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Some results for the Jacobi-Dunkl transform in the space $L^2(\mathbb{R}, A_{\alpha,\beta}(t)dt)$

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Abstract

In this paper, using a generalized Jacobi-Dunkl translation operator, we prove an analog of Titchmarsh's theorem for functions satisfying the Jacobi-Dunkl Lipschitz condition in $L^2(\mathbb{R}, A_{\alpha,\beta}(t)dt)$, $\alpha \geq \beta \geq \frac{-1}{2}$, $\alpha \neq \frac{-1}{2}$.

Keywords: Jacobi-Dunkl operator, Jacobi-Dunkl transform, generalized Jacobi-Dunkl translation.

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1 Introduction

Titchmarsh's [[10], Theorem 85] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have:

Theorem 1.1. *[10] Let $\alpha \in (0,1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following statements are equivalents:*
(a) $\|f(t+h)-f(t)\| = 0(h^\alpha), \quad as \quad h \to 0,$
(b) $\displaystyle\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 d\lambda = O(r^{-2\alpha}), \quad as \quad r \to \infty,$
where $\widehat{f}$ stands for the Fourier transform of f.

In this paper, we prove in analog of Theorem 1.1 for the Jacobi-Dunkl transform for functions satisfying the Jacobi-Dunkl Lipschitz condition in the space $L^2(\mathbb{R}, A_{\alpha,\beta}(t)dt)$. For this purpose, we use the generalized translation operator. Similar results have been established in the context of non compact rank one Riemannian symetric spaces [9].
In section 2 below, we recapitulate from [[1],[2],[3],[5]] some results related to the harmonic analysis associated with Jacobi-Dunkl operator $\Lambda_{\alpha,\beta}$. Section 3 is devoted to the main result after defining the class $Lip(\delta, 2, \alpha, \beta)$ of functions in $L^2_{\alpha,\beta}(\mathbb{R})$ satisfying the Lipschitz condition correspondent to the generalized Jacobi-Dunkl translation.

2 Notation and Preliminaries

The Jacobi-Dunkl function with parameters (α, β), $\alpha \geq \beta \geq \frac{-1}{2}$, $\alpha \neq \frac{-1}{2}$, is defined by the formula:

$$\forall x \in \mathbb{R}, \psi_\lambda^{\alpha,\beta}(x) = \begin{cases} \varphi_\mu^{\alpha,\beta}(x) - \frac{i}{\lambda}\frac{d}{dx}\varphi_\mu^{\alpha,\beta}(x) & \text{if } \lambda \in \mathbb{C}\backslash\{0\}, \\ 1 & \text{if } \lambda = 0, \end{cases}$$

with $\lambda^2 = \mu^2 + \rho^2$, $\rho = \alpha + \beta + 1$ and $\varphi_\mu^{\alpha,\beta}$ is the Jacobi function given by:

$$\varphi_\mu^{\alpha,\beta}(x) = F\left(\frac{\rho + i\mu}{2}, \frac{\rho - i\mu}{2}, \alpha + 1, -(\sinh x)^2\right),$$

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where F is the Gausse hypergeometric function (see [1],[6] and [7]).
$\psi_\lambda^{\alpha,\beta}$ is the unique C^∞-solution on $\mathbb{R}$ of the differentiel-difference equation

$$\begin{cases} \Lambda_{\alpha,\beta}\mathcal{U} = i\lambda\mathcal{U}, \quad \lambda \in \mathbb{C}, \\ \mathcal{U}(0) = 1, \end{cases}$$

where $\Lambda_{\alpha,\beta}$ is the Jacobi-Dunkl operator given by:

$$\Lambda_{\alpha,\beta}\mathcal{U}(x) = \frac{d\mathcal{U}(x)}{dx} + [(2\alpha+1)\coth x + (2\beta+1)\tanh x] \times \frac{\mathcal{U}(x)-\mathcal{U}(-x)}{2}].$$

The operator $\Lambda_{\alpha,\beta}$ is a particular case of the operator D given by

$$D\mathcal{U}(x) = \frac{d\mathcal{U}(x)}{dx} + \frac{A'(x)}{A(x)} \times \left(\frac{\mathcal{U}(x)-\mathcal{U}(-x)}{2}\right),$$

where $A(x) = |x|^{2\alpha+1}B(x)$, and B a function of class C^∞ on $\mathbb{R}$, even and positive. The operator $\Lambda_{\alpha,\beta}$ corresponds to the function

$$A(x) = A_{\alpha,\beta}(x) = 2^\rho(\sinh|x|)^{2\alpha+1}(\cosh|x|)^{2\beta+1}.$$

Using the relation

$$\frac{d}{dx}\varphi_\mu^{\alpha,\beta}(x) = -\frac{\mu^2+\rho^2}{4(\alpha+1)}\sinh(2x)\varphi_\mu^{\alpha+1,\beta+1}(x),$$

the function $\psi_\lambda^{\alpha,\beta}$ can be written in the form below (see [2])

$$\psi_\lambda^{\alpha,\beta}(x) = \varphi_\mu^{\alpha,\beta}(x) + i\frac{\lambda}{4(\alpha+1)}\sinh(2x)\varphi_\mu^{\alpha+1,\beta+1}(x), \quad x \in \mathbb{R},$$

where $\lambda^2 = \mu^2 + \rho^2, \quad \rho = \alpha+\beta+1.$
Denote $L^2_{\alpha,\beta}(\mathbb{R}) = L^2_{\alpha,\beta}(\mathbb{R}, A_{\alpha,\beta}(t)dt)$ the space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{L^2_{\alpha,\beta}(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(t)|^2 A_{\alpha,\beta}(t)dt\right)^{1/2} < +\infty.$$

Using the eigenfunctions $\psi_\lambda^{\alpha,\beta}$ of the operator $\Lambda_{\alpha,\beta}$ called the Jacobi-Dunkl kernels, we define the Jacobi-Dunkl transform of a function $f \in L^2_{\alpha,\beta}(\mathbb{R})$ by:

$$\mathcal{F}_{\alpha,\beta}f(\lambda) = \int_{\mathbb{R}} f(t)\psi_\lambda^{\alpha,\beta}(t)A_{\alpha,\beta}(t)dt, \quad \lambda \in \mathbb{R},$$

and the inversion formula

$$f(t) = \int_{\mathbb{R}} \mathcal{F}_{\alpha,\beta}f(\lambda)\psi_{-\lambda}^{\alpha,\beta}(t)d\sigma(\lambda),$$

where

$$d\sigma(\lambda) = \frac{|\lambda|}{8\pi\sqrt{\lambda^2-\rho^2}|C_{\alpha,\beta}(\sqrt{\lambda^2-\rho^2})|}\mathbb{I}_{\mathbb{R}\backslash]-\rho,\rho[}(\lambda)d\lambda.$$

Here,

$$C_{\alpha,\beta}(\mu) = \frac{2^{\rho-i\mu}\Gamma(\alpha+1)\Gamma(i\mu)}{\Gamma(\frac{1}{2}(\rho+i\mu))\Gamma(\frac{1}{2}(\alpha-\beta+1+i\mu))}, \quad \mu \in \mathbb{C}\backslash(i\mathbb{N}),$$

and $\mathbb{I}_{\mathbb{R}\backslash]-\rho,\rho[}$ is the characteristic function of $\mathbb{R}\backslash]-\rho,\rho[$.
Denote $L^2_\sigma(\mathbb{R}) = L^2(\mathbb{R}, d\sigma(\lambda))$.
The Jacobi-Dunkl transform is a unitary isomorphism from $L^2_{\alpha,\beta}(\mathbb{R})$ onto $L^2_\sigma(\mathbb{R})$, i.e.,

$$\|f\| := \|f\|_{L^2_{\alpha,\beta}(\mathbb{R})} = \|\mathcal{F}_{\alpha,\beta}(f)\|_{L^2_\sigma(\mathbb{R})}. \tag{2.1}$$

The operator of Jacobi-Dunkl translation is defined by:

$$T_x f(y) = \int_{\mathbb{R}} f(z) d\nu_{x,y}^{\alpha,\beta}(z), \quad \forall x, y \in \mathbb{R}, \tag{2.2}$$

where $\nu_{x,y}^{\alpha,\beta}(z), \quad x, y \in \mathbb{R}$, are the signed measures given by

$$d\nu_{x,y}^{\alpha,\beta}(z) = \begin{cases} K_{\alpha,\beta}(x,y,z) A_{\alpha,\beta}(z) dz & \text{if } x, y \in \mathbb{R}^*, \\ \delta_x & \text{if } y = 0, \\ \delta_y & \text{if } x = 0, \end{cases}$$

here, δ_x is the Dirac measure at x. And,

$$\begin{aligned} K_{\alpha,\beta}(x,y,z) &= M_{\alpha,\beta}(\sinh(|x|)\sinh(|y|)\sinh(|z|))^{-2\alpha} \mathbb{I}_{I_{x,y}} \times \int_0^\pi \rho_\theta(x,y,z) \\ &\times (g_\theta(x,y,z))_+^{\alpha-\beta-1} \sin^{2\beta}\theta d\theta, \\ I_{x,y} &= [-|x|-|y|, -||x|-|y||] \cup [||x|-|y||, |x|+|y|], \\ \rho_\theta(x,y,z) &= 1 - \sigma_{x,y,z}^\theta + \sigma_{z,x,y}^\theta + \sigma_{z,y,x}^\theta, \end{aligned}$$

$$\forall z \in \mathbb{R}, \theta \in [0,\pi], \sigma_{x,y,z}^\theta = \begin{cases} \frac{\cosh x + \cosh y - \cosh z \cos\theta}{\sinh x \sinh y}, & \text{if } xy \neq 0, \\ 0, & \text{if } xy = 0, \end{cases}$$

$$g_\theta(x,y,z) = 1 - \cosh^2 x - \cosh^2 y - \cosh^2 z + 2\cosh x \cosh y \cosh z \cos\theta,$$

$$t_+ = \begin{cases} t, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0, \end{cases}$$

and,

$$M_{\alpha,\beta} = \begin{cases} \frac{2^{-2\rho}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} & \text{if } \alpha > \beta, \\ 0 & \text{if } \alpha = \beta. \end{cases}$$

In [2], we have

$$\mathcal{F}_{\alpha,\beta}(T_h f)(\lambda) = \psi_\lambda^{\alpha,\beta}(h) \mathcal{F}_{\alpha,\beta}(f)(\lambda), \quad \lambda, h \in \mathbb{R}. \tag{2.3}$$

For $\alpha \geq \frac{-1}{2}$, we introduce the Bessel normalized function of the first kind defined by:

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n!\Gamma(n+\alpha+1)}, \quad z \in \mathbb{C}.$$

Moreover, we see that

$$\lim_{z \to 0} \frac{j_\alpha(z) - 1}{z^2} \neq 0,$$

by consequence, there exists $C_1 > 0$ and $\eta > 0$ satisfying

$$|z| \leq \eta \Rightarrow |j_\alpha(z) - 1| \geq C_1 |z|^2. \tag{2.4}$$

Lemma 2.1. *The following inequalities are valids for Jacobi functions* $\varphi_\mu^{\alpha,\beta}(t)$*:*
(c) $|\varphi_\mu^{\alpha,\beta}(t)| \leq 1$,
(d) $|1 - \varphi_\mu^{\alpha,\beta}(t)| \leq t^2(\mu^2 + \rho^2)$.
Proof. *(See [8], Lemma 3.1, Lemma 3.2).*

Lemma 2.2. *Let* $\alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$*. Then for* $|\nu| \leq \rho$*, there exists a positive constant* C_2 *such that*

$$|1 - \varphi_{\mu+i\nu}^{\alpha,\beta}(t)| \geq C_2 |1 - j_\alpha(\mu t)|.$$

Proof. *(See [4], Lemma 9).*

3 Main results

In this section we introduce and prove an analog of Theorem 1.1. Firstly we have to define, for functions in $L^2_{\alpha,\beta}(\mathbb{R})$, the conditions of Cauchy-Lipschitz related to the Jacobi-Dunkl translation operator given in 2.2.

Definition 3.1. *Let $\delta \in (0,1)$. A function $f \in L^2_{\alpha,\beta}(\mathbb{R})$ is said to be in the Jacobi-Dunkl-Lipschitz class, denoted by $Lip(\delta, 2, \alpha, \beta)$, if*

$$\|N_h \Lambda^m_{\alpha,\beta} f\| = O(h^\delta), \quad as \quad h \to 0,$$

where $N_h = T_h + T_{-h} - 2I$, I is the unit operator in the space $L^2_{\alpha,\beta}(\mathbb{R})$ and $m = 0, 1, 2, ...$

Lemma 3.3. *For $f \in L^2_{\alpha,\beta}(\mathbb{R})$, then*

$$\|N_h \Lambda^m_{\alpha,\beta} f\|^2 = 4 \int_{\mathbb{R}} \lambda^{2m} |\varphi^{\alpha,\beta}_\mu(h) - 1|^2 |\mathcal{F}_{\alpha,\beta} f(\lambda)|^2 d\sigma(\lambda).$$

Proof. Since $\mathcal{F}_{\alpha,\beta}(\Lambda_{\alpha,\beta} f)(\lambda) = i\lambda \mathcal{F}_{\alpha,\beta}(f)(\lambda)$, we have

$$\mathcal{F}_{\alpha,\beta}(\Lambda^m_{\alpha,\beta} f)(\lambda) = i^m \lambda^m \mathcal{F}_{\alpha,\beta}(f)(\lambda), \quad m = 0, 1, 2, ... \tag{3.5}$$

We us formulas 2.3 and 3.5, we conclude that

$$\mathcal{F}_{\alpha,\beta}(N_h \Lambda^m_{\alpha,\beta} f)(\lambda) = i^m (\psi^{\alpha,\beta}_\lambda(h) + \psi^{\alpha,\beta}_\lambda(-h) - 2) \lambda^m \mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

Since

$$\psi^{\alpha,\beta}_\lambda(h) = \varphi^{\alpha,\beta}_\mu(h) + i \frac{\lambda}{4(\alpha+1)} \sinh(2h) \varphi^{\alpha+1,\beta+1}_\mu(h),$$

$$\psi^{\alpha,\beta}_\lambda(-h) = \varphi^{\alpha,\beta}_\mu(-h) - i \frac{\lambda}{4(\alpha+1)} \sinh(2h) \varphi^{\alpha+1,\beta+1}_\mu(-h),$$

and $\varphi^{\alpha,\beta}_\mu$ is even (see [2]), then

$$\mathcal{F}_{\alpha,\beta}(N_h \Lambda^m_{\alpha,\beta} f)(\lambda) = 2i^m (\varphi^{\alpha,\beta}_\mu(h) - 1) \lambda^m \mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

Now by Parseval's identity (formula 2.1), we have the result.

Theorem 3.1. *Let $f \in L^2_{\alpha,\beta}(\mathbb{R})$. Then the following statements are equivalents:*

(i) $f \in Lip(\delta, 2, \alpha, \beta)$,

(ii) $\displaystyle\int_{|\lambda| \geq r} \lambda^{2m} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = O(r^{-2\delta}), \quad as \quad r \to \infty.$

Proof. $(i) \Rightarrow (ii)$. Assume that $f \in Lip(\delta, 2, \alpha, \beta)$, then we have

$$\|N_h \Lambda^m_{\alpha,\beta} f\| = O(h^\delta), \quad as \quad h \to 0.$$

From Lemma 3.3, we have

$$\|N_h \Lambda^m_{\alpha,\beta} f\|^2 = 4 \int_{\mathbb{R}} \lambda^{2m} |1 - \varphi^{\alpha,\beta}_\mu(h)|^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).$$

By 2.4 and Lemma 2.2, we get:

$$\int_{\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}} \lambda^{2m} |1 - \varphi^{\alpha,\beta}_\mu(h)|^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \geq C_1^2 C_2^2 \int_{\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}} |\mu h|^4 |\lambda^{2m} \mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).$$

From $\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}$ we have

$$\left(\frac{\eta}{2h}\right)^2 - \rho^2 \leq \mu^2 \leq \left(\frac{\eta}{h}\right)^2 - \rho^2$$
$$\Rightarrow \mu^2 h^2 \geq \frac{\eta^2}{4} - \rho^2 h^2.$$

Take $h \leq \frac{\eta}{3\rho}$, then we have $\mu^2 h^2 \geq C_3 = C_3(\eta)$.
So,

$$\int_{\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}} \lambda^{2m} |1-\varphi_\mu^{\alpha,\beta}(h)|^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \geq C_1^2 C_2^2 C_3^2 \int_{\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}} \lambda^{2m} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).$$

There exists then a positive constant C such that

$$\int_{\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}} \lambda^{2m} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq C \int_{\mathbb{R}} \lambda^{2m} |1-\varphi_\mu^{\alpha,\beta}(h)|^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq C h^{2\delta}.$$

For all $0 < h < \frac{\eta}{3\rho}$, then we have

$$\int_{r \leq |\lambda| \leq 2r} \lambda^{2m} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq C r^{-2\delta}, \quad r \to \infty.$$

Furthermore, we obtain

$$\begin{aligned} \int_{|\lambda| \geq r} \lambda^{2m} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) &= \sum_{i=0}^{\infty} \int_{2^i r \leq |\lambda| \leq 2^{i+1} r} \lambda^{2m} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \\ &\leq C \sum_{i=0}^{\infty} (2^i r)^{-2\delta} \\ &\leq C r^{-2\delta}. \end{aligned}$$

This proves that

$$\int_{|\lambda| \geq r} \lambda^{2m} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = O(r^{-2\delta}), \quad as \quad r \to \infty.$$

$(ii) \Rightarrow (i)$. Suppose now that

$$\int_{|\lambda| \geq r} \lambda^{2m} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = O(r^{-2\delta}), \quad as \quad r \to \infty,$$

and write

$$\begin{aligned} \int_{\mathbb{R}} \lambda^{2m} |1-\varphi_\mu^{\alpha,\beta}(h)|^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) &= \int_{|\lambda| < \frac{1}{h}} \lambda^{2m} |1-\varphi_\mu^{\alpha,\beta}(h)|^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \\ &+ \int_{|\lambda| \geq \frac{1}{h}} \lambda^{2m} |1-\varphi_\mu^{\alpha,\beta}(h)|^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda). \end{aligned}$$

Using the inequality (c) of Lemma 2.1, we get

$$\int_{|\lambda| \geq \frac{1}{h}} \lambda^{2m} |1-\varphi_\mu^{\alpha,\beta}(h)|^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq 4 \int_{|\lambda| \geq \frac{1}{h}} \lambda^{2m} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).$$

Then

$$\int_{|\lambda| \geq \frac{1}{h}} \lambda^{2m} |1-\varphi_\mu^{\alpha,\beta}(h)|^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = O(h^{2\delta}), \quad as \quad h \to 0. \tag{3.6}$$

Set

$$\phi(\lambda) = \int_{\lambda}^{\infty} x^{2m} |\mathcal{F}_{\alpha,\beta}(f)(x)|^2 d\sigma(x).$$

An integration by parts gives:

$$\begin{aligned} \int_0^x \lambda^2 \lambda^{2m} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) &= \int_0^x -\lambda^2 \phi'(\lambda) d\lambda \\ &= -x^2 \phi(x) + 2 \int_0^x \lambda \phi(\lambda) d\lambda \\ &\leq 2 \int_0^x O(\lambda^{1-2\delta}) d\lambda \\ &= O(x^{2-2\delta}). \end{aligned}$$

From Lemma 2.1, we get

$$\begin{aligned}
\int_{|\lambda|\leq\frac{1}{h}}\lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda) &\leq \int_{|\lambda|\leq\frac{1}{h}}\lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)||\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda)\\
&\leq \int_{|\lambda|\leq\frac{1}{h}}\lambda^{2m}(\mu^2+\rho^2)h^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda)\\
&\leq h^2\int_{|\lambda|\leq\frac{1}{h}}\lambda^2\lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda)\\
&= O(h^2h^{-2+2\delta}).
\end{aligned}$$

Hence,

$$\int_{|\lambda|\leq\frac{1}{h}}\lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda) = O(h^{2\delta}). \tag{3.7}$$

Finally, we conclude from 3.6 and 3.7 that

$$\begin{aligned}
\int_{\mathbb{R}}\lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda) &= \int_{|\lambda|<\frac{1}{h}}+\int_{|\lambda|\geq\frac{1}{h}}\\
&= O(h^{2\delta})+O(h^{2\delta})\\
&= O(h^{2\delta}).
\end{aligned}$$

And this ends the proof.

Corollary 3.1. *Let $f\in L^2_{\alpha,\beta}(\mathbb{R})$, and let*

$$\|N_h\Lambda^m_{\alpha,\beta}f\| = O(h^\delta), \quad as \quad h\to 0,$$

Then

$$\int_{|\lambda|\geq r}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda) = O(r^{-2m-2\delta}), \quad as \quad r\to\infty.$$

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