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Invariant solutions and conservation laws for a three-dimensional K-S equation

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Abstract

In this paper, we study three-dimensional Kudryashov-Sinelshchikov (K-S) equation, which describes long nonlinear pressure waves in a liquid containing gas bubbles. Firstly, We find the symmetry groups of the K-S equation. Secondly, using the symmetry groups, exact solutions which are invariant under a three-dimensional subalgebra of the symmetry Lie algebra are derived. Finally, by adding Bluman-Anco homotopy formula to the direct method local conservation laws of the K-S equation are obtained.

Keywords: Three-dimensional Kudryashov-Sinelshchikov equation, Lie symmetry analysis, Invariant solution, Conservation laws.

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1 Introduction

A liquid with gas bubbles has many applications in nature, technology and medicine. An extended equation for the description of nonlinear waves in a liquid with gas bubbles was introduced in [1]. Extended models of nonlinear waves in bubbly liquid were considered in [2]. In this study we consider the following equation

$$u_{tx} + u_x^2 + uu_{xx} - \lambda u_{xxx} + u_{xxxx} + \frac{1}{2}(u_{yy} + u_{zz}) = 0, \tag{1.1}$$

where λ is parameter. This equation was introduced by Kudryashov-Sinelshchikov in [3]. This nonlinear equation is for a description of long nonlinear pressure waves. By using Painlevé test, it is shown that the K-S equation is not Painlevé integrable. Bifurcations and phase portraits for the equation were discussed in [4].

To find solutions to nonlinear partial differential equations, the study of their symmetry groups is one of the powerful methods in the theory of nonlinear partial differential equations. Then, the corresponding symmetry groups will be used in construction of exact solutions and mapping solutions to other solutions.

In the study of partial differential equations, the concept of a conservation law plays a very important role in the analyze of essential properties of the solutions, particularly, investigation of existence, uniqueness and stability of the solutions.

This work is organized as follows. In Section 2, we present group classification of the K-S equation. Section 3 is devoted to reductions to ordinary differential equations and exact solutions. In Section 4, the conservation laws associated to the equation are obtained via direct method. The conclusions are presented in Section 5.

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2 Group classification of the K-S equation

In this section we completely classify the Lie point symmetries of the K-S equation in terms of λ. For the non-extended transformations group of equation (1.1) the infinitesimal generator X is given by

$$X = \xi^t(t,x,y,z,u)\partial t + \xi^x(t,x,y,z,u)\partial x + \xi^y(t,x,y,z,u)\partial y + \xi^z(t,x,y,z,u)\partial z + \eta(t,x,y,z,u)\partial u. \tag{2.2}$$

The fourth prolongation of X is

$$X^{(4)} = X + \eta_i^{(1)}\partial u_i + \cdots + \eta^{(4)}_{i_1 i_2 i_3 i_4}\partial u_{i_1 i_2 i_3 i_4}, \tag{2.3}$$

where

$$\eta_i^{(1)} = D_i\eta - (D_i\xi_j)u_j, \qquad i,j = 1,...,4 \tag{2.4}$$

and for $l = 1,2,\cdots,k$ with $k \geq 2$, $i_l = 1,2,\cdots,4$

$$\eta^{(k)}_{i_1 i_2\cdots i_k} = D_{i_k}\eta^{(k-1)}_{i_1 i_2\cdots i_{k-1}} - (D_{i_k}\xi_j)u_{i_1 i_2\cdots i_{k-1} j}, \tag{2.5}$$

where D_i is the total derivative operator defined by

$$D_i = \partial x_i + u_i\partial u + u_{ij}\partial u_j + ..., \qquad i = 1,...,4 \tag{2.6}$$

with summation over a repeated index.

The vector field X generates a one parameter symmetry group of K-S equation if and only if

$$\begin{aligned}&\left(X^{(4)}[u_{tx} + u_x^2 + uu_{xx} - \lambda u_{xxx} + u_{xxxx} + \frac{1}{2}(u_{yy} + u_{zz})]\right)\Big|_{(1.1)} = \\ &\left(\eta u_{xx} + 2u_x\eta_x^{(1)} + \eta_{tx}^{(2)} + u\eta_{xx}^{(2)} + \frac{1}{2}(\eta_{yy}^{(2)} + \eta_{zz}^{(2)}) - \lambda\eta_{xxx}^{(3)} + \eta_{xxxx}^{(4)}\right)\Big|_{(1.1)} = 0.\end{aligned} \tag{2.7}$$

For more details see [5], [6].

Calculating the needed terms in (2.7) and spliting with respect to partial derivatives with respect to t, x, y, and z and various power of u, we can find the determining equations for the symmetry group of the equation (1.1). We study two cases: $\lambda = 0, \lambda \neq 0$.

Case A. $\lambda \neq 0$

Here, we find the following determining equations:

$$\begin{aligned}&\xi^t_t = \xi^t_x = \xi^t_y = \xi^t_z = \xi^t_u = \xi^x_x = \xi^x_{yy} = \xi^x_{zy} = \xi^x_{zz} = \xi^x_u = \xi^y_x = \xi^y_y = \xi^y_{zz} = \xi^y_u = 0,\\ &\xi^z_x = \xi^z_z = \xi^z_u = \eta_x = \eta_{yy} = \eta_{zy} = \eta_{zz} = \eta_u = 0, \xi^x_t = \eta, \xi^y_t = -\xi^x_y, \xi^z_t = -\xi^x_z, \xi^z_y = -\xi^y_z.\end{aligned} \tag{2.8}$$

So we have

$$\begin{aligned}&\xi^t = c_1, \quad \xi^x = -f_1'(t)y - f_2'(t)z + f_3(t), \quad \xi^y = f_1(t) + c_2 z,\\ &\xi^z = f_2(t) - c_2 y, \quad \eta = -f_1''(t)y - f_2''(t)z + f_3'(t),\end{aligned} \tag{2.9}$$

with $f_1(t), f_2(t), f_3(t)$ arbitrary functions and c_1, c_2 arbitrary constants. Thus the K-S equation admits an infinite-dimensional symmetry Lie algebra spanned by

$$\begin{aligned}&X_1 = \partial t, \quad X_2 = -y\partial z + z\partial y, \quad X_\infty = f_3'(t)\partial u + f_3(t)\partial x,\\ &X_\infty = -yf_1''(t)\partial u - yf_1'(t)\partial x + f_1(t)\partial y, \quad X_\infty = -zf_2''(t)\partial u - zf_2'(t)\partial x + f_2(t)\partial z,\end{aligned} \tag{2.10}$$

where $f_1(t), f_2(t), f_3(t)$ are arbitrary functions.

Case B. $\lambda = 0$

Here, we find the following determining equations:

$$\xi^t_x = \xi^t_y = \xi^t_z = \xi^t_u = \xi^x_{yy} = \xi^x_{zy} = \xi^x_{zz} = \xi^x_u = 0,$$
$$\xi^y_x = \xi^y_{zz} = \xi^y_u = \xi^z_x = \xi^z_u = \eta_x = \eta_{tu} = \eta_{yy} = \eta_{yu} = \eta_{zy} = \eta_{zz} = \eta_{zu} = \eta_{uu} = 0,$$
$$\xi^t_t = -\frac{3}{2}\eta_u, \xi^x_t = -\eta_u u + \eta, \xi^x_x = -\frac{1}{2}\eta_u, \xi^y_t = -\xi^x_y, \xi^y_y = \xi^z_z = -\eta_u, \xi^z_t = -\xi^x_z, \xi^z_y = -\xi^y_z. \tag{2.11}$$

So we have

$$\xi^t = c_1 t + c_2, \quad \xi^x = \frac{c_1}{3}x - f_1'(t)y - f_2'(t)z + f_3(t), \quad \xi^y = \frac{2c_1}{3}y + c_3 z + f_1(t),$$
$$\xi^z = -c_3 y + \frac{2c_1}{3}z + f_2(t), \quad \eta = -f_1''(t)y - f_2''(t)z - \frac{2c_1}{3}u + f_3'(t), \tag{2.12}$$

with $f_1(t), f_2(t), f_3(t)$ arbitrary functions and c_1, c_2, c_3 arbitrary constants. Thus the K-S equation admits an infinite-dimensional symmetry Lie algebra spanned by

$$X_1 = \partial t, \quad X_\infty = -zf_2''(t)\partial u - zf_2'(t)\partial x + f_2(t)\partial z, \quad X_\infty = u\partial u - \frac{3t}{2}\partial t - \frac{x}{2}\partial x - y\partial y - z\partial z,$$
$$X_2 = -y\partial z + z\partial y, \quad X_\infty = f_3'(t)\partial u + f_3(t)\partial x, \quad X_\infty = -yf_1''(t)\partial u - yf_1'(t)\partial x + f_1(t)\partial y, \tag{2.13}$$

where $f_1(t), f_2(t), f_3(t)$ are arbitrary functions.

3 Invariant solutions

Here, we use the results of the group classification in the previous section for the construction of exact solutions of the K-S equation. We search for solutions invariant under a three-dimensional subalgebra of the Lie algebra (2.13). Then equation (1.1) is reduced to a fourth-order ordinary differential equation. Solving this equation we find exact solution for the K-S equation[6, 7, 8, 9]. We choose the following three vector fields:

$$X_1 = 2y\partial u + 2ty\partial x - t^2\partial y, \quad X_2 = 2z\partial u + 2tz\partial x - t^2\partial z,$$
$$X_3 = u\partial u - \frac{3t}{2}\partial t - \frac{x}{2}\partial x - y\partial y - z\partial z. \tag{3.14}$$

These vector fields generate a three-dimensional subalgebra of the symmetry Lie algebra (2.13). We construct an exact solution of equation (1) which is invariant under these three vector fields: $X_1(I) = X_2(I) = X_3(I) = 0$. From $X_1(I) = 0$, we obtain four invariants $J_1 = t, J_2 = z, J_3 = u - x/t, J_4 = y^2 + tx$. Now, we rewrite X_2 and X_3 in terms of J_1,J_2,J_3 and J_4:

$$X_2 = -J_1^2\partial J_2 + 2J_1^2 J_2\partial J_4, \quad X_3 = -\frac{3}{2}J_1\partial J_1 - J_2\partial J_2 + J_3\partial J_3 - 2J_4\partial J_4. \tag{3.15}$$

Since the common solution $I(t, x, y, z, u)$ is defined as a function of the invariants J_1,J_2,J_3 and J_4 of X_1, it must be a solution to the differential equations

$$X_2(I) = -J_1^2\frac{\partial I}{\partial J_2} + 2J_1^2 J_2\frac{\partial I}{\partial J_4} = 0, \quad X_3(I) = -\frac{3}{2}J_1\frac{\partial I}{\partial J_1} - J_2\frac{\partial I}{\partial J_2} - J_3\frac{\partial I}{\partial J_3} - 2J_4\frac{\partial I}{\partial J_4} = 0. \tag{3.16}$$

The equation $X_2(I) = 0$ gives the three invariants $K_1 = J_1, K_2 = J_3, K_3 = J_4 + J_2^2$. Again we express these invariants as new variables. Writing X_3 in terms of K_1, K_2, and K_3, we obtain

$$X_3 = -\frac{3}{2}K_1\partial K_1 + K_2\partial K_2 - 2K_3\partial K_3. \tag{3.17}$$

From $X_3(I) = 0$ two invariants $I_1 = K_1^{2/3}K_2, I_2 = K_1^{-4/3}K_3$ are found.

The invariant solution is given by $I_1 = \Phi(I_2)$, where Φ is a function to be determined [7], [8]. Thus

$$t^{\frac{2}{3}}(u - \frac{x}{t}) = \Phi(\frac{y^2 + tx + z^2}{t^{\frac{4}{3}}}). \tag{3.18}$$

From (3.18) we have

$$u = t^{\frac{-2}{3}}\Phi(\delta) + \frac{x}{t}, \tag{3.19}$$

where $\delta = \dfrac{y^2 + tx + z^2}{t^{4/3}}$. Substituting u in the K-S equation (with $\lambda = 0$) we obtain

$$\Phi'''' + \Phi''(\Phi + \frac{2}{3}\delta) + \Phi'^2 + 3\Phi' = 0. \tag{3.20}$$

A solution which arises from the above equation is

$$\Phi = -3\delta = -3(\frac{y^2 + tx + z^2}{t^{\frac{4}{3}}}). \tag{3.21}$$

Therefore the exact solution

$$u = \frac{-3(y^2 + z^2 + \frac{2}{3}tx)}{t^2}, \tag{3.22}$$

for the K-S equation is obtained.

4 Conservation laws

There are many methods to investigate conservation laws, such as Noether's method, the direct method, etc. Here, we present the direct method [10, 11, 12, 13].

Consider a differential equation $P\{x; u\}$ of order k with n independent variables $x = (x^1, \ldots, x^n)$ and one dependent variable u, given by

$$P[u] = P(x, u, \partial u, \ldots, \partial^k u) = 0. \tag{4.23}$$

A multiplier $\Lambda(x, u, \partial u, \ldots, \partial^l u)$ provides a conservation law $\Lambda[u]P[u] = D_i\phi^i[u] = 0$ for the differential equation $P\{x; u\}$ if and only if

$$E_U\Big(\Lambda(x, U, \partial U, \ldots, \partial^l U)P(x, U, \partial U, \ldots, \partial^k U)\Big) \equiv 0, \tag{4.24}$$

for arbitrary functions $U(x)$, where E_U is the Euler operator with respect to U defined as

$$E_U = \partial U - D_i\partial U + \ldots + (-1)^s D_{i_1} \ldots D_{i_s}\partial U_{i_1 \ldots i_s}. \tag{4.25}$$

Since the K-S equation is of *Cauchy-Kovalevskaya form* with respect to x, y, and z, it follows that multipliers providing local conservation laws for equation (1.1) are in the form $\Lambda = \xi(t, x, y, z, U, \partial_t U, \ldots, \partial_t^l U), l = 1, 2, \ldots$ and we can obtain all of its nontrivial local conservation laws from multipliers. Consequently, $\Lambda = \xi(t, x, y, z, U, \partial_t U, \ldots, \partial_t^l U)$ is a conservation law multiplier for the equation(1.1) if and only if

$$E_U\Big[\xi\big(t, x, y, z, U, \partial_t U, \ldots, \partial_t^l U\big)\big(U_{tx} + U_x^2 + UU_{xx} - \lambda U_{xxx} + U_{xxxx} + \frac{1}{2}(U_{yy} + U_{zz})\big)\Big] \equiv 0 \tag{4.26}$$

for an arbitrary function $U(t, x, y, z)$.

We look for all multipliers in the form $\Lambda = \xi(t, x, y, z, U, \partial U_t, \partial U_{tt}, \partial U_{ttt}, \partial U_{tttt})$ for the equation(1). Thus, the Euler operator is taken to be

$$E_U = \partial U - D_i\partial U_i + \ldots + (-1)^4 D_{i_1} \ldots D_{i_4}\partial U_{i_1 \ldots i_4}, \tag{4.27}$$

and the determining equations become

$$E_U[\xi(t, x, y, z, U, \partial U_t, \ldots, \partial U_{tttt})(U_{tx} + U_x^2 + UU_{xx} - \lambda U_{xxx} + U_{xxxx} + \frac{1}{2}(U_{yy} + U_{zz}))] \equiv 0 \tag{4.28}$$

where $U(t, x, y, z)$ is arbitrary function. Equation (4.28) split with respect to $U_x, U_{tx}, \ldots, U_{xxxx}$ to provide the over-determined equations:

$$\xi_{yyyy} = -(2\xi_{zyyz} + \xi_{zzzz}), \quad \xi_{yxy} = -\xi_{zxz}, \quad \xi_{tx} = -\frac{\xi_{yy} + \xi_{zz}}{2}, \quad \xi_{xx} = \xi_U = \xi_{U_t} = \xi_{U_{tt}} = \xi_{U_{ttt}} = \xi_{U_{tttt}} = 0. \tag{4.29}$$

Solving the equations (4.29), we find the infinite set of local multipliers

$$\xi = (f_1(t,z-iy) + f_2(t,z+iy))x + f_3(t,z-iy) + f_4(t,z+iy) -$$
$$2\int^y \int^b (D_1(f_1)(t,-2ib+iy+z) + D_1(f_2)(t,2ia-2ib+iy+z))dadb, \tag{4.30}$$

where f_1, f_2, f_3 and f_4 are arbitrary functions. We study two cases: $f_1(r,s) = f_2(r,s) = f_3(r,s) = f_4(r,s) = r+s$ and $f_1(r,s) = f_2(r,s) = f_3(r,s) = f_4(r,s) = \exp(r+s)$.

Case A

By setting $f_1(r,s) = f_2(r,s) = f_3(r,s) = f_4(r,s) = r+s$ into (4.30), we have $\xi = 2(t+z)(x+1) - 2y^2$. Applying Bluman-Anco homotopy formula [10, 11, 12], we find conserved components Φ^t, Φ^x, Φ^y, and Φ^z with respect to multiplier ξ:

$$\begin{aligned}
\Phi^t &= 2\Big[(t+z)(x+1) - y^2\Big]u_x,\\
\Phi^x &= 3\Big[(t+z)(x+1) - y^2\Big]uu_x - \Big[(t+z)u + ((t+z)(x+1) - y^2)u_x\Big]u - 2\Big[x+1\Big]u -\\
&\quad 2\Big[(t+z)(x+1) - y^2\Big]\lambda u_{xx} + 2\Big[(t+z)(x+1) - y^2\Big]u_{xxx} + 2\Big[t+z\Big]\lambda u_x - 2\Big[t+z\Big]u_{xx},\\
\Phi^y &= 2\Big[y\Big]u + \Big[(t+z)(x+1) - y^2\Big]u_y,\\
\Phi^z &= -\Big[x+1\Big]u + \Big[(t+z)(x+1) - y^2\Big]u_z.
\end{aligned} \tag{4.31}$$

So we obtain the following local conservation law of the K-S equation:

$$\begin{aligned}
&D_t\Big(2[(t+z)(x+1) - y^2]u_x\Big) + D_x\Big(3[(t+z)(x+1) - y^2]uu_x - 2[(t+z)(x+1) - y^2]\lambda u_{xx} -\\
&[(t+z)u + ((t+z)(x+1) - y^2)u_x]u - 2[x+1]u + 2[(t+z)(x+1) - y^2]u_{xxx} +\\
&2[t+z]\lambda u_x - 2[t+z]u_{xx}\Big) + D_y\Big(2[y]u + [(t+z)(x+1) - y^2]u_y\Big) +\\
&D_z\Big(-[x+1]u + [(t+z)(x+1) - y^2]u_z\Big) = 0.
\end{aligned} \tag{4.32}$$

Case B

By setting $f_1(r,s) = f_2(r,s) = f_3(r,s) = f_4(r,s) = \exp(r+s)$ into (4.30), we have:

$$\xi = (x - iy + \frac{1}{2})\exp(t+z-iy) + (x+iy+1)\exp(t+z+iy), \tag{4.33}$$

Applying Bluman-Anco homotopy formula, we find conserved components Φ^t, Φ^x, Φ^y, and Φ^z with respect to multiplier ξ:

$$\begin{aligned}
\Phi^t &= \Big[(x-iy+\frac{1}{2})\exp(t+z-iy)+(x+iy+1)\exp(t+z+iy)\Big]u_x,\\
\Phi^x &= \Big[(x-iy+\frac{1}{2})\exp(t+z-iy)+(x+iy+1)\exp(t+z+iy)\Big]uu_x -\\
&\quad \frac{1}{2}\Big[\exp(t+z-iy)+\exp(t+z+iy)\Big]u^2+\lambda\Big[\exp(t+z-iy)+\exp(t+z+iy)\Big]u_x -\\
&\quad \Big[(x-iy+\frac{1}{2})\exp(t+z-iy)+(x+iy+1)\exp(t+z+iy)\Big]u -\\
&\quad \Big[(\lambda x-\lambda iy+\frac{\lambda}{2}+1)\exp(t+z-iy)+(\lambda x+\lambda iy+\lambda+1)\exp(t+z+iy)\Big]u_{xx} +\\
&\quad \Big[(x-iy+\frac{1}{2})\exp(t+z-iy)+(x+iy+1)\exp(t+z+iy)\Big]u_{xxx},\\
\Phi^y &= \frac{i}{2}\Big[(x-iy+\frac{3}{2})\exp(t+z-iy)-(x+iy+2)\exp(t+z+iy)\Big]u +\\
&\quad \frac{1}{2}\Big[(x-iy+\frac{1}{2})\exp(t+z-iy)+(x+iy+1)\exp(t+z+iy)\Big]u_y,\\
\Phi^z &= -\frac{1}{2}\Big[(x-iy+\frac{1}{2})\exp(t+z-iy)+(x+iy+1)\exp(t+z+iy)\Big]u +\\
&\quad \frac{1}{2}\Big[(x-iy+\frac{1}{2})\exp(t+z-iy)+(x+iy+1)\exp(t+z+iy)\Big]u_z.
\end{aligned} \tag{4.34}$$

So we find the following local conservation law of the equation (1.1):

$$D_t\Phi^t + D_x\Phi^x + D_y\Phi^y + D_z\Phi^z = 0. \tag{4.35}$$

5 Conclusions

In the present paper, we investigated the Lie point symmetries, exact solutions and conservation laws of the K-S equation. We derived exact solutions which are invariant under a three-dimensional subalgebra of the symmetry Lie algebra. We obtained the conservation laws of the K-S equation by adding Bluman-Anco homotopy formula to the direct method.

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