

## Numerical solution of weakly singular integro-differential equations

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### Abstract

In this work, we prove the existence and uniqueness of the solution of weakly singular integro-differential equations. After some transformations direct numerical schemes using collocation methods are proposed for any peicewise closed contours.

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### 1 Introduction

Singular integro-differential equations with logarithmic kernel arise in different problems of elasticity theory, aerodynamics, mechanics, elasticity, this kind of equations has gained a lot of interest in many application fields, in particular their numerical treatment is asked [1]. While several numerical methods for approximating the solution of Volterra integro-differential equations and Fredholm integro-differential equations are known [2, 4]. On the other hand, the singular integro-differential equations have poor documentation.

It is known that, the most effective methods for the approximate solution of weakly singular integro-differential equations consists in their reduction to a system of linear algebraic equations by the replacement of the integral with a proper quadrature sum [5, 6, 7].

Consider the weakly singular integro-differential equation of the form

$$\varphi(t_0) + \frac{1}{\pi i} \int_{\Gamma} \ln(t - t_0) \varphi'(t) dt = f(t_0), \quad (1.1)$$

where  $\Gamma$  designates a smooth-oriented contour;  $t$  and  $t_0$  are points on  $\Gamma$  and  $f(t)$  is a given function on  $\Gamma$ , the density  $\varphi(t)$  is the desired function has to satisfy the Holder condition  $H(\mu)$  [6].

The equation (1) can be put in the form of functional equation

$$\varphi(t_0) + AD\varphi(t_0) = f(t_0), \quad (1.2)$$

with the linear mappings  $A$  and  $D$  respectively given by

$$A\varphi(t_0) = \frac{1}{\pi i} \int_{\Gamma} \ln(t - t_0) \varphi(t) dt, \quad D\varphi(t) = \varphi'(t). \quad (1.3)$$

In this work we prove the existence and the uniqueness of the solution of the equation (1) and solve it numerically.

Let  $\varepsilon > 0$  be a sufficiently small number and describe around  $t_0$  a circle centred at  $t_0$  with a radius  $\varepsilon$  this circle intersects the curve  $\Gamma$  in the two points  $t_1$  and  $t_2$  such that the arc lengths  $t_1 t_0$  and  $t_0 t_2$  are equal to  $\varepsilon$  and denoting by  $\Gamma_\varepsilon$  this part of  $\Gamma$  limited by  $t_1 t_2$ .

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## 2 Main results

**Theorem:** Suppose that the function  $\varphi(t) \in W^1(\Gamma)$ ,  $t$  and  $t_0$  are points on the smooth-oriented contour  $\Gamma$  then, the equation (1) given by

$$\varphi(t_0) + \frac{1}{\pi i} \int_{\Gamma} \ln(t - t_0) \varphi'(t) dt = f(t_0),$$

admits a unique solution for all  $f(t_0)$  in the given space.

**Proof**

The integration by parts for the operator  $AD\varphi(t_0)$  in (2) gives

$$\begin{aligned} \pi i AD\varphi(t_0) &= \int_{\Gamma - \Gamma_\varepsilon} \ln(t - t_0) \varphi'(t) dt \\ &= \varphi(t_1) \ln(t_1 - t_0) - \varphi(t_2) \ln(t_2 - t_0) - \int_{\Gamma - \Gamma_\varepsilon} \frac{\varphi(t)}{t - t_0} dt \\ &= \varphi(t) [\ln(t_1 - t_0) - \ln(t_2 - t_0)] + (\varphi(t_1) - \varphi(t_0)) \ln(t_1 - t_0) \\ &\quad + (\varphi(t_2) - \varphi(t_0)) \ln(t_2 - t_0) - \int_{\Gamma - \Gamma_\varepsilon} \frac{\varphi(t)}{t - t_0} dt. \end{aligned}$$

The expansion  $\varphi(t) [\ln(t_1 - t_0) - \ln(t_2 - t_0)]$  converges to  $\pi i \varphi(t_0)$  as  $\varepsilon$  converges to zero, on the other hand the expansions

$(\varphi(t_1) - \varphi(t_0)) \ln(t_1 - t_0)$  and  $(\varphi(t_2) - \varphi(t_0)) \ln(t_2 - t_0)$  converge to zero as  $\varepsilon$  goes to zero. Hence the integral becomes

$$\begin{aligned} AD\varphi(t_0) &= \frac{1}{\pi i} \int_{\Gamma - \Gamma_\varepsilon} \ln(t - t_0) \varphi'(t) dt \\ &= \varphi(t_0) - \frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(t)}{t - t_0} dt. \end{aligned}$$

Therefore the equation (1)

$$\varphi(t_0) + \frac{1}{\pi i} \int_{\Gamma} \ln(t - t_0) \varphi'(t) dt = f(t_0),$$

is transformed to the following equation

$$2\varphi(t_0) - \frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(t)}{t - t_0} dt = f(t_0). \quad (2.4)$$

The equation (4) admits a unique solution for all second member, that is to say, the equation (1) admits a unique solution or all second member.

## 3 Numerical Experiments

In this section we describe some of the numerical experiments performed in solving the weakly singular integro-differential equations (1), using collocation methods with the approximation technical in [5,7]. In all cases, the curve is taking the unit circle and we chose the right hand side  $f(t)$  in such way that we know the exact solution. This exact solution is used only to show that the numerical solution obtained with the method is correct.

In each table,  $\varphi$  represents the given exact solution of the weakly singular integro-differential equations and  $\tilde{\varphi}$  corresponds to the approximate solution of the equation produced by the approximation method for singular integral with logarithmic kernel in [5,7].

### Example 1

Consider the weakly singular integro-differential equation on the unit circle  $\Gamma$

$$t_0 \varphi(t_0) + \int_{\Gamma} \ln(t - t_0) \varphi'(t) dt = t_0^3 - t_0^2,$$

where the function  $f(t_0)$  is chosen so that the solution  $\varphi(t)$  is given by

$$\varphi(t) = t^2.$$

The approximate solution  $\tilde{\varphi}(t)$  of  $\varphi(t)$  is obtained by the solution of a system of linear algebraic equations by the replacement of the integral with a proper quadrature sum.

Points of t	Exact solution	Approx solution	Error
1.0000	1.0000e+000	1.00e+000 +2.07e-014i	3.15e-014
3.68e-001 +9.29e-001i	-7.28e-001 +6.84e-001i	-7.28e-001 +6.84e-001i	3.37e-014
-7.70e-001 +6.37e-001i	1.87e-001 -9.82e-001i	1.87e-001 -9.82e-001i	3.19e-014
-8.44e-001 -5.35e-001i	4.25e-001 +9.04e-001i	4.25e-001 +9.04e-001i	2.86e-014
3.09e-001 -9.51e-001i	-8.09e-001 -5.87e-001i	-8.09e-001 -5.87e-001i	2.75e-014
9.98e-001 -6.27e-002i	9.92e-001 -1.25e-001i	9.92e-001 -1.25e-001i	3.25e-014

**Table 1.** The exact and approximate solutions of example 1 in some arbitrary points and the error

**Example 2**

Consider the weakly singular integro-differential equation on the unit circle  $\Gamma$

$$\varphi(t_0) + \int_{\Gamma} \ln(t - t_0)\varphi'(t)dt = \frac{1}{t_0 + 2},$$

where the function  $f(t_0)$  is chosen so that the solution  $\varphi(t)$  is given by

$$\varphi(t) = \frac{1}{t + 2}.$$

The approximate solution  $\tilde{\varphi}(t)$  of  $\varphi(t)$  is obtained by the solution of a system of linear algebraic equations by the replacement of the integral with a proper quadrature sum.

Points of t	Exact solution	Approx solution	Error
1.0000	3.3333e-001	3.33e-001 -3.70e-007i	5.13e-007
3.68e-001 +9.29e-001i	3.65e-001 -1.43e-001i	3.65e-001 -1.43e-001i	1.64e-006
-7.70e-001 +6.37e-001i	6.41e-001 -3.32e-001i	6.41e-001 -3.32e-001i	1.73e-005
-8.44e-001 -5.35e-001i	7.12e-001 +3.30e-001i	7.12e-001 +3.30e-001i	2.64e-005
3.09e-001 -9.51e-001i	3.70e-001 +1.52e-001i	3.70e-001 +1.52e-001i	1.00e-006
9.98e-001 -6.27e-002i	3.33e-001 +6.98e-003i	3.33e-001 +6.98e-003i	8.75e-007

**Table 2.** The exact and approximate solutions of example 2 in some arbitrary points and the error

**Example 3**

Consider the weakly singular integro-differential equation on the unit circle  $\Gamma$

$$\varphi(t_0) + \int_{\Gamma} \ln(t - t_0)\varphi'(t)dt = \frac{3}{t_0},$$

where the function  $f(t_0)$  is chosen so that the solution  $\varphi(t)$  is given by

$$\varphi(t) = \frac{1}{t}.$$

The approximate solution  $\tilde{\varphi}(t)$  of  $\varphi(t)$  is obtained by the solution of a system of linear algebraic equations by the replacement of the integral with a proper quadrature sum.

Points of t	Exact solution	Approx solution	Error
1.0000	1.000000e+000	9.99e-001 -1.61e-005i	5.82e-005
3.68e-001 +9.29e-001i	3.68e-001 -9.29e-001i	3.68e-001 -9.29e-001i	5.09e-005
-7.70e-001 +6.37e-001i	-7.70e-001 -6.37e-001i	-7.70e-001 -6.37e-001i	5.09e-005
-8.44e-001 -5.35e-001i	-8.44e-001 +5.35e-001i	-8.44e-001 +5.35e-001i	5.09e-005
3.09e-001 -9.51e-001i	3.09e-001 +9.51e-001i	3.09e-001 +9.51e-001i	5.82e-005
9.98e-001 -6.27e-002i	9.98e-001 +6.27e-002i	9.98e-001 +6.27e-002i	5.09e-005

**Table 3.** The exact and approximate solutions of example 2 in some arbitrary points and the error

## 4 Conclusion

In this work we remark the convergence of the method to the exact solution with a considerable accuracy for the weakly singular integro-differential equations. This numerical results show that the accuracy improves with increasing of the number of points on the curve. Finally we confirm that this method lead us to the good approximation of the exact solution.

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