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Oscillation results for third order nonlinear neutral differential equations of mixed type

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Abstract

Some oscillation results are obtained for the third order nonlinear mixed type neutral differential equations of the form

$$((x(t)+b(t)x(t-\tau_1)+c(t)x(t+\tau_2))^{\alpha})''' = q(t)x^{\beta}(t-\sigma_1)+p(t)x^{\gamma}(t+\sigma_2),\ t \geq t_0$$

where $\alpha,\ \beta$ and γ are ratios of odd positive integers $\tau_1,\ \tau_2,\ \sigma_1$ and σ_2 are positive constants.

Keywords: Oscillation, third order, neutral differential equations.

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1 Introduction

In this paper, we study the oscillatory nature of the third order nonlinear mixed type neutral differential equations of the form

$$((x(t)+b(t)x(t-\tau_1)+c(t)x(t+\tau_2))^{\alpha})''' = q(t)x^{\beta}(t-\sigma_1)+p(t)x^{\gamma}(t+\sigma_2),\ t \geq t_0 \tag{1.1}$$

subject to the following conditions:

(c_1) $\tau_1,\ \tau_2,\ \sigma_1$ and σ_2 are positive constants;

(c_2) $q(t)$ and $p(t)$ are real valued positive continuous functions on $[t_0,\infty)$;

(c_3) $\alpha,\ \beta$ and γ are ratios of odd positive integers;

(c_4) $b(t)$ and $c(t)$ are real valued and thrice continuously differentiable functions with $0 \leq b(t) < b < \infty$ and $0 \leq c(t) < c < \infty$.

Let $\theta = \max\{\tau_1, \sigma_1\}$. By a solution of equation (1.1), we mean a real valued continuous function $x(t)$ defined for all $t \geq t_0 - \theta$ and satisfying the equation (1.1) for all $t \geq t_0$. A nontrivial solution of equation (1.1) is called oscillatory if it has infinitely many zeros on $[t_0,\infty)$, otherwise it is called nonoscillatory.

Recently there has been a great interest in studying the oscillatory and asymptotic behavior of third order differential equations, see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23], and the references cited therein. In [1, 4, 7, 8, 9, 15, 20, 23], the authors studied the oscillatory behavior of solutions of equation (1.1) when $b(t) \equiv 0,\ c(t) \equiv 0$ and $p(t) \equiv 0$. In [5, 6, 10, 11, 17, 18, 19, 21], the authors studied the oscillatory behavior of solutions of equation (1.1) when $c(t) \equiv 0$ and $p(t) \equiv 0$. In [2, 13, 14, 22], the authors discussed the oscillatory behavior of all solutions of equation (1.1) when $\alpha = \beta = \gamma = 1$.

Motivated by this observation, in this paper we study the oscillatory and asymptotic behavior of all solutions of equation (1.1) for different values of α, β and γ. So the purpose of this paper is to present some new oscillatory

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and asymptotic criteria for equation (1.1). In Section 2, we present criteria for equation (1.1) to be either oscillatory or all its nonoscillatory solutions tend to zero as $t \to \infty$. Examples are provided in Section 3 to illustrate the results presented in Section 2.

2 Oscillation results

In this section, we present some new oscillation criteria for the equation (1.1). For convenience we use the following notations:
$Q(t)=\min(q(t), q(t-\tau_1), q(t+\tau_2))$, $P(t)=\min(p(t), p(t-\tau_1), p(t+\tau_2))$,
and $z(t)=[x(t)+b(t)x(t-\tau_1)+c(t)x(t+\tau_2)]^{\alpha}$.

Lemma 2.1. *If $x(t)$ is a positive solution of equation (1.1), then the corresponding function $z(t)$ satisfies only the following two cases*

$$\text{Case (I)} \quad z(t)>0,\ z'(t)>0,\ z''(t)>0, z'''(t)>0; \tag{2.1}$$

$$\text{Case (II)} \quad z(t)>0,\ z'(t)>0,\ z''(t)<0, z'''(t)>0. \tag{2.2}$$

Proof. Assume that $x(t)$ is a positive solution of equation (1.1). Then there exists a $t_1 \geq t_0$ such that $x(t-\theta)>0$ for all $t \geq t_1$. From the definition of $z(t)$, it is clear that $z(t)>0$ for all $t \geq t_1$. From equation (1.1), we have $z'''(t)>0$ for all $t \geq t_1$. Therefore $z''(t)$ is strictly increasing for all $t \geq t_1$ and $z''(t)$ and $z'(t)$ are of one sign for all $t \geq t_1$. We prove that $z'(t)>0$ for all $t \geq t_1$. If not, there exists a $t_2 \geq t_1$ and $M<0$ such that $z'(t)<M$ for all $t \geq t_2$. Integrating the last inequality from t_2 to t, we get

$$z(t)-z(t_2)<M(t-t_2).$$

Letting $t \to \infty$, we see that $z(t) \to -\infty$, which is a contradiction. Hence $z'(t)>0$ for all $t \geq t_1$.This completes the proof of the lemma. □

Lemma 2.2. *If $A \geq 0$, $B \geq 0$ and $0<\delta \leq 1$, then*

$$A^{\delta}+B^{\delta} \geq (A+B)^{\delta} \tag{2.3}$$

If $\delta \geq 1$ then

$$(A^{\delta}+B^{\delta}) \geq \frac{1}{2^{\delta-1}}(A+B)^{\delta}. \tag{2.4}$$

Proof. Proof can be found in [21]. □

Theorem 2.3. *Assume that $0<\beta=\gamma \leq 1$ and $\sigma_2>\sigma_1>\max\{\tau_1,\tau_2\}$. If the second order differential inequality*

$$y''(t) \geq \frac{P(t)(\sigma_1-\tau_2)^{\beta/\alpha}}{(1+b^{\beta}+c^{\beta})^{\beta/\alpha}}\ y^{\beta/\alpha}(t+\sigma_2-\sigma_1) \tag{2.5}$$

has no positive increasing solution, and the second order differential inequality

$$y''(t) \geq \frac{Q(t)(\sigma_1-\tau_1)^{\beta/\alpha}}{(1+b^{\beta}+c^{\beta})^{\beta/\alpha}}\ y^{\beta/\alpha}(t-\sigma_1+\tau_1) \tag{2.6}$$

has no positive decreasing solution, then every solution of equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1) for all $t \geq t_1 \geq t_0$. Without loss of generality, we may assume that $x(t)$ is a positive solution of equation (1.1) for all $t \geq\ t_1 \geq\ t_0$ (since the case $x(t)$ is negative is similar). Then there exists a $t_2 \geq t_1$ such that $x(t-\theta)>0$ for all $t \geq t_2$. By the definition of $z(t)$ we have, $z(t-\theta)>0$ for all $t \geq t_2$. Define a function $y(t)$ by

$$y(t)=z(t)+b^{\beta}z(t-\tau_1)+c^{\beta}z(t+\tau_2), \text{ for all } t \geq t_2. \tag{2.7}$$

Then $y(t) > 0$ for all $t \geq t_2$, and

$$\begin{aligned} y'''(t) &= z'''(t) + b^\beta z'''(t-\tau_1) + c^\beta z'''(t+\tau_2) \\ &= q(t)x^\beta(t-\sigma_1) + p(t)x^\beta(t+\sigma_2) + b^\beta q(t-\tau_1)x^\beta(t-\tau_1-\sigma_1) + \\ &\quad b^\beta p(t-\tau_1)x^\beta(t-\tau_1+\sigma_2) + c^\beta q(t+\tau_2)x^\beta(t+\tau_2-\sigma_1) + \\ &\quad c^\beta p(t+\tau_2)x^\beta(t+\tau_2+\sigma_2) \\ &\geq Q(t)[x^\beta(t-\sigma_1) + b^\beta x^\beta(t-\tau_1-\sigma_1) + c^\beta x^\beta(t+\tau_2-\sigma_1)] + \\ &\quad P(t)[x^\beta(t+\sigma_2) + b^\beta x^\beta(t-\tau_1+\sigma_2) + c^\beta x^\beta(t+\tau_2+\sigma_2)]. \end{aligned}$$

Using (2.3) twice, the above inequality becomes

$$y'''(t) \geq Q(t)z^{\beta/\alpha}(t-\sigma_1) + P(t)z^{\beta/\alpha}(t+\sigma_2). \tag{2.8}$$

Since $x(t)$ is a positive solution of equation (1.1), from Lemma 2.1 we have two cases for $z(t)$.
Case (I): In this case, we have $z'(t) > 0,\ z''(t) > 0$ and $z'''(t) > 0$ for all $t \geq t_2$. Then from (2.7), we have $y'(t) > 0,\ y''(t) > 0$ and $y'''(t) > 0$ for all $t \geq t_2$.

From the inequality (2.8), we have

$$y'''(t) \geq P(t)z^{\beta/\alpha}(t+\sigma_2). \tag{2.9}$$

Since $z'(t)$ is increasing, we have

$$\begin{aligned} y'(t) &= z'(t) + b^\beta z'(t-\tau_1) + c^\beta z'(t+\tau_2) \\ &\leq (1+b^\beta+c^\beta)z'(t+\tau_2) \text{ for all } t \geq t_0. \end{aligned} \tag{2.10}$$

Now

$$z(t+\sigma_1-\tau_2) - z(t) = \int_t^{t+\sigma_1-\tau_2} z'(s)\, ds$$

or

$$z(t+\sigma_1-\tau_2) \geq z'(t)(\sigma_1-\tau_2). \tag{2.11}$$

Using (2.10) and (2.11) in (2.9), we obtain

$$\begin{aligned} y'''(t) &\geq P(t)z^{\beta/\alpha}(t+\sigma_2) \\ &\geq P(t)(\sigma_1-\tau_2)^{\beta/\alpha}(z'(t+\sigma_2-\sigma_1+\tau_2))^{\beta/\alpha} \\ &\geq \frac{P(t)(\sigma_1-\tau_2)^{\beta/\alpha}}{(1+b^\beta+c^\beta)^{\beta/\alpha}}(y'(t+\sigma_2-\sigma_1))^{\beta/\alpha}, t \geq t_2. \end{aligned} \tag{2.12}$$

By setting $y'(t) = w(t)$, we see that $w(t) > 0$ and $w'(t) > 0$ for all $t \geq t_2$. Now inequality (2.9) becomes

$$w''(t) \geq \frac{P(t)}{(1+b^\beta+c^\beta)^{\beta/\alpha}}(\sigma_1-\tau_2)^{\beta/\alpha}w^{\beta/\alpha}(t+\sigma_2-\sigma_1), t \geq t_2. \tag{2.13}$$

That is, $w(t)$ is a positive increasing solution of the second order differential inequality (2.5), which is a contradiction.

Case (II). In this case, we have $z'(t) > 0,\ z''(t) < 0$ and $z'''(t) > 0$ for all $t \geq t_2$. Then $y'(t) > 0,\ y''(t) < 0$ for all $t \geq t_2$. From the inequality (2.8), we have

$$y'''(t) \geq Q(t)z^{\beta/\alpha}(t-\sigma_1). \tag{2.14}$$

Since $z'(t)$ and $y'(t)$ are decreasing, we have

$$\begin{aligned} y'(t) &= z'(t) + b^\beta z'(t-\tau_1) + c^\beta z'(t+\tau_2) \\ &\leq (1+b^\beta+c^\beta)z'(t-\tau_1) \end{aligned}$$

or

$$y'(t-\sigma_1+\tau_1) \leq (1+b^\beta+c^\beta)z'(t-\sigma_1),\ t \geq t_2. \tag{2.15}$$

Now

$$z(t) - z(t - (\sigma_1 - \tau_1)) = \int_{t-(\sigma_1-\tau_1)}^{t} z'(s)\, ds$$

or

$$z(t) \geq z'(t)(\sigma_1 - \tau_1). \tag{2.16}$$

Using (2.15) and (2.16) in (2.14), we obtain

$$\begin{aligned} y'''(t) &\geq Q(t) z^{\beta/\alpha}(t - \sigma_1) \\ &\geq Q(t)(\sigma_1 - \tau_1)^{\beta/\alpha}(z'(t - \sigma_1))^{\beta/\alpha} \\ &\geq \frac{Q(t)(\sigma_1 - \tau_1)^{\beta/\alpha}}{(1 + b^\beta + c^\beta)^{\beta/\alpha}}(y'(t - \sigma_1 + \tau_1))^{\beta/\alpha}, t \geq t_2. \end{aligned}$$

By taking $y'(t) = w(t)$, we see that $w(t) > 0$ and $w'(t) < 0$. Thus, $w(t)$ is a positive decreasing solution of the second order differential inequality

$$w''(t) \geq \frac{Q(t)}{(1 + b^\beta + c^\beta)^{\beta/\alpha}}(\sigma_1 - \tau_1)^{\beta/\alpha} w^{\beta/\alpha}(t - \sigma_1 + \tau_1), \tag{2.17}$$

which is a contradiction to (2.6). This completes the proof. □

Theorem 2.4. *Assume that* $\beta = \gamma \geq 1$ *and* $\sigma_2 > \sigma_1 > \max\{\tau_1, \tau_2\}$. *If the second order differential inequality*

$$y''(t) \geq \frac{P(t)(\sigma_1 - \tau_2)^{\beta/\alpha}}{4^{\beta-1}(1 + b^\beta + \frac{c^\beta}{2^{\beta-1}})^{\beta/\alpha}} y^{\beta/\alpha}(t + \sigma_2 - \sigma_1) \tag{2.18}$$

has no positive increasing solution, and the second order differential inequality

$$y''(t) \geq \frac{Q(t)(\sigma_1 - \tau_1)^{\beta/\alpha}}{4^{\beta-1}(1 + b^\beta + \frac{c^\beta}{2^{\beta-1}})^{\beta/\alpha}} y^{\beta/\alpha}(t + \tau_1 - \sigma_1) \tag{2.19}$$

has no positive decreasing solution, then every solution of equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1) for all $t \geq t_1 \geq t_0$. Without loss of generality, we may assume that $x(t)$ is a positive solution of equation (1.1). Then there exists a $t_2 \geq t_1$ such that $x(t - \theta) > 0$ for all $t \geq t_2$. By the definition of $z(t)$, we have $z(t - \theta) > 0$ for all $t \geq t_2$. Now define a function $y(t)$ by

$$y(t) = z(t) + b^\beta z(t - \tau_1) + \frac{c^\beta}{2^{\beta-1}} z(t + \tau_2), t \geq t_2. \tag{2.20}$$

Then, since $z(t) > 0$, we have $y(t) > 0$ and

$$\begin{aligned} y'''(t) &= z'''(t) + b^\beta z'''(t - \tau_1) + \frac{c^\beta}{2^{\beta-1}} z'''(t + \tau_2) \\ &= q(t)x^\beta(t - \sigma_1) + p(t)x^\beta(t + \sigma_2) + b^\beta q(t - \tau_1)x^\beta(t - \tau_1 - \sigma_1) + \\ &\quad b^\beta p(t - \tau_1)x^\beta(t - \tau_1 + \sigma_2) + \frac{c^\beta}{2^{\beta-1}} q(t + \tau_2)x^\beta(t + \tau_2 - \sigma_1) + \\ &\quad \frac{c^\beta}{2^{\beta-1}} p(t + \tau_2)x^\beta(t + \tau_2 + \sigma_2) \\ &\geq Q(t)[x^\beta(t - \sigma_1) + b^\beta x^\beta(t - \tau_1 - \sigma_1) + \frac{c^\beta}{2^{\beta-1}} x^\beta(t + \tau_2 - \sigma_1)] + \\ &\quad P(t)[x^\beta(t + \sigma_2) + b^\beta x^\beta(t - \tau_1 + \sigma_2) + \frac{c^\beta}{2^{\beta-1}} x^\beta(t + \tau_2 + \sigma_2)],\ t \geq t_2. \end{aligned}$$

Now using (2.4) twice in the last inequality, we obtain

$$y'''(t) \geq \frac{Q(t)}{4^{\beta-1}} z^{\beta/\alpha}(t - \sigma_1) + \frac{P(t)}{4^{\beta-1}} z^{\beta/\alpha}(t + \sigma_2) t \geq t_2. \tag{2.21}$$

Since $x(t)$ is a positive solution of equation (1.1), there are only two cases, as given in Lemma 2.1, for $z(t)$.

Case (I): In this case, we have $z(t) > 0,\ z'(t) > 0,\ z''(t) > 0$ and $z'''(t) > 0$ for all $t \geq t_2$. Then from (2.20), we have $y'(t) > 0,\ y''(t) > 0$ for all $t \geq t_2$. From the inequality (2.21), we have

$$y'''(t) \geq \frac{P(t)}{4^{\beta-1}} z^{\beta/\alpha}(t+\sigma_2), t \geq t_2. \tag{2.22}$$

Since $z'(t)$ is increasing, we have

$$\begin{aligned} y'(t) &= z'(t) + b^\beta z'(t-\tau_1) + \frac{c^\beta}{2^{\beta-1}} z'(t+\tau_2) \\ &\leq (1 + b^\beta + \frac{c^\beta}{2^{\beta-1}}) z'(t+\tau_2), t \geq t_2 \end{aligned}$$

or

$$y'(t) \leq (1 + b^\beta + \frac{c^\beta}{2^{\beta-1}}) z'(t+\sigma_2+\tau_2), t \geq t_2 \tag{2.23}$$

and

$$z(t+\sigma_1-\tau_2) - z(t) = \int_t^{t+\sigma_1-\tau_2} z'(s)\ ds \geq z'(t)(\sigma_1-\tau_2)$$

or

$$z(t+\sigma_1-\tau_2) \geq z'(t)(\sigma_1-\tau_2). \tag{2.24}$$

Now using (2.23) and (2.24) in (2.22), we have

$$\begin{aligned} y'''(t) &\geq \frac{P(t)}{4^{\beta-1}} z^{\beta/\alpha}(t+\sigma_2) \\ &\geq \frac{P(t)}{4^{\beta-1}} (\sigma_1-\tau_2)^{\beta/\alpha} (z'(t+\tau_2-\sigma_1+\sigma_2))^{\beta/\alpha} \end{aligned}$$

$$y'''(t) \geq \frac{P(t)(\sigma_1-\tau_2)^{\beta/\alpha}(y'(t+\sigma_2-\sigma_1))^{\beta/\alpha}}{4^{\beta-1}(1+b^\beta+\frac{c^\beta}{2^{\beta-1}})^{\beta/\alpha}}, t \geq t_2. \tag{2.25}$$

Setting $y'(t) = w(t)$, we see that $w(t) > 0$, $w'(t) = y''(t) > 0$ and

$$w''(t) \geq \frac{P(t)(\sigma_1-\tau_2)^{\beta/\alpha} w(t+\sigma_2-\sigma_1)^{\beta/\alpha}}{4^{\beta-1}(1+b^\beta+c^\beta)^{\beta/\alpha}}, t \geq t_2. \tag{2.26}$$

That is $w(t)$ is a positive increasing solution of the second order differential inequality (2.18), which is a contradiction.

Case (II): In this case we have $z'(t) > 0,\ z''(t) < 0$ and $z'''(t) > 0$ for all $t \geq t_2$. Then from (2.20), we obtain $y'(t) > 0$ and $y''(t) < 0$ for all $t \geq t_2$. From the inequality (2.21), we have

$$y'''(t) \geq \frac{Q(t)}{4^{\beta-1}} z^{\beta/\alpha}(t-\sigma_1), t \geq t_2. \tag{2.27}$$

Using the monotonicity of $z'(t)$ and $y'(t)$, we have

$$\begin{aligned} y'(t) &= z'(t) + b^\beta z'(t-\tau_1) + \frac{c^\beta}{2^{\beta-1}} z'(t+\tau_2) \\ &\leq (1 + b^\beta + \frac{c^\beta}{2^{\beta-1}}) z'(t-\tau_1) \end{aligned}$$

or

$$y'(t+\sigma_1) \leq (1 + b^\beta + \frac{c^\beta}{2^{\beta-1}}) z'(t+\sigma_1-\tau_1), t \geq t_2. \tag{2.28}$$

Also from the monotonicity of $z'(t)$ we have

$$z(t) - z(t-\sigma_1+\tau_1) = \int_{t-(\sigma_1-\tau_1)}^{t} z'(s)\ ds \geq z'(t)(\sigma_1-\tau_1)$$

or

$$z(t) \geq (\sigma_1 - \tau_1)z'(t). \tag{2.29}$$

Using (2.28) and (2.29) in (2.27), we get

$$\begin{aligned} y'''(t) &\geq \frac{Q(t)}{4^{\beta-1}} z^{\beta/\alpha}(t-\sigma_1) \\ &\geq \frac{Q(t)}{4^{\beta-1}}(\sigma_1-\tau_1)^{\beta/\alpha}(z'(t-\sigma_1))^{\beta/\alpha} \\ &\geq \frac{Q(t)}{4^{\beta-1}}(\sigma_1-\tau_1)^{\beta/\alpha}\frac{(y'(t-\sigma_1+\tau_1))^{\beta/\alpha}}{(1+b^\beta+\frac{c^\beta}{2^{\beta-1}})^{\beta/\alpha}} \end{aligned}$$

or

$$y'''(t) \geq \frac{Q(t)}{4^{\beta-1}}(\sigma_1-\tau_1)^{\beta/\alpha}\frac{(y'(t-\sigma_1+\tau_1))^{\beta/\alpha}}{(1+b^\beta+\frac{c^\beta}{2^{\beta-1}})^{\beta/\alpha}}, t \geq t_2.$$

Set $y'(t) = w(t)$. Then $w(t) > 0$ and $w'(t) = y''(t) < 0$ and the last inequality becomes

$$w''(t) \geq \frac{Q(t)(\sigma_1-\tau_1)^{\beta/\alpha} w^{\beta/\alpha}(t-\sigma_1+\tau_1)}{4^{\beta-1}(1+b^\beta+\frac{c^\beta}{2^{\beta-1}})^{\beta/\alpha}}, t \geq t_2. \tag{2.30}$$

Thus, $w(t)$ is a positive decreasing solution of the second order differential inequality (2.19), which is a contradiction. Now the proof is complete. □

Theorem 2.5. *Assume that* $0 < \beta \leq 1,\ \gamma \geq 1,\ b \leq 1, c \leq 1$ *and* $\sigma_2 > \sigma_1 > \max\{\tau_1, \tau_2\}$. *If the second order differential inequality*

$$y''(t) \geq \frac{P(t)(\sigma_1-\tau_2)^{\gamma/\alpha}}{4^{\gamma-1}(1+b^\beta+c^\beta)^{\gamma/\alpha}} y^{\beta/\alpha}(t+\sigma_2-\sigma_1) \tag{2.31}$$

has no positive increasing solution, and the second order differential inequality

$$y''(t) \geq \frac{Q(t)(\sigma_1-\tau_1)^{\beta/\alpha}}{(1+b^\beta+c^\beta)^{\beta/\alpha}} y^{\beta/\alpha}(t-\sigma_1+\tau_1) \tag{2.32}$$

has no positive decreasing solution, then every solution of equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1) for all $t \geq t_1 \geq t_0$. Let us assume that $x(t)$ is a positive solution of (1.1) for all $t \geq t_1 \geq t_0$.. Then there exists a $t_2 \geq t_1$ such that $x(t-\theta) > 0$ for all $t \geq t_2$. By the definition of $z(t)$, we have $z(t-\theta) > 0$ for all $t \geq t_2$. Set

$$y(t) = z(t) + b^\beta z(t-\tau_1) + c^\beta z(t+\tau_2) \text{ for all } t \geq t_2. \tag{2.33}$$

Then, $y(t) > 0$, and

$$\begin{aligned} y'''(t) &= z'''(t) + b^\beta z'''(t-\tau_1) + c^\beta z'''(t+\tau_2) \\ &= q(t)x^\beta(t-\sigma_1) + p(t)x^\gamma(t+\sigma_2) + b^\beta q(t-\tau_1)x^\beta(t-\tau_1-\sigma_1) + \\ &\quad b^\beta p(t-\tau_1)x^\gamma(t-\tau_1+\sigma_2) + c^\beta q(t+\tau_2)x^\beta(t+\tau_2-\sigma_1) + \\ &\quad c^\beta p(t+\tau_2)x^\gamma(t+\tau_2+\sigma_2) \\ &\geq Q(t)[x^\beta(t-\sigma_1) + b^\beta x^\beta(t-\tau_1-\sigma_1) + c^\beta x^\beta(t+\tau_2-\sigma_1)] + \\ &\quad P(t)[x^\gamma(t+\sigma_2) + b^\beta x^\gamma(t-\tau_1+\sigma_2) + c^\beta x^\gamma(t+\tau_2+\sigma_2)], t \geq t_2. \end{aligned}$$

Using (2.3) twice in the first part of righthand side of the last inequality, we have

$$y'''(t) \geq Q(t)z^{\beta/\alpha}(t-\sigma_1) + P(t)[x^\gamma(t+\sigma_2) + b^\beta x^\gamma(t-\tau_1+\sigma_2) + c^\beta x^\gamma(t+\tau_2+\sigma_2)], t \geq t_2. \tag{2.34}$$

Using the fact that $b \leq 1,\ c \leq 1,\ \gamma \geq 1,$ and $0 < \beta \leq 1$, we have

$$\begin{aligned} & x^\gamma(t+\sigma_2) + b^\beta x^\gamma(t-\tau_1+\sigma_2) + c^\beta x^\gamma(t+\tau_2+\sigma_2) \\ \geq\ & x^\gamma(t+\sigma_2) + b^\gamma x^\gamma(t-\tau_1+\sigma_2) + c^\gamma x^\gamma(t+\tau_2+\sigma_2) \end{aligned}$$

$$\geq \quad x^{\gamma}(t+\sigma_2) + b^{\gamma}x^{\gamma}(t-\tau_1+\sigma_2) + \frac{c^{\gamma}}{2^{\gamma-1}}x^{\gamma}(t+\tau_2+\sigma_2),$$

and applying (2.4) twice and simplifying, we obtain

$$x^{\gamma}(t+\sigma_2) + b^{\beta}x^{\gamma}(t-\tau_1+\sigma_2) + c^{\beta}x^{\gamma}(t+\tau_2+\sigma_2) \ \geq \ \frac{1}{4^{\gamma-1}}z^{\frac{\gamma}{\alpha}}(t+\sigma_2). \tag{2.35}$$

Substituting (2.35) in (2.34), we get

$$y'''(t) \geq Q(t)z^{\beta/\alpha}(t-\sigma_1) + \frac{P(t)}{4^{\gamma-1}}z^{\gamma/\alpha}(t+\sigma_2), t \geq t_2. \tag{2.36}$$

Now we consider the following two cases for $z(t)$ as in Lemma 2.1.

Case (I): In this case we have $z'(t) > 0,\ z''(t) > 0$ and $z'''(t) > 0$ for all $t \geq t_2$. Then from (2.33), we have $y'(t) > 0,\ y''(t) > 0$ and $y'''(t) > 0$ for all $t \geq t_2$.

From the inequality (2.36), we have

$$y'''(t) \geq \frac{P(t)}{4^{\gamma-1}}z^{\gamma/\alpha}(t+\sigma_2), t \geq t_2. \tag{2.37}$$

Using the monotonicity of $z'(t)$, we get

$$\begin{aligned} y'(t) &= z'(t) + b^{\beta}z'(t-\tau_1) + c^{\beta}z'(t+\tau_2) \\ &\leq (1+b^{\beta}+c^{\beta})z'(t+\tau_2), t \geq t_2. \end{aligned} \tag{2.38}$$

Again using the monotonicity of $z'(t)$, we obtain

$$z(t+\sigma_1-\tau_2) - z(t) = \int_t^{t+\sigma_1-\tau_2} z'(s)\ ds \geq z'(t)(\sigma_1-\tau_2),$$

or

$$z(t+\sigma_1-\tau_2) \geq (\sigma_1-\tau_2)z'(t). \tag{2.39}$$

Now using (2.38) and (2.39) in (2.37), we obtain

$$\begin{aligned} y'''(t) &= \frac{P(t)}{4^{\gamma-1}}z^{\gamma/\alpha}(t+\sigma_2) \\ &\geq \frac{P(t)}{4^{\gamma-1}}(\sigma_1-\tau_2)^{\gamma/\alpha}(z'(t+\sigma_2-\sigma_1+\tau_2))^{\gamma/\alpha} \\ &\geq \frac{P(t)}{4^{\gamma-1}}\frac{(\sigma_1-\tau_2)^{\gamma/\alpha}(y'(t+\sigma_2-\sigma_1))^{\gamma/\alpha}}{(1+b^{\beta}+c^{\beta})^{\gamma/\alpha}}, t \geq t_2. \end{aligned}$$

By setting $y'(t) = w(t)$, we see that $w(t) = y'(t) > 0$, $w'(t) = y''(t) > 0$ and it satisfies

$$w''(t) \geq \frac{P(t)(\sigma_1-\tau_2)^{\gamma/\alpha}}{4^{\gamma-1}(1+b^{\beta}+c^{\beta})^{\gamma/\alpha}}w^{\gamma/\alpha}(t+\sigma_2-\sigma_1), t \geq t_2.$$

Thus, $w(t)$ is a positive increasing solution of the second order differential inequality (2.31), which is a contradiction.

Case (II): In this case we have $z'(t) > 0,\ z''(t) < 0$ and $z'''(t) > 0$ for all $t \geq t_2$. Therefore $y'(t) > 0,\ y''(t) < 0$ and $y'''(t) > 0$ for all $t \geq t_2$. From the inequality (2.36) we have

$$y'''(t) \geq Q(t)z^{\beta/\alpha}(t-\sigma_1), t \geq t_2. \tag{2.40}$$

Since $z''(t) < 0$, we have $z'(t)$ is decreasing and therefore

$$\begin{aligned} y'(t) &= z'(t) + b^{\beta}z'(t-\tau_1) + c^{\beta}z'(t+\tau_2) \\ &\leq (1+b^{\beta}+c^{\beta})z'(t-\tau_1), \end{aligned} \tag{2.41}$$

or

$$y'(t-\sigma_1) \leq (1+b^{\beta}+c^{\beta})z'(t-\sigma_1-\tau_1), t \geq t_2.$$

Again using the monotonicity of $z'(t)$, we see that

$$z(t) - z(t-\sigma_1+\tau_1) = \int\limits_{t-(\sigma_1-\tau_1)}^{t} z'(s)\ ds \geq (\sigma_1-\tau_1)z'(t),$$

or

$$z(t) \geq (\sigma_1-\tau_1)z'(t). \tag{2.42}$$

Substituting (2.41) and (2.42) in (2.40), we obtain

$$y'''(t) \geq \frac{Q(t)(\sigma_1-\tau_1)^{\beta/\alpha}}{(1+b^\beta+c^\beta)^{\beta/\alpha}}(y'(t-\sigma_1+\tau_1))^{\beta/\alpha}, t \geq t_2. \tag{2.43}$$

By setting $y'(t) = w(t)$, we see that $w(t)$ is a positive decreasing solution of

$$w''(t) \geq \frac{Q(t)(\sigma_1-\tau_1)^{\beta/\alpha}}{(1+b^\beta+c^\beta)^{\beta/\alpha}} w^{\beta/\alpha}(t-\sigma_1+\tau_1), t \geq t_2, \tag{2.44}$$

which is a contradiction to (2.32). This completes the proof. □

Theorem 2.6. *Assume that* $0 < \gamma \leq 1,\ \beta \geq 1,\ b \leq 1, c \leq 1$ *and* $\sigma_2 > \sigma_1 > \max\{\tau_1,\tau_2\}$. *If the second order differential inequality*

$$y''(t) \geq \frac{P(t)(\sigma_1-\tau_2)^{\gamma/\alpha}}{(1+b^\beta+c^\beta)^{\gamma/\alpha}} y^{\beta/\alpha}(t+\sigma_2-\sigma_1) \tag{2.45}$$

has no positive increasing solution and the second order differential inequality

$$y''(t) \geq \frac{Q(t)(\sigma_1-\tau_1)^{\beta/\alpha}}{4^{\beta-1}(1+b^\beta+c^\beta)^{\beta/\alpha}} y^{\beta/\alpha}(t-\sigma_1+\tau_1) \tag{2.46}$$

has no positive decreasing solution, then equation (1.1) *is oscillatory.*

Proof. The proof is similar to that of Theorem 2.5 and hence the details are omitted. □

Theorem 2.7. *Assume that* $\beta \geq 1,\ 0 < \gamma \leq 1,\ b \geq 1, c \geq 1$ *and* $\sigma_2 > \sigma_1 > \max\{\tau_1,\tau_2\}$. *If the second order differential inequality*

$$y''(t) \geq \frac{P(t)(\sigma_1-\tau_2)^{\gamma/\alpha}}{(1+b^\beta+c^\beta)^{\gamma/\alpha}} y^{\gamma/\alpha}(t+\sigma_2-\sigma_1) \tag{2.47}$$

has no positive increasing solution, and the second order differential inequality

$$y''(t) \geq \frac{Q(t)(\sigma_1-\tau_1)^{\beta/\alpha}}{4^{\beta-1}(1+b^\beta+c^\beta)^{\beta/\alpha}} y^{\beta/\alpha}(t+\tau_1-\sigma_1) \tag{2.48}$$

has no positive decreasing solution, then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1) for all $t \geq t_1 \geq t_0$. Without loss of generality, let us assume that $x(t)$ is a positive solution of equation (1.1) for all $t \geq\ t_1 \geq\ t_0$. Then there exists a $t_2 \geq t_1$ such that $x(t-\theta) > 0$ for all $t \geq t_2$. By the definition of $z(t)$ we have $z(t-\theta) > 0$ for all $t \geq t_2$. Set

$$y(t) = z(t) + b^\beta z(t-\tau_1) + \frac{c^\beta}{2^{\gamma-1}} z(t+\tau_2) \text{ for all } t \geq t_2. \tag{2.49}$$

Then, $y'(t) > 0$, and using the fact $b \geq 1,\ c \geq 1,\ \gamma \leq 1,\ \beta \geq 1$, we have

$$\begin{aligned} y'''(t) &= z'''(t) + b^\beta z'''(t-\tau_1) + \frac{c^\beta}{2^{\gamma-1}} z'''(t+\tau_2) \\ &\geq Q(t)[x^\beta(t-\sigma_1) + b^\beta x^\beta(t-\tau_1-\sigma_1) + \frac{c^\beta}{2^{\beta-1}} x^\beta(t+\tau_2-\sigma_1)] + \\ &\quad P(t)[x^\gamma(t+\sigma_2) + b^\beta x^\gamma(t-\tau_1+\sigma_2) + \frac{c^\beta}{2^{\gamma-1}} x^\gamma(t+\tau_2+\sigma_2)] \\ &\geq Q(t)[x^\beta(t-\sigma_1) + b^\beta x^\beta(t-\tau_1-\sigma_1) + \frac{c^\beta}{2^{\beta-1}} x^\beta(t+\tau_2+\sigma_2)] \end{aligned}$$

$$+P(t)[x^\gamma(t+\sigma_2)+b^\gamma x^\gamma(t-\tau_1+\sigma_2)+c^\gamma x^\gamma(t+\tau_2+\sigma_2)], t\ge t_2$$

Now applying (2.4) and (2.3) twice in first and second part of right hand side of last inequality, we get

$$y'''(t)\ge \frac{Q(t)}{4^{\beta-1}}z^{\beta/\alpha}(t-\sigma_1)+P(t)z^{\gamma/\alpha}(t+\sigma_2), t\ge t_2. \tag{2.50}$$

Now we consider the following two cases for $z(t)$ as given in Lemma 2.1.

Case (I): In this case we have $z'(t)>0,\ z''(t)>0$ and $z'''(t)>0$ and therefore $y'(t)>0,\ y''(t)>0$ and $y'''(t)>0$ for all $t\ge t_2$. From the inequality (2.50), we have

$$y'''(t)\ge P(t)z^{\gamma/\alpha}(t+\sigma_2), t\ge t_2. \tag{2.51}$$

Applying monotonicity of $z'(t)$, we get

$$\begin{aligned} y'(t) &= z'(t)+b^\beta z'(t-\tau_1)+c^\beta z'(t+\tau_2)\\ y'(t) &\le (1+b^\beta+c^\beta)z'(t+\tau_2), t\ge t_2. \end{aligned} \tag{2.52}$$

Also using the monotonicity of $z'(t)$, we get

$$z(t+\sigma_1-\tau_2)-z(t)=\int_t^{t+\sigma_1-\tau_2} z'(s)\ ds>z'(t)(\sigma_2-\tau_2)$$

$$z(t+\sigma_1-\tau_2)\ge z'(t)(\sigma_1-\tau_2). \tag{2.53}$$

Combining (2.51), (2.52) and (2.53), we obtain

$$\begin{aligned} y'''(t) &= P(t)z^{\gamma/\alpha}(t+\sigma_2)\\ &\ge P(t)(\sigma_1-\tau_2)^{\gamma/\alpha}z'(t+\tau_2)\\ &\ge \frac{P(t)(\sigma_1-\tau_2)^{\gamma/\alpha}(y'(t+\sigma_2-\sigma_1))^{\gamma/\alpha}}{(1+b^\beta+c^\beta)^{\gamma/\alpha}}, t\ge t_2. \end{aligned}$$

By putting $y'(t)=w(t)$, we see that $w(t)$ is a positive increasing solution of

$$w''(t)\ge \frac{P(t)(\sigma_2-\tau_2)^{\gamma/\alpha}}{(1+b^\beta+c^\beta)^{\gamma/\alpha}}w^{\gamma/\alpha}(t+\sigma_2-\sigma_1), t\ge t_2$$

which is a contradiction (2.47).

Case (II): In this case we have $z''(t)<0$ for all $t\ge t_2$. Therefore $z'(t)$ is decreasing, for all $t\ge t_2$. Since $z'(t)$ is decreasing we have

$$\begin{aligned} y'(t) &= z'(t)+b^\beta z'(t-\tau_1)+c^\beta z'(t+\tau_2)\\ &\le (1+b^\beta+c^\beta)z'(t-\tau_1), t\ge t_2. \end{aligned} \tag{2.54}$$

Also using the monotonicity of $z'(t)$, we get

$$z(t)-z(t-\sigma_1+\tau_1)=\int_{t-(\sigma_1-\tau_1)}^{t} z'(s)\ ds\ge(\sigma_1-\tau_1)z'(t)$$

or

$$z(t)\ge(\sigma_1-\tau_1)z'(t). \tag{2.55}$$

From (2.50), we have

$$y'''(t)\ge\frac{Q(t)}{4^{\beta-1}}z^{\beta/\alpha}(t-\sigma_1), t\ge t_2. \tag{2.56}$$

Combining (2.54), (2.55) and (2.56), we obtain

$$y'''(t)\ge\frac{Q(t)(\sigma_1-\tau_1)^{\beta/\alpha}}{4^{\beta-1}(1+b^\beta+c^\beta)^{\beta/\alpha}}(y'(t-\sigma_1+\tau_1))^{\beta/\alpha}, t\ge t_2. \tag{2.57}$$

By taking $y'(t) = w(t)$, we see that $w(t)$ is a positive decreasing solution of

$$w''(t) \geq \frac{(\sigma_1 - \tau_1)^{\beta/\alpha} Q(t)}{4^{\beta-1}(1 + b^\beta + c^\beta)^{\beta/\alpha}} (w(t - \sigma_1 + \tau_1))^{\beta/\alpha}, t \geq t_2, \tag{2.58}$$

which is a contradiction to (2.48). This completes the proof. □

Theorem 2.8. *Assume that* $\gamma \geq 1,\ 0 < \beta \leq 1,\ b \geq 1, c \geq 1$ *and* $\sigma_2 > \sigma_1 > \max\{\tau_1, \tau_2\}$. *If the second order differential inequality*

$$y''(t) \geq \frac{P(t) y^{\gamma/\alpha}(t + \sigma_2 - \sigma_1)(\sigma_1 - \tau_2)^{\gamma/\alpha}}{4^{\gamma-1}(1 + b^\beta + c^\beta)^{\gamma/\alpha}} \tag{2.59}$$

has no positive increasing solution and the second order differential inequality

$$y''(t) \geq \frac{Q(t) y^{\beta/\alpha}(t - \sigma_1 + \tau_1)(\sigma_1 - \tau_1)^{\beta/\alpha}}{(1 + b^\beta + c^\beta)^{\beta/\alpha}} \tag{2.60}$$

has no positive decreasing solution, then equation (1.1) *is oscillatory.*

Proof. The proof is similar to that of Theorem 2.7 and hence the details are omitted. □

Corollary 2.9. *Assume that* $\alpha = \beta = \gamma \geq 1$, $\sigma_2 > \sigma_1 > \max\{\tau_1, \tau_2\}$. *If*

$$\limsup_{t\to\infty} \int_t^{t+\sigma_2-\tau_2-2} (t + \sigma_2 - \tau_2 - s - 1) P(s)\ ds \geq 4^{\alpha-1}(1 + a^\alpha + \frac{b^\alpha}{2^{\alpha-1}}) \tag{2.61}$$

and

$$\limsup_{t\to\infty} \int_{t-\sigma_1+\tau_1}^{t} (t - s + 1) Q(s)\ ds \geq 4^{\alpha-1}(1 + a^\alpha + \frac{b^\alpha}{2^{\alpha-1}}) \tag{2.62}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Condition (2.61) and (2.62) imply that the differential inequalities (2.59) and (2.60) have no positive increasing and no positive decreasing solutions respectively see [12, 16]. Now the result follows from Theorem 2.8. □

Corollary 2.10. *Let* $\beta < \gamma,\ b \leq 1,\ c \leq 1$, $\sigma_2 > \sigma_1 > \max\{\tau_1, \tau_2\}$. *If*

$$\int_{t_0}^{\infty} \Big(\int_t^{t+\sigma_1-\tau_1} Q(s)\ ds \Big)\ dt \ = \ \infty \tag{2.63}$$

$$\int_{t_0}^{\infty} \Big(\int_{t-\sigma_2+\tau_2+1}^{t} P(s)\ ds \Big)\ dt \ = \ \infty \tag{2.64}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Conditions (2.63) and (2.64) imply that the differential inequalities (2.31) and (2.32) have no positive increasing and no positive decreasing solutions respectively [12, 16]. Now the result follows from Theorem 2.5. □

3 Examples

In this section, we shall see some examples to illustrate main results.

Example 3.1. *Consider the third order differential equation*

$$((x(t) + 2x(t-1) + 3x(t+2))^3)''' = (t+1)x^3(t-3) + tx^3(t+5),\ \ t \geq 1 \tag{3.1}$$

Here $b(t) = 2,\ c(t) = 3,\ \tau_1 = 1,\ \tau_2 = 2,\ \sigma_1 = 3,\ \sigma_2 = 5,\ q(t) = t+1,\ p(t) = t$ *and* $\alpha = \beta = \gamma = 3$. *Then* $Q(t) = t,\ P(t) = t - 1$ *and we can easily see that all the conditions of Corollary* 2.9 *are satisfied. Therefore all the solutions of equation* (3.1) *are oscillatory.*

Example 3.2. *Consider the third order differential equation*

$$((x(t)+\frac{1}{2}x(t-1)+\frac{1}{3}x(t+2))^3)''' = (t+1)x(t-3)+(t+2)^2x^3(t+4),\ t\geq 1 \tag{3.2}$$

Here $b(t)=\frac{1}{2},\ c(t)=\frac{1}{3},\ \tau_1=1,\ \tau_2=2,\ \sigma_1=3,\ \sigma_2=4,\ \alpha=1,\ \beta=1,\ \gamma=3,\ q(t)=t+1,\ p(t)=(t+2)^2.$ *Then* $Q(t)=t,\ P(t)=t^2$ *and we can easily see that all the conditions of Corollary* 2.10 *are satisfied. Therefore all the solutions of equation* (3.2) *are oscillatory.*

References

[1] R. P. Agarwal, M. F. Aktas, and A. Tiryaki, On oscillation criteria for third order nonlinear delay differential equations, *Arch. Math.,* 45(2009), 1-18.

[2] R. P. Agarwal, B. Baculáková, J. Džurina and T.Li, Oscillation of third-order nonlinear functional differential equations with mixed arguments, *Acta Math. Hungar.,* 134(2011), 54-67.

[3] R. P. Agarwal, S. R. Grace and D. O'Regan, *Oscillation Theory for Difference and Functional Differential Equation,* Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[4] B. Baculíková and J. Džurina, Oscillation of third-order functional differential equations, *Elec. J. Qual. Theo. Diff. Equ.,* 43(2010), 1-10.

[5] B. Baculíková and J. Džurina, Oscillation of third-order neutral differential equations, *Math. Comput. Modelling,* 52(2010), 215-226.

[6] B. Baculíková and J. Džurina, On the aymptotic behavior of a class of third order nonlinear neutral differential equations, *Cent. European J. Math.,* 8(2010), 1091-1103.

[7] B. Baculíková and J. Džurina, Oscillation of third-order nonlinear differential equations, *Appl. Math. Lett.,* (2011), 466-470.

[8] B. Baculíková, E. M. Elabbasy, S. H. Saker, and J. Džurina, Oscillation criteria for third-order nonlinear differential euqations, *Math. Slovaca,* 58(2008), 201-220.

[9] T. Candan and R. S. Dahiya, Functional differential equations of third order, *Elec. J. Diff. Equ.,* 12(2005), 47-56.

[10] P. Das, Oscillation criteria for odd order neutral equations, *J. Math. Anal. Appl.,* 188(1994), 245-257.

[11] K. Gopalsamy, B. S. Lalli and B. G. Zhang, Oscillation of odd order neutral differential equations, *Czech. Math. J.,* 42(1992), 313-323.

[12] S.R.Grace, Oscillation criteria for nth order neutral functional differential equations, *J. Math. Anal. Appl.,* 184(1994), 44-55.

[13] S. R. Grace, On oscillation of mixed neutral equations, *J. Math. Anal. Appl.,* 194(1995), 377-388.

[14] S. R. Grace, Oscillation of mixed neutral fractional differential equations, *Appl. Math. Comp.,* 68(1995), 1-13.

[15] S. R. Grace, R. P. Agarwal, R. Pavani, and E. Thandapani, On the oscillation of certain third order nonlinear functional differential equations, *Appl. Math. Comput.,* 202(2008), 102-112.

[16] S.R.Grace and B.S.Lalli, Oscillation theorems for certain neutral functional differential equations with periodic cofficients, *Dynam. Systems Appl.,* 3(1994), 85-93.

[17] J. R. Graef, R. Savithri, and E. Thandapani, Oscillatory properties of third order neutral delay differential equations, *Disc. Cont. Dyn. Sys. A,* 2003, 342-350.

[18] T. Li and E. Thandapani, Oscillation of solutions to odd-order nonlinear neutral functional differential equations, *Elec. J. Diff. Equ.,* 23(2011), 1-12.

[19] T.Li, C.Zhang and G.Xing, Oscillation of third order neutral delay differential equations, *Abstract and Applied Analysis*, 2012(2012), 1-11.

[20] S. H. Saker and J. Džurina, On the oscillation of certain class of third-order nonlinear delay differential equations, *Math. Bohemica,* 135(2010), 225-237.

[21] E. Thandapani and T. Li, On the oscillation of third-order quasi-linear neutral functional differentiall equations, *Arch. Math.,* 47(2011), 181-199.

[22] J. R. Yau, Oscillation of higher order neutral differential equations of mixed type, *Israel J. Math.,* 115(2000), 125-136.

[23] C. Zhang, T. Li, B. Sun and E. Thandapani, On the oscillation of higher-order half-linear delay differential equations, *Appl. Math. Lett.,* 24(2011), 1618-1621.

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