Fuzzy filters in Γ−semirings

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Abstract

We introduce the notion of fuzzy prime ideals and fuzzy filters in gamma semirings and study some of their properties.

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1   Introduction

The notion of semiring was introduced by H. S. Vandiver[6] in 1934. The notion of Γ− ring was introduced by N. Nobusawa [4] as a generalization of ring in 1964. M. Murali Krishna Rao [3] introduced the notion of Γ−semiring which is a generalization of ring, ternary semiring and semiring. After the paper [3] was published , many mathematicians obtained interesting results on Γ−semirings. The theory of fuzzy sets was first introduced by L. A. Zadeh [7] in 1965, many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory etc. In this paper, we introduce the notion of fuzzy prime ideals and fuzzy filters in gamma semirings and study some of their properties. In this section we will recall some of the fundamental concepts and definitions, these are necessary for this paper.

Definition 1.1. *A set R together with two associative binary operations called addition and multiplication (denoted by + and · respectively) will be called a semiring provided*

   (i) Addition is a commutative operation.

  (ii) Multiplication distributes over addition both from the left and from the right.

 (iii) There exists $0 \in R$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for each $x \in R$.

Definition 1.2. *[1] Let M and Γ be additive abelian groups. If there exists a mapping $M \times \Gamma \times M \to M$ (images to be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) satisfying the following conditions for all $x, y, z \in M, \alpha, \beta \in \Gamma$*

   (i) $x\alpha(y\beta z) = (x\alpha y)\beta z$

  (ii) $x\alpha(y + z) = x\alpha y + x\alpha z$

 (iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$

 (iv) $(x + y)\alpha z = x\alpha z + y\alpha z.$

Then M is called a Γ− ring.

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Definition 1.3. *[3] Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. Then we call M as a $\Gamma-$semiring, if there exists a mapping $M \times \Gamma \times M \to M$ written as (x, α, y) as $x\alpha y$ such that it satisfies the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$*

  (i) $x\alpha(y + z) = x\alpha y + x\alpha z$

  (ii) $(x + y)\alpha z = x\alpha z + y\alpha z$

  (iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$

  (iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Definition 1.4. *A non empty subset A of $\Gamma-$semiring M is called a $\Gamma-$subsemiring M if $(A, +)$ is a sub semigroup of $(M, +)$ and $a\alpha b \in A$ for all $a, b \in A$ and $\alpha \in \Gamma$.*

Definition 1.5. *An additive sub semigroup I of a $\Gamma-$semiring M is said to be a left (right) ideal of M if $M\Gamma I \subseteq I$ ($I\Gamma M \subseteq I$).*
  If I is both left and right ideal then I is called an ideal of $\Gamma-$semiring M.

Definition 1.6. *[7] Let S be a non empty set, a mapping $f : S \to [0, 1]$ is called a fuzzy subset of S.*

Definition 1.7. *Let f be a fuzzy subset of a nonempty set S, for $t \in [0, 1]$ the set $f_t = \{x \in S \mid f(x) \geq t\}$ is called level subset of S with respect f.*

Definition 1.8. *A fuzzy subset $\mu : S \to [0, 1]$ is nonempty if μ is not the constant function.*

Definition 1.9. *For any two fuzzy subsets λ and μ of $S, \lambda \subseteq \mu$ means $\lambda(x) \leq \mu(x)$ for all $x \in S$.*

Definition 1.10. *A fuzzy subset μ of M is a proper fuzzy subset if it is a non constant function.*

Definition 1.11. *A fuzzy subset μ is an improper if it is a constant function.*

Definition 1.12. *Let M be a $\Gamma-$semiring and f be a fuzzy subset of M. The mapping $f' : M \to [0, 1]$ is defined by $f'(x) = 1 - f(x)$ is a fuzzy subset of M, called complement of f.*

Definition 1.13. *Let M be a $\Gamma-$semiring . A fuzzy subset μ of M is said to be a fuzzy $\Gamma-$subsemiring of M if it satisfies the following conditions*
(i) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$
(ii) $\mu(x\alpha y) \geq min\{\mu(x), \mu(y)\}$ for all $x, y \in M, \alpha \in \Gamma$.

Definition 1.14. *A fuzzy subset μ of a $\Gamma-$semiring M is called a fuzzy left(right) ideal of M if for all $x, y \in M, \alpha \in \Gamma$*
(i) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$
(ii) $\mu(x\alpha y) \geq \mu(y)\big(\mu(x)\big)$

Definition 1.15. *A fuzzy subset μ of a $\Gamma-$semiring M is called a fuzzy ideal of M if for all $x, y \in M, \alpha \in \Gamma$*
(i) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$
(ii) $\mu(x\alpha y) \geq max\{\mu(x), \mu(y)\}$

2   Main results

In this section, we introduce the notion of fuzzy prime ideals, fuzzy filters in $\Gamma-$semirings and study some of their properties.

Definition 2.16. *Let M be a $\Gamma-$semiring. An ideal P of M is called a prime ideal of Mif for any $a, b \in M$ and $\gamma \in \Gamma, a\gamma b \in P \Rightarrow a$ or $b \in P$.*

Definition 2.17. *A proper fuzzy ideal μ of M is called fuzzy prime ideal if*

$$\mu(x\alpha y) = max \{\mu(x), \mu(y)\}, \forall x, y \in M, \alpha \in \Gamma$$

Definition 2.18. *Let M be a $\Gamma-$semiring. A $\Gamma-$subsemiring F of M is called a filter of Mif for any $a, b \in M$ and $\gamma \in \Gamma, a\gamma b \in F \Rightarrow a$ and $b \in F$.*

Definition 2.19. *Let M be a Γ−semiring. A fuzzy Γ−subsemiring μ of M is called a fuzzy filter of M if*

$$\mu(x\alpha y) = min\{\mu(x), \mu(y)\}, \forall x, y \in M, \alpha \in \Gamma.$$

The following theorems are straight forward.

Theorem 2.1. *Let M be a Γ−semiring, f be a fuzzy subset of M and f′ be the complement of f. Then the following statements are equivalent. Let x, y ∈ M, α ∈ Γ.*

(1). $f(x + y) \geq min\{f(x), f(y)\}$
      $f(x\alpha y) \geq max\{f(x), f(y)\}$

(2). $f'(x + y) \leq max\{f(x), f(y)\}$
      $f'(x\alpha y) \leq min\{f(x), f(y)\}$

Theorem 2.2. *Let M be a Γ−semiring and f be a fuzzy subset of M. Then f is a fuzzy filter of M if and only if f′, the complement of f is a fuzzy prime ideal of M.*

Theorem 2.3. *μ is a fuzzy filter of Γ−semiring M if and only if its level subset $\mu_t \neq \phi$, for any t ∈ [0, 1] is a filter of M.*

Proof. Suppose μ is a fuzzy filter of Γ−semiring M. Let $t \in [0, 1]$ such that μ_t is a Γ−subsemiring of M. Let $a, b \in M, \alpha \in \Gamma$ and $a\alpha b \in \mu_t$

$$\Rightarrow \mu(a\alpha b) \geq t$$
$$\Rightarrow min\{\mu(a), \mu(b)\} \geq t$$
$$\Rightarrow \mu(a) \geq t \text{ and } \mu(b) \geq t$$
$$\Rightarrow a \in \mu_t \text{ and } b \in \mu_t.$$

Hence μ_t is a filter of Γ−semiring M.

Conversely suppose that its level subset $\mu_t \neq \phi$, for any $t \in [0, 1]$ is a filter of M. Let $x, y \in M, \alpha \in \Gamma$.

$$\text{Suppose } t = min\{\mu(x), \mu(y)\}$$
$$\Rightarrow \mu(x) \geq t, \mu(y) \geq t$$
$$\Rightarrow x, y \in \mu_t$$
$$\Rightarrow x + y, x\alpha y \in \mu_t$$
$$\Rightarrow \mu(x + y) \geq t = min\{\mu(x), \mu(y)\}$$
$$\text{and } \mu(x\alpha y) \geq t = min\{\mu(x), \mu(y)\}.$$

Hence μ is fuzzy Γ−subsemiring of M.
Let $x, y \in M, \gamma \in \Gamma$ and $\mu(x\gamma y) = t$.

$$\Rightarrow x\gamma y \in \mu_t$$
$$\Rightarrow x \in \mu_t \text{ and } y \in \mu_t$$
$$\Rightarrow \mu(x) \text{ and } \mu(y) \geq t$$
$$\Rightarrow min\{\mu(x), \mu(y)\} \geq t = \mu(x\gamma y)$$
$$\Rightarrow min\{\mu(x), \mu(y)\} \geq \mu(x\gamma y) \geq min\{\mu(x), \mu(y)\}$$
$$\text{Hence } \mu(x\gamma y) = min\{\mu(x), \mu(y)\}.$$

Therefore μ is a fuzzy filter of Γ−semiring M.   □

Theorem 2.4. *μ is a fuzzy prime ideal of Γ−semiring M if and only if for any t ∈ [0, 1] such that μ_t is a prime ideal of M.*

Proof. Suppose μ is a fuzzy prime ideal of Γ−semiring M. Let $t \in [0, 1]$ such that μ_t is a proper ideal of Γ−semiring M. Let $x, y \in \mu_t$.
$$\Rightarrow \mu(x) \geq t, \ \mu(y) \geq t \Rightarrow \mu(x + y) \geq min\{\mu(x), \mu(y)\} \geq t.$$

Therefore $x + y \in \mu_t$ Let $x \in \mu_t, \alpha \in \Gamma, y \in M \setminus \mu_t \Rightarrow \mu(x) \geq t$ and $\mu(y) < t$
$\Rightarrow \mu(x\alpha y) = max\{\mu(x), \mu(y)\} \geq t \Rightarrow x\alpha y \in \mu_t$. similarly we can prove $y\alpha x \in \mu_t$. Hence μ_t is an ideal of $\Gamma-$semiring M.
Let $a, b \in M, \alpha \in \Gamma$ and $a\alpha b \in \mu_t$.

$$\Rightarrow \mu(a\alpha b) \geq t$$
$$\Rightarrow max\{\mu(a), \mu(b)\} \geq t$$
$$\Rightarrow \mu(a) \geq t \text{ or } \mu(b) \geq t$$
$$\Rightarrow a \in \mu_t \text{ or } b \in \mu_t.$$

Hence μ_t is a prime ideal of M.
     Conversely suppose that μ_t is a prime ideal, for any $t \in [0, 1]$.
Let $x, y \in M, \alpha \in \Gamma$ and $min\{\mu(x), \mu(y)\} = t$

$$\Rightarrow \mu(x) \geq t, \mu(y) \geq t$$
$$\Rightarrow x, y \in \mu_t$$
$$\Rightarrow x + y \in \mu_t$$
$$\text{Therefore} \mu(x + y) \geq t = min\{\mu(x), \mu(y)\}.$$
$$\text{Let } s = max\{\mu(x), \mu(y)\}.$$
$$\Rightarrow \mu(x) = s \text{ or } \mu(y) = s$$
$$\Rightarrow x \in \mu_s \text{ or } y \in \mu_s$$
$$\Rightarrow x\alpha y \in \mu_s$$
$$\Rightarrow \mu(x\alpha y) \geq s = max\{\mu(x), \mu(y)\}.$$

Therefore μ is a fuzzy ideal of $\Gamma-$semiring. Let $x, y \in M, \gamma \in \Gamma$ and $\mu(xry) = t$.

$$\Rightarrow xry \in \mu_t$$
$$\Rightarrow x \in \mu_t \text{ or } y \in \mu_t$$
$$\Rightarrow \mu(x) \geq t \text{ or } \mu(y) \geq t$$
$$\Rightarrow max\{\mu(x), \mu(y)\} \geq t = \mu(xry)$$
$$\Rightarrow max\{\mu(x), \mu(y)\} \geq \mu(xry) \geq max\{\mu(x), \mu(y)\}.$$

Hence μ is a fuzzy prime ideal of M.   □

Theorem 2.5. *Let M be a $\Gamma-$semiring. Then I is a prime ideal of M if and only if the fuzzy subsets χ_I is a fuzzy prime ideal of M.*

Proof. Let I be a prime ideal of $\Gamma-$semiring M. Obviously χ_I is a fuzzy ideal of M.. Let $x, y \in M, \alpha \in \Gamma$ and $x\alpha y \in I$. Since I is a prime ideal, we have $x \in I$ or $y \in I$.

$$\Rightarrow \chi_I(x) = 1 \text{ or } \chi_I(y) = 1.$$
$$\text{Hence } \chi_I(x\alpha y) = max\{\chi_I(x), \chi_I(y)\}.$$

Let $x \in M \setminus I, y \in I, \alpha \in \Gamma$ then $x\alpha y \in I$

$$\Rightarrow \chi_I(x\alpha y) = max\{\chi_I(x), \chi_I(y)\}.$$

Hence χ_I is a fuzzy prime ideal of M.
     Suppose χ_I is a fuzzy prime ideal of M. Let $x, y \in M, \alpha \in \Gamma$ such that $x\alpha y \in I$, we have

$$\chi_I(x\alpha y) = max\{\chi_I(x), \chi_I(y)\}$$
$$\Rightarrow 1 \leq max\{\chi_I(x), \chi_I(y)\}$$
$$\Rightarrow \chi_I(x) = 1 \text{ or } \chi_I(y) = 1$$
$$\Rightarrow x \in I \text{ or } y \in I.$$

Hence I is a prime ideal of $\Gamma-$semiring M.   □

Theorem 2.6. *Let M be a Γ−semiring and $\phi \neq F \subseteq M$. Then F is a filter of M if and only if the fuzzy subset χ_F is a fuzzy filter of M.*

Proof. Suppose F is a filter of Γ−semiring M. Obviously χ_F is a non empty fuzzy subset of M and fuzzy Γ−subsemiring of M. Let $x, y \in M, \alpha \in \Gamma$.

$$\text{Suppose } x\alpha y \notin F \Rightarrow x \notin F, y \in F$$
$$\Rightarrow \chi_F(x\alpha y) = 0, \chi_F(x) = 0, \chi_F(y) = 1$$
$$\Rightarrow \chi_F(x\alpha y) = min\{\chi_F(x), \chi_F(y)\}.$$
$$\text{If } x\alpha y \in F \Rightarrow x \in F \text{ and } y \in F$$
$$\chi_F(x\alpha y) = 1, \ \chi_F(x) = \chi_F(y) = 1.$$
$$\text{Hence } \chi_F(x\alpha y) = min\{\chi_F(x), \chi_F(y)\}.$$

Therefore χ_F is a fuzzy filter of M.

Conversely supposes that χ_F is a fuzzy filter of M. Obviously, F is a non empty Γ−subsemiring of M. Let $x\alpha y \in F, x, y \in M, \alpha \in \Gamma$. Since χ_F is a fuzzy filter of M. We have

$$\chi_F(x\alpha y) = min\{\chi_F(x), \chi_F(y)\}.$$
$$\Rightarrow \chi_F(x) = \chi_F(y) = 1$$
$$\Rightarrow x, y \in F.$$

Hence F is a filter of Γ−semiring M. □

Theorem 2.7. *If μ is a proper and maximal fuzzy ideal of Γ−semiring M. Then μ is a fuzzy prime ideal of Γ−semiring M.*

Proof. Suppose μ is a proper and maximal fuzzy ideal of Γ−semiring M. Let $t \in [0, 1]$ such that μ_t is a proper ideal of M. Let J be an ideal of M such that $\mu_t \subseteq J$. Suppose $J \neq M$. Then there exist $a \in M$ such that $a \notin J$. Therefore $a \notin \mu_t \Rightarrow \mu(a) < t$. Let γ be the fuzzy subset of M defined by $\gamma(x) = \mu(x)$ if $x \neq a, \gamma(a) = t$. Then $\mu \leq \gamma \leq \langle \gamma \rangle$ (The fuzzy ideal generated by γ). This is a contradiction to the fact that, μ is a maximal. Then $J = M$. Therefore μ_t is a maximal ideal $\Rightarrow \mu_t$ is a prime ideal.
Let $x, y \in M, \alpha \in \Gamma$ such that $\mu(x\alpha y) = c$

$$\Rightarrow x\alpha y \in \mu_c$$
$$\Rightarrow x \in \mu_c \text{ or } y \in \mu_c$$
$$\Rightarrow \mu(x) \geq c \text{ or } \mu(y) \geq c$$
$$\Rightarrow max\{\mu(x), \mu(y)\} \geq \mu(x\alpha y)$$
$$\Rightarrow \mu(x\alpha y) = max\{\mu(x), \mu(y)\}$$

Hence μ is a fuzzy prime ideal of Γ−semiring M □

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