s,y_1(s),D^\delta y_1(s)\right)\end{array}\right|ds\\&+\tfrac{t}{(\lambda_1\eta-1)\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\left|\begin{array}{c}f\left(s,y(s),D^\delta y(s)\right)\\-f\left(s,y_1(s),D^\delta y_1(s)\right)\end{array}\right|ds.\end{aligned}$$

Thanks to $(H1)$, we can write

$$\begin{aligned}|T_1y(t)-T_1y_1(t)|\leq{}&\tfrac{k_1}{\Gamma(\alpha+1)}\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right)\\&+\tfrac{k_1(|\lambda_1|\eta^\alpha+1)}{|\lambda_1\eta-1|\Gamma(\alpha+1)}\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right).\end{aligned}$$

Consequently,

$$\|T_1(y)-T_1(y_1)\|\leq\tfrac{k_1[|\lambda_1\eta-1|+|\lambda_1|\eta^\alpha+1]}{|\lambda_1\eta-1|\Gamma(\alpha+1)}\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right),\tag{3.38}$$

and

$$\begin{aligned}|D^\sigma T_1y(t)-D^\sigma T_1y_1(t)|\leq{}&\tfrac{1}{\Gamma(\alpha-\sigma)}\int_0^t(t-s)^{\alpha-\sigma-1}\left|\begin{array}{c}f\left(s,y(s),D^\delta y(s)\right)\\-f\left(s,y_1(s),D^\delta y_1(s)\right)\end{array}\right|ds\\&+\tfrac{|\lambda_1|t^{1-\sigma}}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^\eta(\eta-s)^{\alpha-1}\left|\begin{array}{c}f\left(s,y(s),D^\delta y(s)\right)\\-f\left(s,y_1(s),D^\delta y_1(s)\right)\end{array}\right|ds\\&+\tfrac{t^{1-\sigma}}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^1(1-s)^{\alpha-1}\left|\begin{array}{c}f\left(s,y(s),D^\delta y(s)\right)\\-f\left(s,y_1(s),D^\delta y_1(s)\right)\end{array}\right|ds.\end{aligned}$$

By $(H1)$, yields

$$\begin{aligned}\left|D^{\sigma}T_1y(t)-D^{\sigma}T_1y_1(t)\right| &\leq \tfrac{k_1}{\Gamma(\alpha-\sigma+1)}\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right)\\ &+\tfrac{k_1|\lambda_1|\eta^{\alpha}}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right)\\ &+\tfrac{k_1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right).\end{aligned}$$

Hence,

$$\left\|D^{\sigma}T_1(y)-D^{\sigma}T_1(y_1)\right\|\leq k_1\left[\begin{array}{c}\frac{1}{\Gamma(\alpha-\sigma+1)}\\ +\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\end{array}\right]\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right). \tag{3.39}$$

By (3.38) and (3.39), we can write

$$\|T_1(y)-T_1(y_1)\|_X\leq k_1\left[\begin{array}{c}\frac{|\lambda_1\eta-1|+|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha-\sigma+1)}\\ +\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\end{array}\right]\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right).$$

Thus,

$$\|T_1(y)-T_1(y_1)\|_X\leq k_1\theta_1\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right). \tag{3.40}$$

Analogously, we can get

$$\|T_2(x)-T_2(x_1)\|_Y\leq k_2\theta_2\left(\|x-x_1\|+\|D^{\sigma}x-D^{\sigma}x_1\|\right). \tag{3.41}$$

It follows from (3.40) and (3.41) that

$$\|T(x,y)-T(x_1,y_1)\|_{X\times Y}\leq[k_1\theta_1+k_2\theta_2]\left(\|(x-x_1,y-y_1)\|_{X\times Y}\right).$$

Using (3.15), we conclude that T is a contraction mapping.

As a consequence of Krasnoselskii's fixed point theorem we deduce that ϕ has a fixed point which is a solution of (1.1). □

4 Examples

In this section we give some examples to illustrate our main results.

Example 4.1. *Let us consider the following system:*

$$\begin{cases}D^{\frac{7}{2}}x(t)+\frac{\sqrt{\pi}e^{-\pi t^2}\cos(\pi t)\left(y(t)+D^{\frac{5}{2}}y(t)\right)}{\left(5\sqrt{\pi}+7e^t\right)\left(1+y(t)+D^{\frac{5}{2}}y(t)\right)}+\ln\left(1+t^2\right)=0, t\in J,\\ D^{\frac{11}{3}}y(t)+\frac{\sqrt{\pi}e^{-\pi t^2}\cos(\pi t)\left(x(t)+D^{\frac{9}{4}}x(t)\right)}{\left(5\sqrt{\pi}+7e^t\right)\left(1+x(t)+D^{\frac{9}{4}}x(t)\right)}+\ln\left(1+t^2\right)=0, t\in J,\\ x(0)=0, x(1)-\frac{3}{4}x\left(\frac{1}{3}\right)=0, y(0)=0, y(1)-\frac{3}{4}y\left(\frac{1}{3}\right)=0,\\ x''(0)=0, x''(1)-\frac{4}{5}x''\left(\frac{2}{3}\right)=0, y''(0)=0, y''(1)-\frac{4}{5}y''\left(\frac{2}{3}\right)=0.\end{cases}$$

Set

$$f(t,x,y)=g(t,x,y)=\frac{\sqrt{\pi}e^{-\pi t}\left|\cos(\pi t)\right|(|x|+|y|)}{\left(5\sqrt{\pi}+7e^t\right)^2(1+|x|+|y|)}+\ln\left(1+t^2\right), t\in[0,1], x,y\in[0,\infty).$$

For $t \in J = [0,1]$ and $x_1, y_1, x_2, y_2 \in [0, \infty)$, we have:

$$\begin{aligned}
|f(t,x,y) - f(t,x_1,y_1)| &= \frac{\sqrt{\pi} e^{-\pi t} |\cos(\pi t)|}{\left(5\sqrt{\pi} + 7e^t\right)^2} \left| \frac{x+y}{(1+|x|+|y|)} - \frac{x_1+y_1}{(1+|x_1|+|y_1|)} \right| \\
&\leq \frac{\sqrt{\pi} e^{-\pi t} |\cos(\pi t)| \left(|x-x_1| + |y-y_1|\right)}{\left(5\sqrt{\pi}+7e^t\right)^2 (1+|x|+|y|)(1+|x_1|+|y_1|)} \\
&\leq \frac{\sqrt{\pi} e^{-\pi t} |\cos(\pi t)| \left(|x-x_1| + |y-y_1|\right)}{\left(5\sqrt{\pi}+7e^t\right)^2} \\
&\leq \frac{\sqrt{\pi}}{\left(5\sqrt{\pi}+7\right)^2} \left(|x-x_1| + |y-y_1|\right).
\end{aligned}$$

The condition $(H1)$ holds with $k_1 = k_2 = \frac{\sqrt{\pi}}{(5\sqrt{\pi}+7)^2}$.
For $\alpha = \frac{7}{2}, \beta = \frac{11}{3}, \sigma = \frac{9}{4}, \delta = \frac{5}{2}$ and $\lambda_1 = \frac{3}{4}, \lambda_2 = \frac{4}{5} = \eta = \frac{1}{3}, \xi = \frac{2}{3}$, we have:

$$M_1 = 1,089, M_2 = 3,503, M_3 = 0,909, M_4 = 3,089,$$

and,

$$k_1 (M_1 + M_2) + k_2 (M_3 + M_4) = 0,0605075.$$

Therefore,

$$k_1 (M_1 + M_2) + k_2 (M_3 + M_4) < 1.$$

Hence, by Theorem 3.1, the problem (1.1) has a unique solution.

Example 4.2. *Consider the following system:*

$$\begin{cases}
D^{\frac{7}{2}} x(t) + \frac{\left|D^{\frac{7}{3}} y(t)\right|}{5\pi\left(\sqrt{\pi}+2e^t\right)} + \frac{e^{-t^2}|y(t)|}{5\pi\left(\sqrt{\pi}e^t+2\right)^2(1+|y(t)|)} = 0, t \in J, \\
D^{\frac{11}{3}} y(t) + \frac{|x(t)|}{14\sqrt{\pi}(1+|x(t)|)} + \frac{|\cos(\pi t)|\left|D^{\frac{5}{2}} x(t)\right|}{7\sqrt{\pi}(t+1)^2} = 0, t \in J, \\
x(0) = 0, x(1) - \frac{2}{3} x\left(\frac{1}{5}\right) = 0, y(0) = 0, y(1) - \frac{2}{3} y\left(\frac{1}{5}\right) \\
x''(0) = 0, x''(1) - \frac{1}{2} x''\left(\frac{1}{4}\right) = 0, y''(0) = 0, y''(1) - \frac{1}{2} y''\left(\frac{1}{4}\right) = 0.
\end{cases}$$

For this example, we have

$$\begin{aligned}
f(t,x,y) &= \frac{|x|}{5\pi\left(\sqrt{\pi}+2e^t\right)} + \frac{e^{-t^2}|y|}{5\pi\left(\sqrt{\pi}e^t+2\right)^2(1+|y|)}, t \in [0,1], x, y \in [0,\infty), \\
g(t,x,y) &= \frac{|x|}{14\sqrt{\pi}(1+|x|)} + \frac{|\cos(\pi t)||y|}{7\sqrt{\pi}(t+1)^2}, t \in [0,1], x, y \in [0,\infty).
\end{aligned}$$

For $t \in J = [0,1]$ and $x, y, x_1, y_1 \in [0,\infty)$, we have:

$$\begin{aligned}
|f(t,x,y) - f(t,x_1,y_1)| &= \frac{e^{-t^2}|x-x_1|}{5\left(\sqrt{\pi}e^t+2\right)^2(1+|x|)(1+|x_1|)} + \frac{|y-y_1|}{5\pi\left(\sqrt{\pi}+2e^t\right)} \\
&\leq \frac{e^{-t^2}}{5\pi\left(\sqrt{\pi}e^t+2\right)^2}|x-x_1| + \frac{1}{5\pi\left(\sqrt{\pi}+2e^t\right)}|y-y_1| \\
&\leq \frac{1}{5\pi\left(\sqrt{\pi}+2\right)^2}\left(|x-x_1| + |y-y_1|\right),
\end{aligned}$$

and

$$\begin{aligned}|g(t,x,y)-g(t,x_1,y_1)| &= \frac{|x-x_1|}{14\sqrt{\pi}\,(1+|x|)\,(1+|x_1|)}+\frac{|\cos(\pi t)|\,|y-y_1|}{7\sqrt{\pi}\,(t+1)^2}\\ &\leq \frac{1}{14\sqrt{\pi}}\,|x-x_1|+\frac{|\cos(\pi t)|}{7\sqrt{\pi}\,(t+1)^2}\,|y-y_1|\\ &\leq \frac{1}{14\sqrt{\pi}}\,(|x-x_1|+|y-y_1|)\,.\end{aligned}$$

By Theorem 3.2, *the problem* (1.1) *has at leat one solution.*

References

[1] B. Ahmad, N. Juan, J. A. Aalsaedi: *Existence and uniqueness of solutions for nonlinear fractional differential equations with non-separated type integral boundary conditions*. *Acta Mathematica Scientia.* 31B(6), pp. 2122-2130, 2011.

[2] B. Ahmad, J. Nieto: *Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions*: Computers & Mathematics with Applications. 58(9), pp. 1838-1843, 2009.

[3] C.Z. Bai, J.-X. Fang: *The existence of a positive solution for a singular coupled system of nonlinear fractional differential equations.* Applied Mathematics and Computation. 150(3), pp. 611-621, 2004.

[4] Z. Cui, P. Yu and Z. Mao: *Existence of solutions for nonlocal boundary value problems of nonlinear fractional differential equations*. *Advances in Dynamical Systems and Applications.* 7(1), pp. 31-40, 2012.

[5] Z. Dahmani, L. Tabharit: *Fractional order differential equations involving Caputo derivative*. *Theory and Applications of Mathematics Computer Science.* 4(1), pp. 40-55, 2014.

[6] D. Delbosco, L. Rodino: *Existence and uniqueness for a nonlinear fractional differential equation*. *J. Math. Anal. Appl.* 204 (3-4), pp. 429-440, 1996.

[7] K. Diethelm, N.J. Ford: *Analysis of fractiona differential equations*. *J. Math. Anal. Appl.* 265(2), pp. 229-248, 2002.

[8] K. Diethelm, G. Walz: *Numerical solution of fraction order differential equations by extrapolation*. *Numer. Algorithms.* 16(3), pp. 231-253, 1998.

[9] A.M.A. El-Sayed: *Nonlinear functional differential equations of arbitrary orders*. *Nonlinear Anal.* 33(2), pp. 181-186, 1998.

[10] W. Feng, S. Sun, Z. Han, y. Zhao: *Existence of solutions for a singular system of nonlinear fractional differential equations*, Comput. Math. Appl. 62, pp. 1370-1378, 2011.

[11] V. Gafiychuk, B. Datsko, and V. Meleshko: *Mathematical modeling of time fractional reaction-diffusion systems*. Journal of Computational and Applied Mathematics. 220(1-2), pp. 215-225, 2008.

[12] M. Houas, Z. Dahmani: New fractional results for a boundary value problem with caputo derivative. Paper Accepted in IJOPCM Journal of Open Problems, 2013.

[13] A.A. Kilbas, S.A. Marzan: *Nonlinear differential equation with the Caputo fraction derivative in the space of continuously differentiable functions*. *Differ. Equ.* 41(1), pp. 84-89, 2005.

[14] V. Lakshmikantham, A.S. Vatsala: *Basic theory of fractional differential equations*. *Nonlinear Anal.* 69(8), pp. 2677-2682, 2008.

[15] F. Mainardi, *Fractional calculus: some basic problem in continuum and statistical mechanics. Fractals and fractional calculus in continuum mechanics*, *Springer, Vienna.* 1997.

[16] D.X. Ma, X.Z. Yang: *Upper and lower solution method for fourth-order four-point boundary value problems*. Journal of Computational and Applied Mathematics 223, pp. 543-551, 2009.

[17] S.K. Ntouyas: *Existence results for first order boundary value problems for fractional differential equations and inclusions with fractional integral boundary conditions*. *Journal of Fractional Calculus and Applications.* 3(9), pp. 1-14, 2012.

[18] I. Podlubny, I. Petras, B.M. Vinagre, P. O'leary, L. Dorcak: *Analogue realizations of fractional-order controllers. Fractional order calculus and its applications*. Nonlinear Dynam. 29(4), pp. 281-296, 2002.

[19] M.U. Rehman ,R.A Khan and N.A Asif :*Three point boundary value problems for nonlinear fractional differential equations*. Acta Mathematica Scientia. 31B(4), pp.1337-1346, 2011.

[20] M. Rehman, R. Khan: *A Note on boundary value problems for a coupled system of fractional differential equations*. Comput. Math. Appl. 61, pp. 2630-2637, 2011.

[21] X. Su: *Boundary value problem for a coupled system of nonlinear fractional differential equations*. Applied Mathematics Letters. 22(1), pp. 64-69, 2009.

[22] G. Wang, R. P. Agarwal, and A. Cabada: *Existence results and monotone iterative technique for systems of nonlinear fractional differential equations*. Applied Mathematics Letters. 25, pp. 1019-1024, 2012.

[23] J. Wang, H. Xiang, and Z. Liu: *Positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations*. International Journal of Differential Equations. Article ID 186928, 12 pages, 2010.

[24] W. Yang: *Positive solutions for a coupled system of nonlinear fractional differential equations with integral boundary conditions.* Comput. Math. Appl. 63, pp. 88-297, 2012.

[25] W. Yang: *Three-point boundary value problems for a coupled system of nonlinear fractional differential equations*. J. Appl. Math. & Informatics. 30(5-6), pp. 773-785, 2012.

[26] Y. Zhang, Z. Bai, T. Feng: *Existence results for a coupled system of nonlinear fractional three-point boundary value problems at resonance*. Comput. Math. Appl. 61, pp. 1032-1047, 2011.

[27] S. Zhang: *Positive solutions for boundary value problems of nonlinear fractional differential equations*. *Electron. J. Differential Equations*. 2(36), pp. 12-19, 2006.

Received: February 12, 2014; *Accepted*: October 15, 2014

UNIVERSITY PRESS

Website: http://www.malayajournal.org/