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A new classes of open mappings

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Abstract

The aim of this paper is to introduce new classes of mappings namely $\hat{\Omega}$-open mappings, somewhat $\hat{\Omega}$ open functions and hardly $\hat{\Omega}$-open mappings by utilizing $\hat{\Omega}$-closed sets. Also investigate some of their properties.

Keywords: $\hat{\Omega}$-closed sets, $\hat{\Omega}$ dense sets, $\hat{\Omega}$-open mappings, somewhat $\hat{\Omega}$ open mappings, hardly $\hat{\Omega}$-open mappings.

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1 Introduction

In 1969, Karl R. et al.[6] introduced the concept of Somewhat continuous and somewhat open function and investigated their properties.These functions are nothing but Frolik functions such that the condition onto was just dropped. These notions are also related to the idea of weakly equivalent topologies which was first introduced by Yougslova [11]. In this paper we study the concept of somewhat $\hat{\Omega}$ continuous and somewhat $\hat{\Omega}$ open function and investigated their properties by giving suitable examples on it. More over,we introduce and study two more kinds of open mappings via $\hat{\Omega}$-closed sets. Also we investigate their properties.

2 Preliminaries

Throughout this paper (X,τ) (or briefly X) represent a topological space with no separation axioms assumed unless otherwise explicitly stated. For a subset A of (X,τ) , we denote the closure of A, the interior of A and the complement of A as $cl(A), int(A)$ and A^c respectively. The following notations are used in this paper. The family of all open (resp.δ-open,$\hat{\Omega}$-open) sets on X are denoted by $O(X)$ (resp.$\delta O(X), \hat{\Omega}O(X)$). The family of all $\hat{\Omega}$-closed sets on X are denoted by $\hat{\Omega}C(X)$.

- $O(X,x) = \{U \in X / x \in U \in O(X)\}$
- $\delta O(X,x) = \{U \in X / x \in U \in \delta O(X)\}$
- $\hat{\Omega}O(X,x) = \{U \in X / x \in U \in \hat{\Omega}O(X)\}$

Let us sketch some existing definitions,which are useful in the sequel as follows.

Definition 2.1. *[5] A subset A of $X$is called δ-closed in a topological space (X,τ) if $A = \delta cl(A)$, where $\delta cl(A) = \{x \in X : int(cl(U)) \cap A \neq \emptyset, U \in O(X,x)\}$.The complement of δ-closed set in (X,τ) is called δ-open set in (X,τ). From [5],lemma 3, $\delta cl(A) = \cap\{F \in \delta C(X) : A \subseteq F\}$ and from corollary 4, $\delta cl(A)$ is a δ-closed for a subset A in a topological space (X,τ).*

Definition 2.2. *A subset A of a topological space (X,τ) is called*

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(i) *semiopen set in* (X,τ) *if* $A \subseteq cl(int(A))$.

(ii) $\hat{\Omega}$*-closed set [7] if* $\delta cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is semi open in* (X,τ).

The complement of $\hat{\Omega}$*-closed set is called* $\hat{\Omega}$*-open.*

Definition 2.3. *A function* $f\colon (X,\tau) \to (Y,\sigma)$ *is called*

(i) *somewhat open [6] if* $U \in \tau$ *and* $U \neq \emptyset$, *then there exists* $V \in \sigma$ *such that* $V \neq \emptyset$ *and* $U \subseteq f(U)$.

(ii) *somewhat b open [3] if* $U \in \tau$ *and* $U \neq \emptyset$, *then there exists a b-open set* $V \in \sigma$ *such that* $V \neq \emptyset$ *and* $U \subseteq f(U)$.

(iii) *somewhat sg open [2] if* $U \in \tau$ *and* $U \neq \emptyset$, *then there exists a sg-open set* $V \in \sigma$ *such that* $V \neq \emptyset$ *and* $U \subseteq f(U)$.

(iv) *perfectly continuous [10] if the inverse image of open set in* Y *is clopen set in* X.

(v) *completely continuous [1] if the inverse image of open set in* Y *is regular open set in* X.

(vi) *super continuous [9] if the inverse image of open set in* Y *is* δ *open set in* X.

(vii) *somewhat continuous [6] if* $U \in \sigma$ *and* $f^{-1}(U) \neq \emptyset$, *then there exists a non empty set*$V \in \tau$ *such that* $V \subseteq f^{-1}(U)$.

(viii) *somewhat b continuous [3] if* $U \in \sigma$ *and* $f^{-1}(U) \neq \emptyset$, *then there exists a non empty b-open set*V *in* (X,τ) *such that* $V \subseteq f^{-1}(U)$.

(ix) *somewhat sg continuous [2] if* $U \in \sigma$ *and* $f^{-1}(U) \neq \emptyset$, *then there exists a non empty sg-open set*V *in* (X,τ) *such that* $V \subseteq f^{-1}(U)$.

Definition 2.4. *A space* (X,τ) *is said to be* $T_{\frac{3}{4}}$ *[4] if every* δg*-open set is* δ*-open set in* X.

Definition 2.5. *A space* (X,τ) *is said to be* T_1 *if for every two different point* x *and* y, *there exists open sets* U *and* V *such that* $x \in U, y \notin U$ *and* $y \in V, x \notin V$. *Also every proper set is contained in a proper open set.*

Theorem 2.6. *[8] A space* (X,τ) *is* ${}_{\omega}T_{\hat{\Omega}}$*-space if and only if every closed set is* $\hat{\Omega}$*-closed in* (X,τ).

Theorem 2.7. *[8] A space* (X,τ) *is semi-*$T_{\frac{1}{2}}$ *if and only if every* $\hat{\Omega}$*-open set is open in* (X,τ).

3 $\hat{\Omega}$-open mappings

Definition 3.1. *A map* $f\colon (X,\tau) \to (Y,\sigma)$ *is said to be* $\hat{\Omega}$*-open function if the image of every open set in* X *is* $\hat{\Omega}$*-open set in* Y.

Example 3.2. *Let* $X = Y = \{a,b,c\}, \tau = \{\emptyset,\{a\},\{a,b\},\{a,c\},X\}, \sigma = \{\emptyset,\{a\},\{b\},\{a,b\},Y\}$. *Define* $f\colon (X,\tau) \to (Y,\sigma)$ *by* $f(a) = a, f(b) = a, f(c) = b$. *Then* f *is* $\hat{\Omega}$*-open function.*

Remark 3.3. *The notion of* $\hat{\Omega}$*-open function and open mappings are independent from the following examples.*

Example 3.4. *Let* $X = Y = \{a,b,c\}, \tau = \{\emptyset,\{a\},\{b\},\{a,b\},,X\}, \sigma = \{\emptyset,\{a,b\},Y\}$. *Define* $f\colon (X,\tau) \to (Y,\sigma)$ *by* $f(a) = a, f(b) = b, f(c) = c$. *Then* f *is* $\hat{\Omega}$*-open but not open function.*

Example 3.5. *Let* $X = Y = \{a,b,c\}, \tau = \{\emptyset,\{a\},\{b,c\},X\}, \sigma = \{\emptyset,\{a\},\{a,b\},Y\}$. *Define* $f\colon (X,\tau) \to (Y,\sigma)$ *by* $f(a) = a, f(b) = a, f(c) = b$. *Then* f *is open but not* $\hat{\Omega}$*-open function.*

Let us characterize $\hat{\Omega}$-open function in the following theorems.

Theorem 3.6. *A function* $f\colon (X,\tau) \to (Y,\sigma)$ *is* $\hat{\Omega}$*-open function if and only if for any subset* A *of* Y *and for any closed set* F *in* X *such that* $f^{-1}(A) \subseteq F$, *there exists a* $\hat{\Omega}$*-closed set* F^1 *in* Y *such that* $A \subseteq F^1$ *and* $f^{-1}(F^{-1}) \subseteq F$.

Proof. **Necessity-** Let A be any subset in Y and F be any closed set in X such that $f^{-1}(A) \subseteq F$. Then $(X \setminus F)$ is open in X. By hypothesis, $f((X \setminus F))$ is $\hat{\Omega}$-open in Y and hence $Y \setminus f((X \setminus F))$ is $\hat{\Omega}$-closed in Y. Since $f^{-1}(A) \subseteq F, (X \setminus F) \subseteq (X \setminus f^{-1}(A)) = f^{-1}(Y \setminus A)$. Therefore, $f(X \setminus F) \subseteq (Y \setminus A)$ and hence $A \subseteq (Y \setminus f(X \setminus F))$. Now $f^{-1}(Y \setminus f(X \setminus F)) = (X \setminus f^{-1}f(X \setminus F)) \subseteq F$. If we take $F^1 = (Y \setminus f(X \setminus F))$,then F^1 is a $\hat{\Omega}$-closed set in Y such that $f^{-1}(F^1) \subseteq F$.
Sufficiency- Suppose that U is any open set in X. Then $(X \setminus U)$ is closed in X and $f^{-1}(Y \setminus f(U)) \subseteq (X \setminus U)$ By hypothesis, there exists $\hat{\Omega}$-closed set F in Y such that $(Y \setminus f(U)) \subseteq F$ and $f^{-1}(F) \subseteq (X \setminus U)$. Therefore, $(Y \setminus F) \subseteq f(U)$ and $U \subseteq (X \setminus f^{-1}(F)) = f^{-1}(Y \setminus F)$. Therefore,$(Y \setminus F) \subseteq f(U) \subseteq (Y \setminus F)$ and hence $(Y \setminus F) = f(U)$. Thus $f(U)$ is $\hat{\Omega}$-open set in Y. □

Theorem 3.7. *A function $f\colon (X, \tau) \to (Y, \sigma)$ is $\hat{\Omega}$-open function if and only if for any subset B of Y,$f^{-1}(\hat{\Omega}cl(B)) \subseteq cl(f^{-1}(B))$.*

Proof. **Necessity-** For any subset B of $Y, f^{-1}(B) \subseteq cl(f^{-1}(B))$. By theorem 3.6,there exists a $\hat{\Omega}$-closed set A in Y such that $B \subseteq A$ and $f^{-1}(A) \subseteq cl(f^{-1}(B))$. By[7] the definition of $\hat{\Omega}$ closure, $\hat{\Omega}cl(B) \subseteq A$. Then $f^{-1}(\hat{\Omega}cl(B)) \subseteq f^{-1}(A) \subseteq cl(f^{-1}(B))$. Thus, $f^{-1}(\hat{\Omega}cl(B)) \subseteq cl(f^{-1}(B))$.
Sufficiency- Let A be any set in Y and F be any closed set in X such that $f^{-1}(A) \subseteq F$. If $F^1 = \hat{\Omega}cl(A)$,then [7] theorem 5.3, F^1 is $\hat{\Omega}$-closed set in Y containing A. By hypothesis, $f^{-1}(F^1) = f^{-1}(\hat{\Omega}cl(A)) \subseteq cl(f^{-1}(A)) \subseteq cl(F) \subseteq F$. By theorem 3.6, f is $\hat{\Omega}$-open function. □

Theorem 3.8. *For any function $f\colon (X, \tau) \to (Y, \sigma)$, the following statements are true.*

(i) *f is $\hat{\Omega}$-open mapping.*

(ii) *$f(\delta int(A)) \subseteq \hat{\Omega}int(f(A))$ for any subset A in X.*

(iii) *For every $x \in X$ and for every δ-open set U in X containing x,there exists a $\hat{\Omega}$-open set W in Y containing $f(x)$ such that $W \subseteq f(U)$.*

Proof. $(i) \Rightarrow (ii)$ Suppose that A is any subset of X.Then $\delta int(A)$ is open in X and $\delta int(A) \subseteq A$. By hypothesis, $f(\delta int(A))$ is $\hat{\Omega}$-open set in Y and $f(\delta int(A)) \subseteq f(A)$. By the definition of $\hat{\Omega}$ interior,$\hat{\Omega}int(f(A))$ is the largest $\hat{\Omega}$-open set contained in $f(A)$. Therefore, $f(\delta int(A)) \subseteq \delta int(f(A))$.

$(ii) \Rightarrow (iii)$ Let $x \in X$ and U be any δ-open set in X containing x. Then there exists δ-open set V in X such that $x \in V \subseteq U$. By hypothesis, $f(V) = f(\delta int(V)) \subseteq \hat{\Omega}int(f(V))$. Then $f(V)$ is $\hat{\Omega}$-open in Y containing $f(x)$ such that $f(V) \subseteq f(U)$. If we take $W = f(V)$,then W satisfies our requirement.

$(iii) \Rightarrow (i)$ Suppose that U is any δ-open set in X and y is any point in $f(U)$. By hypothesis, there exists an $\hat{\Omega}$-open set W_y in Y containing y such that $W_y \subseteq f(U)$.Therefore,$f(U) = \bigcup\{W_y : y \in f(U)\}$. By [7] theorem 4.16,$f(U)$ is $\hat{\Omega}$-open set in Y. □

Theorem 3.9. *A surjective function $f\colon (X, \tau) \to (Y, \sigma)$ is $\hat{\Omega}$-open function if and only if $f^{-1}\colon Y \to X$ is $\hat{\Omega}$-continuous.*

Proof. **Necessity-** If U is any open set in X then by hypothesis, $(f^{-1})^{-1}(U) = f(U)$ is $\hat{\Omega}$-open in Y.Hence $f^{-1}\colon Y \to X$ is $\hat{\Omega}$-continuous.
Sufficiency- If U is any open set in X, then by hypothesis,$f(U) = (f^{-1})^{-1}(U)$ is $\hat{\Omega}$-open in Y. Hence $f\colon (X, \tau) \to (Y, \sigma)$ is $\hat{\Omega}$-open function. □

Remark 3.10. *In general, composition of any two $\hat{\Omega}$-open functions is not a $\hat{\Omega}$-open function from the following example.*

Example 3.11. *$X = Y = \{a, b, c, d\}$ and $Z = \{a, b, c\}, \tau = \{\emptyset, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}$, $\sigma = \{\emptyset, \{a\}, \{b, c, d\}, Y\}, \eta = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Z\}$. Then $\hat{\Omega}O(Y) = P(X), \hat{\Omega}O(Z) = \eta$. If $f\colon (X, \tau) \to (Y, \sigma)$ is defined by $f(a) = a, f(b) = c, f(c) = d, f(d) = a$. Then f is $\hat{\Omega}$-open function. If $g\colon (Y, \sigma) \to (Z, \eta)$ is defined by $g(a) = a, g(b) = a, g(c) = b, g(d) = c$. Then g is $\hat{\Omega}$-open function.But $g \circ f\colon (X, \tau) \to (Z, \eta)$ defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$ is not $\hat{\Omega}$-open function because $(g \circ f)(\{b, c\}) = \{b, c\}$ not belongs to $\hat{\Omega}O(Z)$.*

Theorem 3.12. *If $f\colon (X,\tau) \to (Y,\sigma)$ is open function and $g\colon (Y,\sigma) \to (Z,\eta)$ is $\hat{\Omega}$-open function, then $g \circ f\colon (X,\tau) \to (Z,\eta)$ is $\hat{\Omega}$-open function.*

Proof. It follows from their definitions. □

Theorems on Composition

Theorem 3.13. *Let (Y,σ) be a semi-$T_{\frac{1}{2}}$-space. If $f\colon (X,\tau) \to (Y,\sigma)$ and $g\colon (Y,\sigma) \to (Z,\eta)$ are $\hat{\Omega}$-open functions, then $g \circ f\colon (X,\tau) \to (Z,\eta)$ is $\hat{\Omega}$-open function.*

Proof. It follows from their definitions. □

Theorem 3.14. *If $f\colon (X,\tau) \to (Y,\sigma)$ and $g\colon (Y,\sigma) \to (Z,\eta)$ are any two functions such that $g \circ f\colon (X,\tau) \to (Z,\eta)$ is $\hat{\Omega}$-open function then,*

(i) f is $\hat{\Omega}$-open mapping if g is $\hat{\Omega}$-irresolute and injective.

(ii) g is $\hat{\Omega}$-open mapping if f is continuous and surjective.

Proof. (i) If U is any open set in X, $g(f(U))$ is $\hat{\Omega}$-open in Z. Since g is $\hat{\Omega}$-irresolute, $g^{-1}(g(f(U)))$ is $\hat{\Omega}$-open in Y. Since g is injective, $g^{-1}(g(f(U))) = f(U)$ is $\hat{\Omega}$-open in Y. Thus f is $\hat{\Omega}$-open mapping.

(ii) If U is any open set in Y, then $f^{-1}(U)$ is open set in X. Since $g \circ f\colon (X,\tau) \to (Z,\eta)$ is $\hat{\Omega}$-open function, $g(f(f^{-1}(U)))$ is $\hat{\Omega}$-open in Z. Since f is surjective, $g(f(f^{-1}(U))) = g(U)$ is $\hat{\Omega}$-open in Z.

□

4 Somewhat $\hat{\Omega}$-open, Hardly $\hat{\Omega}$-open mappings

Definition 4.1. *A subset A of a space X is said to be $\hat{\Omega}$-dense in X if $\hat{\Omega}cl(A) = X$. Or, there is no $\hat{\Omega}$-closed between A and X.*

Example 4.2. *Let $X = \{a,b,c\}, \tau = \{\emptyset, \{a\}, X\}$. Then $\hat{\Omega}$-dense sets in X are $\{\{a\},\{a,b\},\{a,c\}\}$.*

Definition 4.3. *A function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be somewhat $\hat{\Omega}$-open if for each non empty set $U \in O(X)$, there exists a non empty set $V \in \hat{\Omega}O(Y)$ such that $V \subseteq f(U)$.*

Example 4.4. *Let $X = Y = \{a,b,c\}, \tau = \{\emptyset, \{a,b\}, X\}$ and $\sigma = \{\emptyset, \{b\}, \{c\}, \{b,c\}, Y\}$. Define $f\colon (X,\tau) \to (Y,\sigma)$ by $f(a) = b, f(b) = c, f(c) = a$. Then f is somewhat $\hat{\Omega}$-open mapping.*

Theorem 4.5. *Every somewhat $\hat{\Omega}$-open mapping is somewhat b-(resp.sg-) open mapping.*

Proof. Assume that $f\colon (X,\tau) \to (Y,\sigma)$ is somewhat $\hat{\Omega}$-open mapping and suppose that U is any non empty set in X. By hypothesis,there exists a non empty set $V \in \hat{\Omega}O(Y)$ such that $V \subseteq f(U)$. By [7] remark 3.13, V is b-(resp.sg-)open set in Y. Hence f is somewhat b open mapping. □

Remark 4.6. *The following example shows that the reversible implication is not true in general.*

Example 4.7. *Let $X = Y = \{a,b,c\}, \tau = \{\emptyset, \{a,b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a,b\}, Y\}$. Define $f\colon (X,\tau) \to (Y,\sigma)$ by $f(a) = b, f(b) = c, f(c) = a$. Then f is somewhat b-(resp.sg-) open mapping but not somewhat $\hat{\Omega}$-open mapping.*

Remark 4.8. *The notions, somewhat open (resp.somewhat semi open)mapping and somewhat $\hat{\Omega}$-open mapping are independent from the following examples.*

Question: Is there any example on a mapping which is somewhat open but not somewhat $\hat{\Omega}$-open?

Example 4.9. *Let $X = Y = \{a,b,c\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$, and $\sigma = \{\emptyset, \{a\}, \{b,c\}, Y\}$. Define $f\colon (X,\tau) \to (Y,\sigma)$ by $f(a) = b, f(b) = c, f(c) = a$.Then f is somewhat $\hat{\Omega}$-open mapping but not somewhat open mapping.*

Theorem 4.10. *If (Y,σ) is semi-$T_{\frac{1}{2}}$, then every somewhat $\hat{\Omega}$-open mapping $f\colon (X,\tau) \to (Y,\sigma)$ is somewhat open mapping.*

Proof. Let $U \in O(X)$ be any non empty set in X. By hypothesis,there exists a non empty set $V \in \hat{\Omega}O(Y)$ such that $V \subseteq f(U)$. Since in a *semi*-$T_{\frac{1}{2}}$ space, every $\hat{\Omega}$-open set is open,$V \in O(Y)$. Hence f is somewhat open mapping. □

Theorem 4.11. *If (Y, σ) is ${}_{\omega}T_{\hat{\Omega}}$, then every somewhat open mapping $f\colon (X, \tau) \to (Y, \sigma)$ is somewhat $\hat{\Omega}$-open mapping.*

Proof. Let $U \in O(X)$ be any non empty set in X. By hypothesis,there exists a non empty set $V \in O(Y)$ such that $V \subseteq f(U)$.Since in a ${}_{\omega}T_{\hat{\Omega}}$ space,every open set is $\hat{\Omega}$-open, $V \in \hat{\Omega}O(Y)$. Hence f is somewhat $\hat{\Omega}$-open mapping. □

Let us prove a characterization of somewhat $\hat{\Omega}$ open mapping.

Theorem 4.12. *A mapping $f\colon (X, \tau) \to (Y, \sigma)$ is somewhat $\hat{\Omega}$ open if and only if inverse image of a $\hat{\Omega}$-dense set in Y is dense in X.*

Proof. **Necessity-** Suppose that D is $\hat{\Omega}$-dense set in Y and suppose $f^{-1}(D)$ is not dense in X. Therefore, there exists a proper closed set F in X such that $f^{-1}(D) \subseteq F \subseteq X$. Then $X \setminus F$ is a non empty open set in X. By hypothesis, there exists a non empty set $V \in \hat{\Omega}O(Y)$ such that $V \subseteq f(X \setminus F)$ or $Y \setminus f(X \setminus F) \subseteq Y \setminus V$. Moreover, $X \setminus F \subseteq X \setminus f^{-1}(D) = f^{-1}(Y \setminus D)$ implies that $f(X \setminus F) \subseteq Y \setminus D$. Then $D \subseteq Y \setminus f(X \setminus F) \subseteq Y \setminus V$. We have some proper $\hat{\Omega}$-closed set $Y \setminus V$ in Y such that $D \subseteq Y \setminus V \subseteq Y$ a contradiction to D is $\hat{\Omega}$-dense set in Y. Therefore,$f^{-1}(D)$ is dense in X.

Sufficiency- If f is not somewhat $\hat{\Omega}$-open mapping,for every non empty open set U in X, no non empty $\hat{\Omega}$-open set in Y is such that $V \subseteq f(U)$. Then no proper $\hat{\Omega}$ closed set $Y \setminus V$ is such that $Y \setminus f(U) \subseteq Y \setminus V \subseteq Y$. Therefore, $Y \setminus f(U)$ is $\hat{\Omega}$-dense in Y. By hypothesis, $f^{-1}(Y \setminus f(U))$ is dense in X or $X \setminus (f^{-1}(f(U))$ is dense in X. Therefore,$cl(X \setminus (f^{-1}(f(U))) = X$. Moreover, $U \subseteq (f^{-1}(f(U))$ implies that $X \setminus (f^{-1}(f(U)) \subseteq X \setminus U$. Then $X = cl(X \setminus (f^{-1}(f(U))) \subseteq cl(X \setminus U) = X \setminus \setminus int(U)$ and hence $int(U) = \emptyset$, a contradiction to U is a non empty set in X. □

Theorem 4.13. *Suppose that $f\colon (X, \tau) \to (Y, \sigma)$ is a bijective mapping. f is somewhat $\hat{\Omega}$-open mapping if and only if for every closed set F in X such that $f(F) \neq Y$, there exists a proper set $D \in \hat{\Omega}C(X)$ such that $f(F) \subseteq D$.*

Proof. **Necessity-** Suppose that F is any closed set in X such that $f(F) \neq Y$. Then $X \setminus F$ is a non empty open set in X.By hypothesis, there exists a non empty set $V \in \hat{\Omega}O(Y)$ such that $V \subseteq f(X \setminus F)$ or $Y \setminus f(X \setminus F) \subseteq Y \setminus V$. Since f is bijective,$f(F) \subset Y \setminus V$. If we define, $D = Y \setminus V$,then $D \neq \emptyset, D \in \hat{\Omega}C(Y)$ such that $f(F) \subseteq D$.

Sufficiency- Suppose that U is any non empty open set in X.Then $X \setminus U$ is a proper closed set in X. If $f(X \setminus U) = Y$,then it is easily seen that $U = \emptyset$, a contradiction.Therefore,$f(X \setminus U) \neq Y$. By hypothesis,there exists a proper $\hat{\Omega}$-closed set D in Y such that $f(X \setminus U) \subseteq D$. That is,$Y \setminus D \subseteq Y \setminus f(X \setminus U) = f(U)$, where $Y \setminus D \neq \emptyset, Y \setminus D \in \hat{\Omega}O(Y)$.Thus f is somewhat $\hat{\Omega}$-open mapping. □

Theorem 4.14. *Suppose that A is any open set in a topological space (X, τ). If $f\colon (X, \tau) \to (Y, \sigma)$ is somewhat $\hat{\Omega}$-open mapping, then $f\mid_A\colon (A, \tau\mid_A) \to (Y, \sigma)$ is also somewhat $\hat{\Omega}$-open mapping on the subspace $(A, \tau\mid_A)$.*

Proof. Suppose that $U \in \tau\mid_A, U \neq \emptyset$. Since U is open in $(A, \tau\mid_A)$ and A is open in X, U is open in X. By hypothesis,there exists a non empty $\hat{\Omega}$-open set V in Y such that $V \subseteq f(U)$. Therefore,$f\mid_A$ is somewhat $\hat{\Omega}$-open mapping. □

Theorem 4.15. *Suppose that (X, τ) and (Y, σ) are any two topological spaces and suppose $X = A \cup B$, where A and B are open in X. If $f\colon (X, \tau) \to (Y, \sigma)$ is any function such that $f\mid_A$ and $f\mid_B$ are somewhat $\hat{\Omega}$-open mappings,then f is a somewhat $\hat{\Omega}$-open mapping.*

Proof. Let U be any open set in X. Then $U \cap A$ and $U \cap B$ are open sets in the subspaces $(A, \tau\mid_A)$ and $(A, \tau\mid_B)$ respectively. Since $X = A \cup B$, either $A \cap U \neq \emptyset$ or $B \cap U \neq \emptyset$ or both $A \cap U \neq \emptyset$ and $B \cap U \neq \emptyset$.
case(i) If $U \cap A \neq \emptyset$.
Since $f\mid_A$ is somewhat $\hat{\Omega}$-open mapping,there exists a non empty $V \in \hat{\Omega}O(Y)$ such that $V \subseteq f(U \cap A) \subseteq f(U)$.

It follows that f is somewhat $\hat{\Omega}$-open mapping.
case(ii) If $U \cap B \neq \emptyset$.
Since $f\mid_B$ is somewhat $\hat{\Omega}$-open mapping, there exists a non empty $V \in \hat{\Omega}O(Y)$ such that $V \subseteq f(U \cap B) \subseteq f(U)$.
It follows that f is somewhat $\hat{\Omega}$-open mapping.
case(iii) If both $U \cap A \neq \emptyset$ and $U \cap B \neq \emptyset$. It follows from case(i) or case(ii). □

Remark 4.16. *Composition of two somewhat $\hat{\Omega}$-open mappings is not always somewhat $\hat{\Omega}$-open mapping from the following example.*

Example 4.17. *$X = Y = \{a, b, c, d\}$ and $Z = \{a, b, c\}, \tau = \{\emptyset, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}, \sigma = \{\emptyset, \{a\}, \{b, c, d\}, Y\}$, $\eta = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Z\}$. Then $\hat{\Omega}O(Y) = P(X), \hat{\Omega}O(Z) = \eta$. If $f : (X, \tau) \to (Y, \sigma)$ is defined by $f(a) = a, f(b) = c, f(c) = d, f(d) = a$. Then f is somewhat $\hat{\Omega}$-open function. If $g : (Y, \sigma) \to (Z, \eta)$ is defined by $g(a) = a, g(b) = a, g(c) = b, g(d) = c$. Then g is somewhat $\hat{\Omega}$-open function.But $g \circ f : (X, \tau) \to (Z, \eta)$ defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$ is not a somewhat $\hat{\Omega}$-open function because $(g \circ f)(\{b, c\}) = \{b, c\}$ does not contain any $\hat{\Omega}$-open set in Z.*

The following theorem states the condition under which the composition of two somewhat $\hat{\Omega}$-open mappings is again a somewhat $\hat{\Omega}$-open mappings.

Theorem 4.18. *If $f : (X, \tau) \to (Y, \sigma)$ is an open mapping and $g : (Y, \sigma) \to (Z, \eta)$ is a somewhat $\hat{\Omega}$-open mapping,then $g \circ f : (X, \tau) \to (Z, \eta)$ is a somewhat $\hat{\Omega}$-open mapping.*

Proof. Suppose $U \in O(X)$ is any non empty set in X. Since f is an open mapping, $f(U)$ is an open set in Y. Since g is somewhat $\hat{\Omega}$-open mapping,there exists a non empty set $V \in \hat{\Omega}O(Z)$ such that $V \subseteq g(f(U)) = g \circ f(U)$. Hence $g \circ f$ is somewhat $\hat{\Omega}$-open mapping. □

Definition 4.19. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be hardly $\hat{\Omega}$-open if for each $\hat{\Omega}$ dense set A in Y that is contained in a proper $\hat{\Omega}$-open set in $Y, f^{-1}(A)$ is $\hat{\Omega}$-dense in X.*

Example 4.20. *Let $X = Y = \{a, b, c\}, \tau = \sigma = \{\emptyset, \{a\}, X\}$. Then $\hat{\Omega}$ dense sets in X are $\{\{a\},\{a,b\},\{a,c\},X\}$.Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a, f(b) = a$ and $f(c) = c$. Then f is hardly $\hat{\Omega}$-open mapping.*

Theorem 4.21. *Let Y be a T_1 space. A function $f : (X, \tau) \to (Y, \sigma)$ is hardly open function if and only if for each $\hat{\Omega}$-dense set A in $Y, f^{-1}(A)$ is $\hat{\Omega}$-dense in X.*

Proof. Since in a T_1 space,every set is properly contained in a proper open set,it follows. □

Theorem 4.22. *[4] A topological space is $T_{\frac{3}{4}}$ if and only if $\{x\}$ is either δ-open or closed.*

Theorem 4.23. *If Y is a $T_{\frac{3}{4}}$ space,then $f : (X, \tau) \to (Y, \sigma)$ is hardly $\hat{\Omega}$-open function if and only if for each $\hat{\Omega}$-dense set D in $Y, f^{-1}(D)$ is $\hat{\Omega}$-dense in X.*

Proof. **Necessity-** Assume that f is hardly $\hat{\Omega}$-open function and D is any $\hat{\Omega}$-dense set in Y. Let $y \in Y \setminus D$ be an arbitrary point. Since D is $\hat{\Omega}$-dense in $Y, \hat{\Omega}cl(D) = Y$. That is,$Y \setminus \hat{\Omega}cl(D) = \emptyset$. By [7] theorem 5.3 (vii),$Y \setminus \delta cl(D) = \emptyset$ or $\delta int(Y \setminus D) = \emptyset$. Therefore,$\{y\}$ is not a δ open in a $T_{\frac{3}{4}}$ space Y.By the theorem 4.22, $\{y\}$ is a closed set in Y and hence $Y \setminus \{y\}$ is a proper open set in Y. Therefore,D is contained in a proper open set $Y \setminus \{y\}$. By hypothesis,$f^{-1}(D)$ is $\hat{\Omega}$-dense in X.

Sufficiency- From the given hypothesis, f is hardly $\hat{\Omega}$-open function. □

Theorem 4.24. *$f : (X, \tau) \to (Y, \sigma)$ is hardly $\hat{\Omega}$-open function if and only if $\hat{\Omega}int(f^{-1}(A)) = \emptyset$ for each subset A in Y such that $\hat{\Omega}int(A) = \emptyset$ and A contains a nonempty closed set.*

Proof. **Necessity-** Assume that f is hardly $\hat{\Omega}$-open function and $A \subseteq Y$ such that $\hat{\Omega}int(A) = \emptyset$ and F, a nonempty closed set in Y such that $F \subseteq A$. Then, $\hat{\Omega}cl(Y \setminus A) = Y \setminus \hat{\Omega}int(A) = Y$. Since $F \subseteq A, Y \setminus A \subseteq Y \setminus F \neq Y$. Therefore, $Y \setminus A$ is a $\hat{\Omega}$-dense in Y which is contained in a proper open set $Y \setminus F$. By hypothesis,$f^{-1}(Y \setminus A)$ is $\hat{\Omega}$-dense in X. Therefore, $X = \hat{\Omega}cl(f^{-1}(Y \setminus A)) = X \setminus \hat{\Omega}int(f^{-1}(A))$. Thus, $X \setminus \hat{\Omega}int(f^{-1}(A)) = X$ and hence $\hat{\Omega}int(f^{-1}(A)) = \emptyset$.

Sufficiency- Suppose that D is any $\hat{\Omega}$-dense in Y such that it is contained in a proper open set U. Since $U \neq \emptyset, Y \setminus U$ is a non empty closed set contained in $Y \setminus D$. By hypothesis, $\hat{\Omega}int(f^{-1}(Y \setminus D)) = \emptyset$.Then,$X \setminus \hat{\Omega}cl(f^{-1}(D)) = \emptyset$ and hence $\hat{\Omega}cl(f^{-1}(D)) = X$. Thus,$f^{-1}(D)$ is $\hat{\Omega}$ dense in X. □

Theorem 4.25. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be any function. If $\hat{\Omega}int(f(A)) \neq \emptyset$ for every subset A of X having the property that $\hat{\Omega}int(A) \neq \emptyset$ and there exists a non empty closed set F in X such that $f^{-1}(F) \subseteq A$,then f is hardly $\hat{\Omega}$-open function.*

Proof. Suppose that D is any $\hat{\Omega}$-dense in Y which is contained in a proper open set U. Since $U \neq \emptyset, Y \setminus U \neq \emptyset$ and hence $Y \setminus U$ is a non empty closed set contained in $Y \setminus D$.If we define $A = f^{-1}(Y \setminus D), F = Y \setminus U$, then $f^{-1}(F) \subseteq A$. Moreover, $\hat{\Omega}int(f(A)) = \hat{\Omega}int(f(f^{-1}(Y \setminus D))) \subseteq \hat{\Omega}int(Y \setminus D) = \emptyset$. By hypothesis, we should have $\hat{\Omega}int(A) = \emptyset$. That is, $\hat{\Omega}int(f^{-1}(Y \setminus D)) = \emptyset$. Therefore,$X \setminus \hat{\Omega}cl(f^{-1}(D)) = \emptyset$ and hence $\hat{\Omega}cl(f^{-1}(D)) = X$. Thus, $f^{-1}(D)$ is $\hat{\Omega}$ dense in X. Therefore, f is hardly $\hat{\Omega}$-open function. □

Theorem 4.26. *If $f\colon (X,\tau) \to (Y,\sigma)$ is hardly $\hat{\Omega}$-open function, then $\hat{\Omega}int(f(A)) \neq \emptyset$ for every subset A of X having the property that $\hat{\Omega}int(A) \neq \emptyset$ and $f(A)$ contains a non empty closed set.*

Proof. Suppose that A is any set in X such that $\hat{\Omega}int(A) \neq \emptyset$ and F is any non empty closed set in Y such that $F \subseteq f(A)$. If $\hat{\Omega}int(f(A)) = \emptyset$,then $Y \setminus f(A)$ is $\hat{\Omega}$-dense in Y such that $Y \setminus f(A)$ is contained in a proper open set $Y \setminus F$. Since f is hardly $\hat{\Omega}$-open function, $f^{-1}(Y \setminus f(A))$ is $\hat{\Omega}$ dense in X. That is, $\hat{\Omega}cl(f^{-1}(Y \setminus f(A))) = X$ or $X \setminus \hat{\Omega}int(f^{-1}(f(A))) = X$. Then, $\hat{\Omega}int(f^{-1}(f(A))) = \emptyset$ and hence $\hat{\Omega}int(A) = \emptyset$, a contradiction.Therefore,our assumption is wrong and thus $\hat{\Omega}int(f(A)) \neq \emptyset$. □

Theorem 4.27. *If $f\colon (X,\tau) \to (Y,\sigma)$ is surjective,then the following statements are true.*

(i) *f is hardly $\hat{\Omega}$-open function.*

(ii) *$\hat{\Omega}int(f(A)) \neq \emptyset$ for every subset A of X having the property that $\hat{\Omega}int(A) \neq \emptyset$ and there exists a non empty closed set F in Y such that $F \subseteq f(A)$*

(iii) *$\hat{\Omega}int(f(A)) \neq \emptyset$ for every subset A of X having the property that $\hat{\Omega}int(A) \neq \emptyset$ and there exists a non empty closed set F in Y such that $f^{-1}(F) \subseteq A$*

Proof. $(i) \Rightarrow (ii)$ It's nothing but the theorem 4.10.

$(ii) \Rightarrow (iii)$ Since f is surjective,$f^{-1}(F) \subseteq f^{-1}(f(A)) = A$. Hence it holds.

$(iii) \Rightarrow (i)$ It follows from the theorem 4.9.

□

5 Somewhat $\hat{\Omega}$-Continuous functions

Definition 5.1. *A function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be somewhat $\hat{\Omega}$-continuous if for each non empty set $U \in O(Y)$ and $f^{-1}(U) \neq \emptyset$, there exists a non empty set $V \in \hat{\Omega}O(X)$ such that $V \subseteq f^{-1}(U)$.*

Example 5.2. *Let $X = Y = \{a,b,c\}, \tau = \{\emptyset, \{b,c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, Y\}$. Define $f\colon (X,\tau) \to (Y,\sigma)$ by $f(a) = b, f(b) = a, f(c) = a$. Then f is somewhat $\hat{\Omega}$-continuous.*

Theorem 5.3. *Every somewhat $\hat{\Omega}$-continuous is somewhat b (resp.sg)continuous*

Proof. Assume that $f\colon (X,\tau) \to (Y,\sigma)$ is somewhat $\hat{\Omega}$-continuous and suppose that U is any non empty set in Y such that $f^{-1}(U) \neq \emptyset$. By hypothesis,there exists a non empty set $V \in \hat{\Omega}O(Y)$ such that $V \subseteq f^{-1}(U)$. By [7] figure-1,V is b (resp.sg)open set in Y.Hence f is somewhat b continuous. □

Remark 5.4. *The following example shows that the reversible implication is not true in general.*

Example 5.5. *Let $X = Y = \{a,b,c,d\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, Y\}$. Define $f\colon (X,\tau) \to (Y,\sigma)$ by $f(a) = c, f(b) = a, f(c) = a, f(d) = a$. Then f is both somewhat b continuous and somewhat sg continuous but not somewhat $\hat{\Omega}$-continuous.*

Remark 5.6. *The notions, somewhat continuous and somewhat $\hat{\Omega}$-continuous are independent from the following examples.*

Example 5.7. *Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, Y\}$. If $f\colon (X, \tau) \to (Y, \sigma)$ is defined by $f(a) = b, f(b) = a, f(c) = c$ then f is somewhat $\hat{\Omega}$-continuous but not somewhat continuous.*

Question 2:Is there any example on a mapping which is somewhat continuous but not $\hat{somewhat}\Omega$-continuous?

Example 5.8. *There is no example on another one.*

Theorem 5.9. *If (Y, σ) is semi-$T_{\frac{1}{2}}$, then every somewhat $\hat{\Omega}$-continuous $f\colon (X, \tau) \to (Y, \sigma)$ is somewhat continuous.*

Proof. Let $U \in O(X)$ be any non empty set in X. By hypothesis,there exists a non empty set $V \in \hat{\Omega}O(Y)$ such that $V \subseteq f(U)$. Since in a *semi-$T_{\frac{1}{2}}$* space, every $\hat{\Omega}$-open set is open,$V \in O(Y)$.Hence f is somewhat continuous. □

Let us prove a characterization of somewhat $\hat{\Omega}$-continuous.

Theorem 5.10. *Let (X, τ) and (Y, σ) be any two topological spaces. Then the following are equivalent statements.*

(i) f is somewhat $\hat{\Omega}$-continuous.

(ii) If F is a closed subset of Y such that $f^{-1}(F) \neq X$, then there exists a proper set $G \in \hat{\Omega}C(X)$, such that $f^{-1}(F) \subseteq G$.

(iii) Image of a $\hat{\Omega}$-dense set in X is dense in Y.

Proof. (**i**) $\Rightarrow$ (**ii**). Suppose that F is any closed set in Y such that $f^{-1}(F) \neq X$. Then $Y \setminus F$ is a non empty open set in Y such that $f^{-1}(F^c) = (f^{-1}(F))^c \neq \emptyset$. By hypothesis, there exists a non empty set $V \in \hat{\Omega}O(X)$ such that $V \subseteq f^{-1}(F^c) = (f^{-1}(F))^c$. Then,$f^{-1}(F) \subseteq V^c$. If we define, $G = V^c$, then $G \neq \emptyset, G \in \hat{\Omega}C(X)$ such that $f^(F) \subseteq G$.

(**ii**) $\Rightarrow$ (**i**). Suppose that U is any non empty open set in Y such that $f^{-1}(U) \neq \emptyset$. Then $Y \setminus U$ is a proper closed set in Y such that $f^{-1}(U^c) = (f^{-1}(U))^c \neq X$. By hypothesis,there exists a proper set $G \in \hat{\Omega}C(X)$ such that $f^{-1}(U^c) = (f^{-1}(U))^c \subseteq G$. Then,$G^c \neq \emptyset, G^c \in \hat{\Omega}O(X)$ and $G^c \subseteq f^{-1}(U)$. Therefore, f is somewhat $\hat{\Omega}$-continuous.

(**ii**) $\Rightarrow$ (**iii**). Suppose that D is any $\hat{\Omega}$-dense set in X and assume that $f(D)$ is not dense in Y. Then, there exists a proper closed set F in Y such that $f(D) \subseteq F \subseteq Y$. Since $F \neq Y, f^{-1}(F) \neq f^{-1}(Y) \neq X$. By hypothesis,there exists a proper set $G \in \hat{\Omega}C(X)$ such that $f^{-1}(F) \subseteq G$. Therefore, $D \subseteq f^{-1}f(D) \subseteq f^{-1}(F) \subseteq G$. We have a proper $\hat{\Omega}$-closed set G in X such that $D \subseteq G \subseteq X$,a contradiction to D is $\hat{\Omega}$-dense in X. Therefore,$f(D)$ is dense in Y.

(**iii**) $\Rightarrow$ (**ii**). If (ii) not holds, then there exists a closed set F in Y such that $f^{-1}(F) \neq X$and there is no proper set $G \in \hat{\Omega}C(X)$, such that $f^{-1}(F) \subseteq G \subseteq X$. Then, $f^{-1}(F)$ is $\hat{\Omega}$-dense in X and hence by hypothesis, $f(f^{-1}(F))$ is $\hat{\Omega}$-dense in Y. Moreover, F vis dense in Y,a contraction to the choice of F. □

Remark 5.11. *The following example reveals that composition of two somewhat $\hat{\Omega}$-continuous functions is not always the somewhat $\hat{\Omega}$-continuous.*

Example 5.12. *$X = Y = Z = \{a, b, c\}, \tau = \{\emptyset, \{a, b\}, X\}, \sigma = \{\emptyset, \{a\}, \{b, c\}, Y\}$, $\eta = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Z\}$. If $f\colon (X, \tau) \to (Y, \sigma)$ is defined by $f(a) = a, f(b) = b, f(c) = c$ and $g\colon (Y, \sigma) \to (Z, \eta)$ is defined by $g(a) = b, g(b) = c, g(c) = a$. Then $f$and g are somewhat $\hat{\Omega}$-continuous functions. But $g \circ f\colon (X, \tau) \to (Z, \eta)$ defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$ is not a somewhat $\hat{\Omega}$-continuous because $(g \circ f)(\{a\}) = \{c\}$ is not containing any non empty $\hat{\Omega}$-open set in X.*

Composition Theorems

Theorem 5.13. *Suppose that $(X, \tau), (Y, \sigma)$ and (Z, η) are three topological spaces.*

(i) If $f\colon (X, \tau) \to (Y, \sigma)$ is a somewhat $\hat{\Omega}$-continuous function and $g\colon (Y, \sigma) \to (Z, \eta)$ is surjective continuous, then $g \circ f\colon (X, \tau) \to (Z, \eta)$ is a somewhat $\hat{\Omega}$-continuous mapping.

(ii) If $f\colon (X,\tau) \to (Y,\sigma)$ is a somewhat $\hat{\Omega}$-continuous function and $g\colon (Y,\sigma) \to (Z,\eta)$ is surjective super continuous, then $g \circ f\colon (X,\tau) \to (Z,\eta)$ is a somewhat $\hat{\Omega}$-continuous.

(iii) If $f\colon (X,\tau) \to (Y,\sigma)$ is a somewhat $\hat{\Omega}$-continuous function and $g\colon (Y,\sigma) \to (Z,\eta)$ is surjective completely continuous, then $g \circ f\colon (X,\tau) \to (Z,\eta)$ is a somewhat $\hat{\Omega}$-continuous.

(iv) If $f\colon (X,\tau) \to (Y,\sigma)$ is a somewhat $\hat{\Omega}$-continuous function and $g\colon (Y,\sigma) \to (Z,\eta)$ is surjective perfectly continuous, then $g \circ f\colon (X,\tau) \to (Z,\eta)$ is a somewhat $\hat{\Omega}$-continuous.

Proof. (i) Suppose that U is any open set in Z such that $(g \circ f)^{-1}(U) \neq \emptyset$.Since g is surjective continuous, $g^{-1}(U)$ is a non empty open set in Y. Since f is a somewhat $\hat{\Omega}$-continuous,there exists a non empty $\hat{\Omega}$-open set Vin X such that $V \subseteq (g \circ f)^{-1}(U)$. Therefore, $g \circ f$ is somewhat $\hat{\Omega}$-continuous.

(ii) Suppose that U is any open set in Z such that $(g{\circ}f)^{-1}(U) \neq \emptyset$. Since g is surjective super continuous,$g^{-1}(U)$ is a non empty δ-open and hence open set in Y. Since f is a somewhat $\hat{\Omega}$-continuous, there exists a non empty $\hat{\Omega}$-open set V in X such that $V \subseteq (g \circ f)^{-1}(U)$. Therefore, $g \circ f$ is somewhat $\hat{\Omega}$-continuous.

- •The proofs of **(iii)** and **(iv)** are similar to **(ii)**.

□

Theorem 5.14. *Suppose that A is any open pre closed and $\hat{\Omega}$-dense set in a topological space (X,τ). If $f\colon (X,\tau) \to (Y,\sigma)$ is somewhat $\hat{\Omega}$-continuous, then $f \mid A : (A, \tau \mid_A) \to (Y,\sigma)$ is also somewhat $\hat{\Omega}$-continuous on the subspace $(A, \tau \mid A)$.*

Proof. Suppose that $U \in O(Y)$ such that $(f \mid A)^{-1}(U) \neq A$.If $f^{-1}(U) = X$,then $f^{-1}(U) \cap A = X \cap A = A$,a contradiction to $(f \mid A)^{-1}(U) \neq A$. Therefore, $f^{-1}(U) \neq X$. By hypothesis,there exists a non empty set $V \in \hat{\Omega}O(X)$ such that $V \subseteq f^{-1}(U)$. Then, $V \cap A \subseteq f^{-1}(U) \cap A = (f \mid A)^{-1}(U)$. Since A is $\hat{\Omega}$-dense set in $X, A \cap V \neq \emptyset$. By[7] theorem 6.8, $A \cap V$ is $\hat{\Omega}$-open in the subspace $(A, \tau \mid A)$. Therefore,$f \mid A$ is somewhat $\hat{\Omega}$-continuous on the subspace $(A, \tau \mid A)$. □

Theorem 5.15. *Suppose that (X,τ) and (Y,σ) are any two topological spaces and suppose $X = A \cup B$, where A and B are both δ open and pre closed in X. If $f\colon (X,\tau) \to (Y,\sigma)$ is any function such that $f \mid A$ and $f \mid B$ are somewhat $\hat{\Omega}$-continuous functions,then f is a somewhat $\hat{\Omega}$-continuous.*

Proof. Let U be any open set in Y such that $f^{-1}(U) \neq \emptyset$. If both $(f \mid_A)^{-1}(U) = f^{-1}(U) \cap A, (f \mid_B)^{-1}(U) = f^{-1}(U) \cap B$ are empty, then $f^{-1}(U) = \emptyset$, a contradiction.Therefore, the possible cases are either $f^{-1}(U) \cap A \neq \emptyset$ or $f^{-1}(U) \cap B \neq \emptyset$ or both $f^{-1}(U) \cap A$ and $f^{-1}(U) \cap B$ are nonempty. It is enough to prove only for the case either $f^{-1}(U) \cap A \neq \emptyset$ or $f^{-1}(U) \cap B \neq \emptyset$.Then automatically second one follows.
Suppose that either $f^{-1}(U) \cap A \neq \emptyset$ or $f^{-1}(U) \cap B \neq \emptyset$. If $f^{-1}(U) \cap A \neq \emptyset$, by hypothesis,there exists a non empty $\hat{\Omega}$-open set $V \in (A, \tau \mid_A)$ such that $V \subseteq f^{-1}(U) \cap A \subseteq f^{-1}(U)$. By[7] theorem 6.9,$V$ is $\hat{\Omega}$-open in X.Therefore.f is a somewhat $\hat{\Omega}$-continuous. □

Definition 5.16. *Let τ and σ are two topologies on a set X. Then τ is said to be equivalent (resp.$\hat{\Omega}$-equivalent) to σ if for every non empty $U \in \tau$ there exists an non empty open (resp.$\hat{\Omega}$-open) set V in (X,σ) such that $V \subseteq U$ and if for every non empty $U \in \sigma$ there exists an non empty open (resp.$\hat{\Omega}$-open) set V in (X,τ) such that $V \subseteq U$.*

Theorem 5.17. *Let τ^* be a topology on X which is $\hat{\Omega}$-equivalent to a topology τ on X. If $f\colon (X,\tau) \to (Y,\sigma)$ is a somewhat continuous, then $f : (X, \tau^*) \to (Y,\sigma)$ is somewhat $\hat{\Omega}$-continuous.*

Proof. Suppose that U is any open set in (X,σ) such that $f^{-1}(U) \neq \emptyset$. Since f is somewhat continuous, there exists a non empty open set V in (X,τ) such that $V \subseteq f^{-1}(U)$. Since τ^* $\hat{\Omega}$-equivalent to τ, there exists $\hat{\Omega}$-open set V_1 in (X,τ^*) such that $V_1 \subseteq f^{-1}(U)$. Hence f is somewhat $\hat{\Omega}$-continuous. □

Theorem 5.18. *Let σ^* be a topology on Y which is equivalent to a topology σ on Y. If $f\colon (X,\tau) \to (Y,\sigma)$ is a somewhat $\hat{\Omega}$-continuous surjective function, then $f : (X,\tau) \to (Y,\sigma^*)$ is somewhat $\hat{\Omega}$-continuous.*

Proof. Suppose that U is any open set in (Y,σ^*) such that $f^{-1}(U) \neq \emptyset$. Since σ^* is equivalent to σ, there exists a non empty open set V in (Y,σ) such that $V \subseteq U$. Since $f\colon (X,\tau) \to (Y,\sigma)$ is surjective, $f^{-1}(V) \neq \emptyset$.Since $f\colon (X,\tau) \to (Y,\sigma)$ is somewhat $\hat{\Omega}$-continuous, there exists a non empty $\hat{\Omega}$-open set G in (X,τ) such that $G \subseteq f^{-1}(V)$. Hence $f : (X,\tau) \to (Y,\sigma^*)$ is somewhat $\hat{\Omega}$-continuous. □

6 $\hat{\Omega}$-irresolvable spaces

In this section we establish the definition of $\hat{\Omega}$-resolvable spaces and it's properties.

Definition 6.1. *A space (X, τ) is said to be $\hat{\Omega}$-resolvable,if there exists a subset A of X such that both A and A^c are $\hat{\Omega}$-dense in X.Otherwise it is known as $\hat{\Omega}$-irresolvable space.*

Example 6.2. *Example 3.2 is $\hat{\Omega}$-irresolvable space.*

Let us prove a characterization of $\hat{\Omega}$-resolvable space.

Theorem 6.3. *A space (X, τ) is $\hat{\Omega}$-resolvable if and only if it has a pair of disjoint $\hat{\Omega}$ dense sets in X.*

Proof. **Necessity-** Suppose that X is $\hat{\Omega}$-resolvable. Therefore,there exists a subset A of X such that both A and A^c are $\hat{\Omega}$-dense in X.If we define $B = A^c$, then we get a pair of disjoint $\hat{\Omega}$-dense sets in X.

Sufficiency- By hypothesis,we can choose a disjoint pair of $\hat{\Omega}$-dense sets namely A and B in X. Then $\hat{\Omega}cl(A) = \hat{\Omega}cl(B) = X$ such that $A \subseteq B^c$ or $B \subseteq A^c$. If $A \subseteq B^c$, then by [7] theorem 5.3 (ii), $\hat{\Omega}cl(A) \subseteq \hat{\Omega}cl(B^c)$. Then $X \subseteq \hat{\Omega}cl(B^c)$ and hence $X = \hat{\Omega}cl(B^c)$. Therefore,we have a subset B in X such that B and B^c are both $\hat{\Omega}$-dense in X.If $B \subseteq A^c$, then $\hat{\Omega}cl(B) \subseteq \hat{\Omega}cl(A^c)$. Then $X \subseteq \hat{\Omega}cl(B^c)$ and hence $X = \hat{\Omega}cl(B^c)$. Therefore,we have a subset A in X such that A and A^c are both $\hat{\Omega}$-dense in X. Therefore, X is $\hat{\Omega}$-resolvable. □

Theorem 6.4. *A space (X, τ) is $\hat{\Omega}$-irresolvable if and only if $\hat{\Omega}int(A) \neq \emptyset$ for every $\hat{\Omega}$-dense set A in X.*

Proof. **Necessity-** Suppose that A is any $\hat{\Omega}$-dense set in X.By hypothesis,$\hat{\Omega}cl(A^c) \neq X$ and hence $(\hat{\Omega}int(A))^c \neq \emptyset^c$. Therefore, $\hat{\Omega}int(A) \neq \emptyset$.

Sufficiency- Suppose that X is $\hat{\Omega}$-resolvable. Then,there exists a subset A of X such that both A and A^c are $\hat{\Omega}$-dense in X. Then $\hat{\Omega}cl(A^c) = X$ and hence $[\hat{\Omega}int(A)]^c = [\emptyset]^c$. Therefore, $\hat{\Omega}int(A) = \emptyset$,a contradiction. □

Theorem 6.5. *If $X = A \cup B$, where A and B are such that $\hat{\Omega}int(A) = \emptyset, \hat{\Omega}int(B) = \emptyset$. Then X is $\hat{\Omega}$-resolvable.*

Proof. Given that $X = A \cup B, A$ and B are such that $\hat{\Omega}int(A) = \emptyset, \hat{\Omega}int(B) = \emptyset$. Therefore,$\hat{\Omega}cl(A^c) = X, \hat{\Omega}cl(B^c) = X$.Moreover,$X \setminus (A \cup B) = \emptyset$, or $[X \setminus A] \cap [X \setminus B] = \emptyset$. Then $X \setminus A \subseteq [X \setminus B]^c$. Therefore, $\hat{\Omega}cl(A^c) \subseteq \hat{\Omega}cl(B)$ and hence $X \subseteq \hat{\Omega}cl(B)$.Thus we get a subset B in X such that both B and B^c are $\hat{\Omega}$-dense in X.Therefore, X is $\hat{\Omega}$-resolvable. □

Remark 6.6. *The above theorem can be extended to any finite number. That is, if $X = \bigcup_{i=1}^{i=n} A_i$ for any finite number of empty $\hat{\Omega}$ interior sets $A_1, A_2, ...A_n$, then X is $\hat{\Omega}$-resolvable.*

Theorem 6.7. *If $f: (X, \tau) \to (Y, \sigma)$ is a somewhat $\hat{\Omega}$-open mapping on a irresolvable space X, then Y is $\hat{\Omega}$ irresolvable space.*

Proof. Suppose that A is any non empty $\hat{\Omega}$ dense set in Y. Assume that $\hat{\Omega}(int(A)) = \emptyset$. Then $\hat{\Omega}cl(Y \setminus A) = Y$. Since f is somewhat $\hat{\Omega}$-open by theorem 4.13,$f^{-1}(Y \setminus A)$ is dense in X. Then,$cl(f^{-1}(Y \setminus A)) = X$ and hence $cl(X \setminus f^{-1}(A)) = X$. Thus,$int(f^{-1}(A)) = \emptyset$.Again by hypothesis, $f^{-1}(A)$ is a dense set in X with a empty interior, a contradiction to X is irresolvable. Therefore, our assumption is wrong and hence $\hat{\Omega}(int(A)) \neq \emptyset$. By theorem 6.4,$Y$ is a $\hat{\Omega}$ irresolvable space. □

Theorem 6.8. *Let Y be irresolvable space.If $f: (X, \tau) \to (Y, \sigma)$ is a somewhat $\hat{\Omega}$-continuous bijective mapping,then X is $\hat{\Omega}$ irresolvable space.*

Proof. Suppose that A is any non empty $\hat{\Omega}$ dense set in X. Assume that $\hat{\Omega}(int(A)) = \emptyset$. Then $\hat{\Omega}cl(X \setminus A) = X$. Since f is somewhat $\hat{\Omega}$-continuous by theorem 5.10 (ii), $f(X \setminus A)$ is dense in Y. Then,$cl(f(X \setminus A)) = Y$. Since f is bijective, $cl(Y \setminus f(A)) = Y$. Thus, $int(f(A)) = \emptyset$. Again by hypothesis, $f(A)$ is a dense set in Y with a empty interior, a contradiction to Y is irresolvable. Therefore, our assumption is wrong and hence $\hat{\Omega}(int(A)) \neq \emptyset$. By theorem 6.4, X is a $\hat{\Omega}$ irresolvable space. □

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