



Iterative solution of quadratic fractional integral equation involving generalized Mittag Leffler function

SAYYED JALIL^{1*}, Mohammed Mazhar UI Haque² and Md. Indraman Khan³

Abstract

The present paper verifies and validates the factual and estimated outcomes of a certain quadratic fractional integral equation involving the generalized Mittag-Leffler function via algorithm that representatively embodies successive estimations under fragile partial fractional Lipschitz and compactness type circumstances. The paper also validate the existence and convergence of a nonlinear quadratic fractional integral equation with the generalized Mittag Leffler function which is the generalization of Mittag-Leffler function, on a closed and bounded interval of the real line with the help of some conditions.

Keywords

Quadratic fractional integral equation; Fractional derivatives and Integrals.

AMS Subject Classification

45G10, 47H09, 47H10.

¹ Department of Mathematics , Hutatma Jaywantrao Patil Mahavidyalaya, Himayatnagar, Nanded, M.S., India.

² Department of Mathematics, Gramin College of Engineering, Vishnupuri, Nanded, Maharashtra, India.

³ Department of Mathematics, PETTIGREW College, Ukhrul, Manipur University, Manipur, India.

*Corresponding author: ¹ drsayyedjalil@gmail.com;

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1. Introduction

Linear and nonlinear integral equations establish an indispensable class of problems in mathematics. The concept of integral operators and integral equations is an essential part of nonlinear analysis. It is commenced by the fact that this concept is frequently applicable in other branches of mathematics and some equations describe mathematical models in physics, engineering or biology also in describing problems connected by real world. Numerous researchers have showed applications of fractional calculus in the nonlinear oscillation of earthquakes [13], fluid-dynamic traffic model [14], to demonstrate frequency dependent damping performance of countless viscoelastic materials [15, 16], continuum and statistical mechanics [17], colored noise [18], solid mechan-

ics [19], economics [20], bioengineering [21–23], anomalous transport [24], and dynamics of interfaces between nanoparticles and substrates[25]. There are correspondingly such equations whose relevance rests in another branch of pure mathematics. Integral equations of fractional order generate a thought-provoking and significant branch of the theory of integral equations. The theory of such integral equations is established intensively in recent years collectively with the theory of differential equations of fractional order ([1–7]). In contrast the theory of quadratic integral equations is also rigorously researched and observes various applications in describing real world problems ([8–11]). Let us indicate that this theory was began by considering a quadratic integral equation of Chandrasekhar type ([2, 11, 12]).In this paper we demonstrate the existence along with approximations of the solutions of a certain generalized quadratic integral equation via an algorithm built on successive approximations under weak partial Lipschitz and compactness type conditions. On a closed and bounded interval $J = [0, T]$ of the real line \mathbb{R} for some $T > 0$, we take into account the quadratic fractional

integral equation

$$x(t) = x(t^{q-1})E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^{q-1}) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{(q-1)} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^{q-1})f(s,x(s))ds \tag{1.1}$$

where $f : J \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : J \rightarrow \mathbb{R}$ stand continuous functions, $1 \leq q < 2$ and Γ is the Euler gamma function, and $E_{\alpha,\beta}^{\gamma,\delta,q}(x)$ is generalized mittag leffler function.

By a *solution* of the QFIE (1.1) we necessitate a function $x \in C(J, \mathbb{R})$ that satisfies the equation (1.1) on J , where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J .

2. Preliminaries

The entire paper delineates the significant meaning of E as a partially well-ordered real normed linear space with an order relation \preceq and the norm $\|\cdot\|$. It is recognized as E is **regular** if $\{x_n\}_{n \in \mathbb{N}}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \rightarrow x^*$ as $n \rightarrow \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. Obviously, the partially ordered Banach space $C(J, \mathbb{R})$ is regular and the statistics prove its Regularity of any partially ordered normed linear space which can be easily notified and the relative references therein.

In this section, we exhibit some basic definitions and preliminaries which are effective in further discussion.

Definition 2.1. (Mittag-Leffler Function) [28] The Mittag - Leffler function of one parameter is expressed by $E_\alpha(z)$ and defined as,

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} z^k \tag{2.1}$$

where $z, \alpha \in C, Re(\alpha) > 0$.

If we place $\alpha = 1$, then the above equation turn out to be

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z. \tag{2.2}$$

Definition 2.2. (Mittag-Leffler Function for two parameters) The generalization of $E_\alpha(z)$ was revised by Wiman (1905) [32], Agarwal [26] and Humbert and Agarwal [29] the function as,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} z^k \tag{2.3}$$

where $z, \alpha, \beta \in C, Re(\alpha) > 0, Re(\beta) > 0$,

In 1971, The additional generalized function is presented by Prabhakar [37] as

$$E_{\alpha,\beta}^\gamma(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k + \beta)}. \tag{2.4}$$

where $z, \alpha, \beta, \gamma \in C, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$, where $\gamma \neq 0, (\gamma)_k = \gamma(\gamma+1)(\gamma+2)\dots(\gamma+k-1)$ is the Pochhammer symbol [30], and

$$(\gamma)_k = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$$

In 2007, Shulka and Prajapati [30] presented the function which is defined as,

$$E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk} z^k}{k! \Gamma(\alpha k + \beta)}. \tag{2.5}$$

where $z, \alpha, \beta, \gamma \in C, \min\{Re(\alpha), Re(\beta), Re(\gamma)\} > 0$, and $q \in (0, 1) \cup \mathbb{N}$

In 2012, supplementary generalization of Mittag - Leffler function was stated by Salim [31] and Chauhan [27] as,

$$E_{\alpha,\beta}^{\gamma,\delta,q}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk} z^k}{(\delta)_{(qk)} \Gamma(\alpha k + \beta)}. \tag{2.6}$$

where $z, \alpha, \beta, \gamma \in C, \min\{Re(\alpha), Re(\beta), Re(\gamma)\} > 0$, and $q \in (0, 1) \cup \mathbb{N}$

$$(\gamma)_{qk} = \frac{\Gamma(\gamma+qk)}{\Gamma(\gamma)} \text{ and } (\delta)_{qk} = \frac{\Gamma(\delta+qk)}{\Gamma(\delta)}$$

represent the generalized Pochhammer symbol [30],

Definition 2.3. A mapping $\mathcal{T} : E \rightarrow E$ is named **isotone or nondecreasing** if it preserves the order relation \preceq , explicitly, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$.

Definition 2.4 ([35]). A mapping $\mathcal{T} : E \rightarrow E$ is termed **partially continuous** at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x\| < \delta$. \mathcal{T} called **partially continuous on E** if it is partially continuous at every point of it. It is certain that if \mathcal{T} is partially continuous on E , then it is continuous on every chain C comprised in E .

Definition 2.5. A mapping $\mathcal{T} : E \rightarrow E$ is named **partially bounded** if $\mathcal{T}(C)$ is bounded for every chain C in E . \mathcal{T} is called **uniformly partially bounded** if all chains $\mathcal{T}(C)$ in E are bounded by a unique constant. \mathcal{T} is known as **bounded** if $\mathcal{T}(E)$ is a bounded subset of E .

Definition 2.6. A mapping $\mathcal{T} : E \rightarrow E$ is identified **partially compact** if $\mathcal{T}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E . \mathcal{T} is known as **uniformly partially compact** if $\mathcal{T}(C)$ is a uniformly partially bounded and partially compact on E . \mathcal{T} is called **partially totally bounded** if for any totally ordered and bounded subset C of E , $\mathcal{T}(C)$ is a relatively compact subset of E . If \mathcal{T} is partially continuous and partially totally bounded, then it is called **partially completely continuous** on E .

Definition 2.7 ([35]). The order relation \preceq and the metric d on a non-empty set E are said to be **compatible** if $\{x_n\}_{n \in \mathbb{N}}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}_{n \in \mathbb{N}}$



of $\{x_n\}_{n \in \mathbb{N}}$ converges to x^* implies that the original sequence $\{x_n\}_{n \in \mathbb{N}}$ converges to x^* . Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation \preceq and the norm $\|\cdot\|$ are said to be compatible if \preceq and the metric d defined through the norm $\|\cdot\|$ are compatible.

Definition 2.8 ([33]). A upper semi-continuous and monotone nondecreasing function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called a \mathcal{D} -function provided $\psi(r) = 0$ iff $r = 0$. Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathcal{T} : E \rightarrow E$ is called **partially nonlinear \mathcal{D} -Lipschitz** if there exists a \mathcal{D} -function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x - y\|) \tag{2.7}$$

for all comparable elements $x, y \in E$. If $\psi(r) = kr, k > 0$, then \mathcal{T} is called a partially Lipschitz with a Lipschitz constant k .

Consider $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \{x \in E \mid x \succeq \theta, \text{ where } \theta \text{ is the zero element of } E\}$$

and

$$\mathcal{K} = \{E^+ \subset E \mid uv \in E^+ \text{ for all } u, v \in E^+\}. \tag{2.8}$$

The elements of \mathcal{K} are called the positive vectors of the normed linear algebra E . The following lemma pursues instantaneously from the definition of the set \mathcal{K} and which is frequently used in the applications of hybrid fixed point theory in Banach algebras.

Lemma 2.9 ([34]). If $u_1, u_2, v_1, v_2 \in \mathcal{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1 u_2 \preceq v_1 v_2$.

Definition 2.10. An operator $\mathcal{T} : E \rightarrow E$ is supposed to be positive if the range $R(\mathcal{T})$ of \mathcal{T} is such that $R(\mathcal{T}) \subseteq \mathcal{K}$.

Theorem 2.11 ([36]). Let $(E, \preceq, \|\cdot\|)$ be a regular partially ordered complete normed linear algebra such that the order relation \preceq and the norm $\|\cdot\|$ in E are compatible in every compact chain of E . Let $\mathcal{A}, \mathcal{B} : E \rightarrow \mathcal{K}$ be two nondecreasing operators such that

- (a) \mathcal{A} is partially bounded and partially nonlinear \mathcal{D} -Lipschitz with \mathcal{D} -functions $\psi_{\mathcal{A}}$,
- (b) \mathcal{B} is partially continuous and uniformly partially compact, and
- (c) $M\psi_{\mathcal{A}}(r) < r, r > 0$, where $M = \sup\{\|\mathcal{B}(C)\| : C \text{ is a chain in } E\}$, and
- (d) there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{A}x_0 + \mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0 + \mathcal{B}x_0$.

Then the operator equation

$$\mathcal{A}x + \mathcal{B}x = x \tag{2.9}$$

has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n, n = 0, 1, \dots$, converges monotonically to x^* .

3. Main Results

The Fractional integral equation (1.1) is counted in the function space $C(J, \mathbb{R})$ of continuous real-valued functions on J . We classify a norm $\|\cdot\|$ and the order relation \leq in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{3.1}$$

and

$$x \leq y \iff x(t) \leq y(t) \tag{3.2}$$

for all $t \in J$ correspondingly. Evidently, $C(J, \mathbb{R})$ is a Banach algebra in connection with above supremum norm and is also partially ordered w.r.t. the above partially order relation \leq . The following lemma in this connection follows by an application of Arzelá scoli theorem.

Lemma 3.1. Let $(C(J, \mathbb{R}), \leq, \|\cdot\|)$ be a partially ordered Banach space along with the norm $\|\cdot\|$ and the order relation \leq stated by (3.1) and (3.2) respectively. Then $\|\cdot\|$ and \leq are compatible in each partially compact subset of $C(J, \mathbb{R})$.

Definition 3.2. A function $v \in C(J, \mathbb{R})$ is supposed to be a lower solution of the fractional integral equation (1.1) if it fulfills

$$v(t) \leq v(t^{q-1})E_{\alpha, \beta}^{\gamma, \delta, q}((t-s)^{q-1}) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{(q-1)} E_{\alpha, \beta}^{\gamma, \delta, q}((t-s)^{q-1}) f(s, v(s)) ds$$

for all $t \in J$. In the same way, a function $u \in C(J, \mathbb{R})$ is supposed to be an upper solution of the fractional integral equation (1.1) if it satisfies the above inequalities with opposite sign.

We consider the following set of statements in what follows:

- (A₁) The functions $f : J \times \mathbb{R} \rightarrow \mathbb{R}_+, q : J \rightarrow \mathbb{R}_+$ where q is continuous function.
- (A₂) There are constants $M_f, M > 0$ such that $0 \leq f(t, x) \leq M_f$ and $x(t)E_{\alpha, \beta}^{\gamma, \delta, q}((t-s)^{q-1}) < M$ for all $t \in J$ and $x \in \mathbb{R}$.
- (A₃) There exists a \mathcal{D} -function ψ_f such that

$$0 \leq f(t, x) - f(t, y) \leq \psi_f(x - y)$$

for all $t \in J$ and $x, y \in \mathbb{R}, x \leq y$.

- (A₄) $f(t, x)$ is nondecreasing in x for all $t \in J$.
- (A₅) The FIE (1.1) consumes a lower solution $v \in C(J, \mathbb{R})$.



Theorem 3.3. Suppose that hypotheses (A₁)-(A₅) holds then the FIE (1.1) has a solution x^* on J and the sequence $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ of successive approximations defined by

$$x_{n+1}(t) = x_n(t^{q-1})E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^{q-1}) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{(q-1)} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^{q-1}) f(s, x_n(s)) ds \quad (3.3)$$

for all $t \in J$, where $x_0 = v$, converges monotonically to x^* .

Proof. Put $E = C(J, \mathbb{R})$. Then, from Lemma 3.1 it obeys that each compact chain in E possesses the compatibility property regarding the norm $\|\cdot\|$ and the order relation \leq in E .

Consider two operators \mathcal{A} and \mathcal{B} on E by

$$\mathcal{A}x(t) = x(t^{q-1})E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^{q-1}), \quad t \in J, \quad (3.4)$$

$$\mathcal{B}x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^{q-1}) f(s, x(s)) ds, \quad t \in J, \quad (3.5)$$

since the continuity of the integral and the hypotheses (A₁)-(A₅), it ensues that \mathcal{A} and \mathcal{B} define the maps $\mathcal{A}, \mathcal{B} : E \rightarrow \mathcal{H}$. Now by characterizations of the operators \mathcal{A} and \mathcal{B} , the FIE (1.1) is equivalent to the operator equation

$$\mathcal{A}x(t) + \mathcal{B}x(t) = x(t), \quad t \in J. \quad (3.6)$$

We intend to demonstrate that the operators \mathcal{A} and \mathcal{B} persuade all the conditions of Theorem 2.11. This is attained in the series of subsequent steps.

Step I: \mathcal{A} and \mathcal{B} are nondecreasing on E .

Let $x, y \in E$ be such that $x \leq y$. Then by hypothesis (A₃) and (A₄), we acquire

$$\begin{aligned} \mathcal{A}x(t) &= x(t^{q-1})E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^{q-1}) \\ &\leq y(t^{q-1})E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^{q-1}) = \mathcal{A}y(t), \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}x(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^{q-1}) f(s, x(s)) ds, \\ &\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^{q-1}) f(s, y(s)) ds, \\ &= \mathcal{B}y(t) \quad t \in J, \end{aligned}$$

for all $t \in J$. This illustrates that \mathcal{A} and \mathcal{B} are nondecreasing operators on E into E . Hence, \mathcal{A} and \mathcal{B} stand nondecreasing positive operators on E into itself.

Step II: \mathcal{A} is partially bounded and partially \mathcal{D} -Lipschitz on E .

Permit $x \in E$ be arbitrary. Then by (A₂),

$$|\mathcal{A}x(t)| \leq |x_n(t^{q-1})E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^{q-1})| \leq M,$$

for all $t \in J$. Captivating supremum over t , we get $\|\mathcal{A}x\| \leq M$ and consequently, \mathcal{A} is bounded. This added that \mathcal{A} is partially bounded on E . Now, let $x, y \in E$ be such that $x \leq y$. Therefore, by hypothesis,

$$\begin{aligned} |\mathcal{A}x(t) - \mathcal{A}y(t)| &= \\ |x(t^{q-1})E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^{q-1}) - y(t^{q-1})E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^{q-1})| \\ &\leq E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^{q-1}) |x(t^{q-1}) - y(t^{q-1})| \\ &\leq M(|x - y|), \end{aligned}$$

for all $t \in J$. Taking supremum over t , we attain

$$\|\mathcal{A}x - \mathcal{A}y\| \leq M(\|x - y\|)$$

for all $x, y \in E$ with $x \leq y$. Hence \mathcal{A} is partially nonlinear \mathcal{D} -Lipschitz operators on E which further means that it is also a partially continuous on E into itself.

Step III: \mathcal{B} is a partially continuous operator on E .

Suppose $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in a chain C of E such that $x_n \rightarrow x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{B}x_n(t) &= \lim_{n \rightarrow \infty} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^{q-1}) f(s, x_n(s)) ds, \\ &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^{q-1}) \left[\lim_{n \rightarrow \infty} f(s, x_n(s)) \right] ds \\ &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^{q-1}) f(s, x(s)) ds \\ &= \mathcal{B}x(t), \end{aligned}$$

for all $t \in J$. This indicates that $\mathcal{B}x_n$ converges monotonically to $\mathcal{B}x$ pointwise on J . Afterwards, we will express that $\{\mathcal{B}x_n\}_{n \in \mathbb{N}}$ is an equicontinuous sequence of functions in E . Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then

$$\begin{aligned} &| \mathcal{B}x(t_2) - \mathcal{B}x(t_1) | \\ &\leq \left| \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_2-s)^{q-1}) f(s, x(s)) ds \right. \\ &\quad \left. - \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_1-s)^{q-1}) f(s, x(s)) ds \right| \end{aligned}$$



$$\begin{aligned}
 &\leq \frac{1}{\Gamma(q)} \left| \int_0^{t_2} (t_2 - s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_2 - s)^{q-1}) f(s, x(s)) ds \right. \\
 &\quad \left. - \int_0^{t_1} (t_2 - s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) f(s, x(s)) ds \right| \\
 &+ \frac{1}{\Gamma(q)} \left| \int_0^{t_2} (t_2 - s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) f(s, x(s)) ds \right. \\
 &\quad \left. - \int_0^{t_1} (t_2 - s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) f(s, x(s)) ds \right| \\
 &+ \frac{1}{\Gamma(q)} \left| \int_0^{t_1} (t_2 - s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) f(s, x(s)) ds \right. \\
 &\quad \left. - \int_0^{t_1} (t_1 - s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) f(s, x(s)) ds \right| \\
 &\leq \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} \dots \\
 &\left| E_{\alpha,\beta}^{\gamma,\delta,q}((t_2 - s)^{q-1}) - E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) \right| |f(s, x(s))| ds \\
 &+ \frac{1}{\Gamma(q)} \left| \int_{t_1}^{t_2} (t_2 - s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) f(s, x(s)) ds \right| \\
 &+ \frac{1}{\Gamma(q)} \int_0^{t_1} |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}| \dots \\
 &\quad E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) |f(s, x(s))| ds \\
 &\leq \frac{1}{\Gamma(q)} \int_0^T (t_2 - s)^{q-1} \dots \\
 &\dots \left| E_{\alpha,\beta}^{\gamma,\delta,q}((t_2 - s)^{q-1}) - E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) \right| M_f ds \\
 &+ \frac{1}{\Gamma(q)} \int_0^T |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}| \dots \\
 &\dots E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) M_f ds \\
 &+ \frac{1}{\Gamma(q)} \int_0^T |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}| \dots \\
 &\dots E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) M_f ds \\
 &\leq \frac{M_f}{\Gamma(q)} \left(\int_0^T |(t_2 - s)^{q-1}|^2 ds \right)^{1/2} \dots \\
 &\left(\int_0^T \left| E_{\alpha,\beta}^{\gamma,\delta,q}((t_2 - s)^{q-1}) - E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) \right|^2 ds \right)^{1/2} \\
 &+ 2 \left(\int_0^T |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}|^2 ds \right)^{1/2} \dots \\
 &\quad \left(\int_0^T \left| E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) \right|^2 ds \right)^{1/2} \frac{M_f}{\Gamma(q)} \\
 &\quad \rightarrow 0 \quad \text{as } t_1 \rightarrow t_2,
 \end{aligned}$$

Meanwhile the functions $E_{\alpha,\beta}^{\gamma,\delta,q}$, q are continuous on compact interval J and interval is continuous on compact set $J \times J$, they

are uniformly continuous there. Therefore, from the above inequality it follows that

$$\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{B}x_n \rightarrow \mathcal{B}x$ is uniform and hence \mathcal{B} is partially continuous on E .

Step IV: \mathcal{B} is uniformly partially compact operator on E .

Consider C be an arbitrary chain in E . We demonstrate that $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set in E . Initially we express that $\mathcal{B}(C)$ is uniformly bounded. So let $y \in \mathcal{B}(C)$ be any element. Then there is an element $x \in C$ be such that $y = \mathcal{B}x$. Now, by hypothesis (A₁),

$$\begin{aligned}
 |y(t)| &\leq \left| \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t - s)^{q-1}) f(s, x(s)) ds \right| \\
 &\leq r
 \end{aligned}$$

for all $t \in J$. Selecting the supremum over t , we get $\|y\| \leq \|\mathcal{B}x\| \leq r$ for all $y \in \mathcal{B}(C)$. Hence, $\mathcal{B}(C)$ is a uniformly bounded subset of E . Moreover, $\|\mathcal{B}(C)\| \leq r$ for all chains C in E . Consequently, \mathcal{B} is a uniformly partially bounded operator on E .

Thereafter, we will express that $\mathcal{B}(C)$ is an equicontinuous set in E . Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then, for any $y \in \mathcal{B}(C)$, one has

$$\begin{aligned}
 &\left| \mathcal{B}x(t_2) - \mathcal{B}x(t_1) \right| \\
 &\leq \left| \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_2 - s)^{q-1}) f(s, x(s)) ds \right. \\
 &\quad \left. - \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) f(s, x(s)) ds \right| \\
 &\leq \frac{1}{\Gamma(q)} \left| \int_0^{t_2} (t_2 - s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_2 - s)^{q-1}) f(s, x(s)) ds \right. \\
 &\quad \left. - \int_0^{t_2} (t_2 - s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) f(s, x(s)) ds \right| \\
 &\quad + \frac{1}{\Gamma(q)} \left| \int_0^{t_2} (t_2 - s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) f(s, x(s)) ds \right. \\
 &\quad \left. - \int_0^{t_1} (t_2 - s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) f(s, x(s)) ds \right| \\
 &\quad + \frac{1}{\Gamma(q)} \left| \int_0^{t_1} (t_2 - s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) f(s, x(s)) ds \right. \\
 &\quad \left. - \int_0^{t_1} (t_1 - s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) f(s, x(s)) ds \right| \\
 &\leq \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} \dots \\
 &\left| E_{\alpha,\beta}^{\gamma,\delta,q}((t_2 - s)^{q-1}) - E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) \right| |f(s, x(s))| ds
 \end{aligned}$$



$$\begin{aligned}
 & + \frac{1}{\Gamma(q)} \left| \int_{t_1}^{t_2} (t_2 - s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) f(s, x(s)) ds \right| \\
 & + \frac{1}{\Gamma(q)} \int_0^{t_1} |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}| \dots \\
 & \quad E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) |f(s, x(s))| ds \\
 & \leq \frac{1}{\Gamma(q)} \int_0^T (t_2 - s)^{q-1} \dots \\
 & \quad \left| E_{\alpha,\beta}^{\gamma,\delta,q}((t_2 - s)^{q-1}) - E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) \right| M_f ds \\
 & + \frac{1}{\Gamma(q)} \int_0^T |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}| \dots \\
 & \quad E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) M_f ds \\
 & + \frac{1}{\Gamma(q)} \int_0^T |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}| \dots \\
 & \quad E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) M_f ds \\
 & \leq \frac{M_f}{\Gamma(q)} \left(\int_0^T |(t_2 - s)^{q-1}|^2 ds \right)^{1/2} \dots \\
 & \quad \left(\int_0^T \left| E_{\alpha,\beta}^{\gamma,\delta,q}((t_2 - s)^{q-1}) - E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) \right|^2 ds \right)^{1/2} \\
 & + 2 \left(\int_0^T |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}|^2 ds \right)^{1/2} \dots \\
 & \quad \left(\int_0^T \left| E_{\alpha,\beta}^{\gamma,\delta,q}((t_1 - s)^{q-1}) \right|^2 ds \right)^{1/2} \frac{M_f}{\Gamma(q)} \\
 & \rightarrow 0 \quad \text{as } t_1 \rightarrow t_2,
 \end{aligned}$$

uniformly for all $y \in \mathcal{B}(C)$. Thus $\mathcal{B}(C)$ is an equicontinuous subset of E . Now, $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set of functions in E , as a result it is compact. Accordingly, \mathcal{B} is a uniformly partially compact operator on E into itself.

Step V: v satisfies the operator inequality $v \leq \mathcal{A}v + \mathcal{B}v$.

By supposition (A₅), the FIE (1.1) consumes a lower solution v on J . Then, we possess

$$\begin{aligned}
 v(t) & \leq v(t^{q-1}) E_{\alpha,\beta}^{\gamma,\delta,q}((t - s)^{q-1}) \dots \\
 & \quad + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{(q-1)} E_{\alpha,\beta}^{\gamma,\delta,q}((t - s)^{q-1}) f(s, v(s)) ds,
 \end{aligned} \tag{3.7}$$

for all $t \in J$. Since the definitions of the operators \mathcal{A} , \mathcal{B} and \mathcal{C} it gives that $v(t) \leq \mathcal{A}v(t) + \mathcal{B}v(t)$ for all $t \in J$. Thus $v \leq \mathcal{A}v + \mathcal{B}v$.

Step VI: The \mathcal{D} -functions $\psi_{\mathcal{A}}$ meet the growth condition $M\psi_{\mathcal{A}}(r) < r$, for $r > 0$.

Lastly, the \mathcal{D} -function $\psi_{\mathcal{A}}$ of the operator \mathcal{A} meet the

inequality given in hypothesis (d) of Theorem 2.11, viz.,

$$M\psi_{\mathcal{A}}(r) < r$$

for all $r > 0$.

Hence \mathcal{A} and \mathcal{B} fulfills all the conditions of Theorem 2.11 and we conclude that the operator equation $\mathcal{A}x + \mathcal{B}x = x$ holds a solution. Therefore the FIE (1.1) has a solution x^* defined on J . Moreover, the sequence $\{x_n\}_{n \in \mathbb{N}}$ of successive approximations described by (3.3) converges monotonically to x^* . This completes the proof. \square

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