On boundary value problems for fractional integro-differential equations in Banach spaces

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Abstract

This paper aims to study the existence and uniqueness of solutions of fractional integro-differential equations in Banach spaces by applying a new generalized singular type Gronwall's inequality, fixed point theorems and Hölder inequality. Example is provided to illustrate the main results.

Keywords: Fractional integro-differential equations, boundary value problems, existence and uniqueness, generalized singular type Gronwall's inequality, fixed point theorems.

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1 Introduction

This paper deals with the existence and uniqueness of solutions of boundary value problems (for short BVP) for fractional integro-differential equations given by

$$\begin{cases} {}^c D^\alpha x(t) = f\big(t, x(t), (Sx)(t)\big), t \in J = [0, T], \alpha \in (0, 1], \\ ax(0) + bx(T) = c, \end{cases} \tag{1.1}$$

where ${}^c D^\alpha$ is the Caputo fractional derivative of order α, $f : J \times X \times X \to X$ is a given function satisfying some assumptions that will be specified later and a, b, c are real numbers with $a + b \neq 0$ and S is a nonlinear integral operator given by $(Sx)(t) = \int_0^t k(t, s, x(s)) ds$, where $k \in C(J \times J \times X, X)$.

The ordinary differential equations is considered the basis of the fractional differential equations. In the last few decades, fractional order models are found to be more adequate than integer order models for some real world problems. For more details about fractional calculus and its applications we refer the reader to the monographs of Hilfer [6], Kilbas et al. [8], Miller and Ross [9], Podlubny [10], Samko et al. [11] and the references given therein. Recently, some fractional differential equations and optimal controls in Banach spaces were studied by Balachandran and Park [2], El-Borai [3], Henderson and Ouahab [4], Hernandez et al. [5], Wang et al. [14] and Wang et al. [15, 16]. Very recently, Karthikeyan and Trujillo [7] and Wang et al. [13] have extended the work in [1] from real line $\mathbb{R}$ to the abstract Banach space X by using more general assumptions on the nonlinear function f. Our attempt is to generalize the results proved in [1, 7, 13].

This paper is organized as follows. In Section 2, we set forth some preliminaries. Section 3 introduces a new generalized singular type Gronwall inequality to establish the estimate for priori bounds. In Section 4, we prove our main results by applying Banach contraction principle and Schaefer's fixed point theorem. Finally, in Section 5, application of the main results is exhibited.

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2 Preliminaries

Before proceeding to the statement of our main results, we set forth some preliminaries. Let the Banach space of all continuous functions from J into X with the supremum norm $\|x\|_\infty := \sup\{\|x(t)\| : t \in J\}$ is denoted by $C(J,X)$. For measurable functions $m : J \to \mathbb{R}$, define the norm $\|m\|_{L^p(J,\mathbb{R})} = \left(\int_J |m(t)|^p dt\right)^{\frac{1}{p}}, 1 \leq p < \infty$, where $L^p(J,\mathbb{R})$ the Banach space of all Lebesgue measurable functions m with $\|m\|_{L^p(J,\mathbb{R})} < \infty$.

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a suitable function h is defined by*

$$I_{a+}^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds,$$

where $a \in \mathbb{R}$ and Γ is the Gamma function.

Definition 2.2. *For a suitable function h given on the interval $[a,b]$, the Riemann-Liouville fractional derivative of order $\alpha > 0$ of h, is defined by*

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s) ds,$$

where $n = [\alpha] + 1, [\alpha]$ denotes the integer part of α.

Definition 2.3. *For a suitable function h given on the interval $[a,b]$, the Caputo fractional order derivative of order $\alpha > 0$ of h, is defined by*

$$({}^c D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds,$$

where $n = [\alpha] + 1, [\alpha]$ denotes the integer part of α.

Lemma 2.1. *([8, 17]) Let $\alpha > 0$; then the differential equation ${}^c D^\alpha h(t) = 0$, has the following general solution $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, where $c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n-1$, where $n = [\alpha] + 1$.*

Lemma 2.2. *([8, 17]) Let $\alpha > 0$; then*

$$I^\alpha({}^c D^\alpha h)(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n-1$, where $n = [\alpha] + 1$.

Definition 2.4. *A function $x \in C^1(J,X)$ is said to be a solution of the fractional BVP (1.1) if x satisfies the equation ${}^c D^\alpha x(t) = f(t, x(t), (Sx)(t))$ a.e. on J, and the condition $ax(0) + bx(T) = c$.*

For the existence of solutions for the fractional BVP (1.1), we need the following auxiliary lemma.

Lemma 2.3. *Let $\overline{f} : J \to X$ be continuous. A function $x \in C(J,X)$ is solution of the fractional integral equation*

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \overline{f}(s) ds - \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \overline{f}(s) ds - c\right], \tag{2.2}$$

if and only if x is a solution of the following fractional BVP

$$\begin{cases} {}^c D^\alpha x(t) = \overline{f}(t), t \in J = [0,T], \alpha \in (0,1], \\ ax(0) + bx(T) = c. \end{cases} \tag{2.3}$$

Proof. Assume that x satisfies fractional BVP (2.3); then by using Lemma 2.2 and Def. 2.1, we get

$$x(t) + c_0 = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \overline{f}(s) ds,$$

where $c_0 \in \mathbb{R}$, that is:

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \overline{f}(s) ds - c_0. \tag{2.4}$$

By applying boundary condition $ax(0) + bx(T) = c$, we have

$$c_0 = \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \overline{f}(s) ds - \frac{c}{a+b}.$$

Now, by substituting the value of c_0 in (2.4), we obtain

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \overline{f}(s) ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \overline{f}(s) ds + \frac{c}{a+b}.$$

Conversely, it is clear that if x satisfies fractional integral equation (2.2), then fractional BVP (2.3) is also satisfied.   □

As a consequence of lemma 2.3, we have the following result which is useful in what follows.

Lemma 2.4. *Let $f : J \times X \times X \to X$ be continuous function. Then, $x \in C(J, X)$ is a solution of the fractional integral equation*

$$\begin{aligned}
x(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\big(s, x(s), (Sx)(s)\big) ds \\
& - \frac{1}{a+b} \Big[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f\big(s, x(s), (Sx)(s)\big) ds - c \Big],
\end{aligned} \tag{2.5}$$

if and only if x is solution of the fractional BVP (1.1).

Lemma 2.5. *(Bochner theorem) A measurable function $f : J \to X$ is Bochner integrable if $\|f\|$ is Lebesgue integrable.*

Lemma 2.6. *(Mazur theorem, [12]) Let X be a Banach space. If $U \subset X$ is relatively compact, then $\mathrm{conv}(U)$ is relatively compact and $\overline{\mathrm{conv}}(U)$ is compact.*

Lemma 2.7. *(Ascoli-Arzela theorem) Let $S = \{s(t)\}$ is a function family of continuous mappings $s : [a, b] \to X$. If S is uniformly bounded and equicontinuous, and for any $t^* \in [a, b]$, the set $\{s(t^*)\}$ is relatively compact, then, there exists a uniformly convergent function sequence $\{s_n(t)\}(n = 1, 2, \ldots, t \in [a, b])$ in S.*

Lemma 2.8. *(Schaefer's fixed point theorem) Let $F : X \to X$ be a completely continuous operator. If the set $E(F) = \{x \in X : x = \eta Fx$ for some $\eta \in [0, 1]\}$ is bounded, then, F has fixed points.*

3   A generalized singular type Gronwall's inequality

Before dealing with the main reslts, we need to introduce a new generalized singular Gronwall type inequality with mixed type singular integral operator.

We, first, state a generalized Gronwall inequality from [15].

Lemma 3.9. *(Lemma 3.2, [15]) Let $x \in C(J, X)$ satisfies the following inequality:*

$$\begin{aligned}
\|x(t)\| \leq {} & a + b \int_0^t \|x(\theta)\|^{\lambda_1} d\theta + c \int_0^T \|x(\theta)\|^{\lambda_2} d\theta + d \int_0^t \|x_\theta\|_B^{\lambda_3} d\theta \\
& + e \int_0^T \|x_\theta\|_B^{\lambda_4} d\theta, t \in J,
\end{aligned}$$

where $\lambda_1, \lambda_3 \in [0, 1], \lambda_2, \lambda_4 \in [0, 1), a, b, c, d, e \geq 0$ are constants and $\|x_\theta\|_B = \sup_{0 \leq s \leq \theta} \|x(s)\|$. Then there exists a constant $L > 0$ such that

$$\|x(t)\| \leq L.$$

Using the above generalized Gronwall inequality, we can obtain the following new generalized singular type Gronwall inequality.

Lemma 3.10. *Let $x \in C(J, X)$ satisfies the following inequality:*

$$
\begin{aligned}
\|x(t)\| \quad \leq \quad & a + b \int_0^t (t-s)^{\alpha-1} \|x(s)\|^\lambda ds + c \int_0^T (T-s)^{\alpha-1} \|x(s)\|^\lambda ds \\
& + d \int_0^t (t-s)^{\alpha-1} \|x_s\|_B^\lambda ds + e \int_0^T (T-s)^{\alpha-1} \|x_s\|_B^\lambda ds,
\end{aligned}
$$
(3.6)

where $\alpha \in (0,1], \lambda \in [0, 1 - \frac{1}{p})$ for some $1 < p < \frac{1}{1-\alpha}, \|x_s\|_B = \sup_{0 \leq \tau \leq s} \|x(\tau)\|$ and $a, b, c, d, e \geq 0$ are constants. Then, there exists a constant $L > 0$, such that

$$\|x(t)\| \leq L.$$

Proof. Let

$$
y(t) = \begin{cases} 1, & \|x(t)\| \leq 1, \\ x(t), & \|x(t)\| > 1. \end{cases}
$$

Using (3.6) and Hölder inequality, we get

$$
\begin{aligned}
\|x(t)\| \leq{}& \|y(t)\| \\
\leq{}& (a+1) + b \int_0^t (t-s)^{\alpha-1} \|y(s)\|^\lambda ds + c \int_0^T (T-s)^{\alpha-1} \|y(s)\|^\lambda ds \\
& + d \int_0^t (t-s)^{\alpha-1} \|y_s\|_B^\lambda ds + e \int_0^T (T-s)^{\alpha-1} \|y_s\|_B^\lambda ds \\
\leq{}& (a+1) + b \left(\int_0^t (t-s)^{p(\alpha-1)} ds \right)^{\frac{1}{p}} \left(\int_0^t \|y(s)\|^{\frac{\lambda p}{p-1}} ds \right)^{\frac{p-1}{p}} \\
& + c \left(\int_0^T (T-s)^{p(\alpha-1)} ds \right)^{\frac{1}{p}} \left(\int_0^T \|y(s)\|^{\frac{\lambda p}{p-1}} ds \right)^{\frac{p-1}{p}} \\
& + d \|y_s\|_B^\lambda \int_0^t (t-s)^{\alpha-1} ds + e \|y_s\|_B^\lambda \int_0^T (T-s)^{\alpha-1} ds \\
\leq{}& (a+1) + b \left(\frac{t^{p(\alpha-1)+1}}{p(\alpha-1)+1} \right)^{\frac{1}{p}} \int_0^t \|y(s)\|^{\frac{\lambda p}{p-1}} ds \\
& + c \left(\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1} \right)^{\frac{1}{p}} \int_0^T \|y(s)\|^{\frac{\lambda p}{p-1}} ds \\
& + d \|y_s\|_B^\lambda \frac{t^\alpha}{\alpha} + e \|y_s\|_B^\lambda \frac{T^\alpha}{(\alpha)} \\
\leq{}& (a+1) + d \|y_s\|_B^\lambda \frac{T^\alpha}{\alpha} + e \|y_s\|_B^\lambda \frac{T^\alpha}{(\alpha)} \\
& + b \left(\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1} \right)^{\frac{1}{p}} \int_0^t \|y(s)\|^{\frac{\lambda p}{p-1}} ds \\
& + c \left(\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1} \right)^{\frac{1}{p}} \int_0^T \|y(s)\|^{\frac{\lambda p}{p-1}} ds,
\end{aligned}
$$

where $0 < \frac{\lambda p}{p-1} < 1$.

Hence, by lemma 3.9 there exists a constant $L > 0$, such that $\|x(t)\| \leq L$.

$\square$

4 Main results

For convenience, we list hypotheses that will be used in our further discussion.

● (H1) The function $f : J \times X \times X \rightarrow X$ is measurable with respect to t on J and is continuous with respect to x on X.

● (H2) There exists a constant $\alpha_1 \in (0, \alpha)$ and real-valued functions $m_1(t), m_2(t) \in L^{\frac{1}{\alpha_1}}(J, \mathbb{R})$, such that

$$\|f(t, x(t), (Sx)(t)) - f(t, y(t), (Sy)(t))\| \leq m_1(t) (\|x(t) - y(t)\| + \|Sx(t) - Sy(t)\|),$$

$$\|k(t, s, x(s)) - k(t, s, y(s))\| \leq m_2(t)\|x(s) - y(s)\|,$$

for each $s \in [0, t], t \in J$ and all $x, y \in X$.

● (H3) There exists a constant $\alpha_2 \in (0, \alpha)$ and real-valued function $h(t) \in L^{\frac{1}{\alpha_2}}(J, \mathbb{R})$, such that $\|f(t, x(t), (Sx)(t))\| \leq h(t)$, for each $t \in J$, and all $x \in X$.

For brevity, let $M = \|m_1 + m_1 m_2 T\|_{L^{\frac{1}{\alpha_1}}(J, \mathbb{R})}$ and $H = \|h\|_{L^{\frac{1}{\alpha_2}}(J, \mathbb{R})}$.

● (H4) There exist constants $\lambda \in [0, 1 - \frac{1}{p})$ for some $1 < p < \frac{1}{1-\alpha}$ and $N_f, N_k > 0$, such that

$$\|f(t, x(t), (Sx)(t))\| \leq N_f(1 + \|x(t)\|^\lambda + \|(Sx)(t)\|),$$

$$\|k(t, s, x(s))\| \leq N_k(1 + \|x(s)\|^\lambda),$$

for each $s \in [0, t], t \in J$ and all $x \in X$.

● (H5) For every $t \in J$, the set
$K_1 = \{(t - s)^{\alpha-1} f(s, x(s), (Sx)(s)) : x \in C(J, X), s \in [0, t]\}$ is relatively compact.

Now, we are in position to deal with our main results.

Theorem 4.1. *Assume that (H1)-(H3) hold. If*

$$\Omega_{\alpha, T} = \frac{M}{\Gamma(\alpha)} \frac{T^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} \left(1 + \frac{|b|}{|a+b|}\right) < 1. \tag{4.7}$$

Then, the fractional BVP (1.1) has a unique solution on J.

Proof. By making use of hypothesis (H3) and Hölder inequality, for each $t \in J$, we have

$$\int_0^t \|(t-s)^{\alpha-1} f(s, x(s), (Sx)(s))\| ds \leq \int_0^t (t-s)^{\alpha-1} \|f(s, x(s), (Sx)(s))\| ds$$

$$\leq \int_0^t (t-s)^{\alpha-1} h(s) ds$$

$$\leq \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}} ds\right)^{1-\alpha_2} \left(\int_o^t (h(s))^{\frac{1}{\alpha_2}} ds\right)^{\alpha_2}$$

$$\leq H \left(\left[\frac{-(t-s)^{\frac{\alpha-1}{1-\alpha_2}+1}}{\frac{\alpha-1}{1-\alpha_2}+1}\right]_0^t\right)^{1-\alpha_2} \leq H \frac{T^{\alpha-\alpha_2}}{(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}.$$

Thus, $\|(t-s)^{\alpha-1} f(s, x(s), (Sx)(s))\|$ is Lebesgue integrable with respect to $s \in [0, t]$ for all $t \in J$ and $x \in C(J, X)$. Then, $(t-s)^{\alpha-1} f(s, x(s), (Sx)(s))$ is Bochner integrable with respect to $s \in [0, t]$ for all $t \in J$ due to lemma 2.5.

Hence, the fractional BVP (1.1) is equivalent to the following fractional integral equation

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), (Sx)(s)) ds$$

$$- \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x(s), (Sx)(s)) ds - c\right], \tag{4.8}$$

Now, let $B_r = \{x \in C(J, X) : \|x\|_\infty \leq r\}$, where

$$r \geq \frac{HT^{\alpha-\alpha_2}}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{|b|}{|a+b|} \times \frac{HT^{\alpha-\alpha_2}}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{|c|}{|a+b|}. \tag{4.9}$$

Define the operator F on B_r as follows:

$$(F(x))(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), (Sx)(s)) ds$$
$$- \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x(s), (Sx)(s)) ds - c \right], t \in J. \tag{4.10}$$

Clearly, the solution of the fractional BVP (1.1) is the fixed point of the operator F on B_r. We shall use the Banach contraction principle to prove that F has a fixed point. The proof is divided into two steps.

Step 1. $F(x) \in B_r$ for every $x \in B_r$.
For every $x \in B_r$ and $\delta > 0$, by (H3) and Hölder inequality, we have

$$\| (F(x))(t+\delta) - (F(x))(t) \|$$
$$\leq \left\| \frac{1}{\Gamma(\alpha)} \int_0^{t+\delta} (t+\delta-s)^{\alpha-1} f(s, x(s), (Sx)(s)) ds \right.$$
$$\left. - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), (Sx)(s)) ds \right\|$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t \left[(t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1} \right] \| f(s, x(s), (Sx)(s)) \| ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_t^{t+\delta} (t+\delta-s)^{\alpha-1} \| f(s, x(s), (Sx)(s)) \| ds$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t [(t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1}] h(s) ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_t^{t+\delta} (t+\delta-s)^{\alpha-1} h(s) ds$$
$$\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t+\delta-s)^{\frac{\alpha-1}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_0^t (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2}$$
$$- \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_0^t (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2}$$
$$+ \frac{1}{\Gamma(\alpha)} \left(\int_t^{t+\delta} (t+\delta-s)^{\frac{\alpha-1}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_t^{t+\delta} (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2}$$
$$\leq \frac{H}{\Gamma(\alpha)} \left(\frac{-\delta^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}} + \frac{(t+\delta)^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}} \right)^{1-\alpha_2} - \frac{H}{\Gamma(\alpha)} \left(\frac{t^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}} \right)^{1-\alpha_2} + \frac{H}{\Gamma(\alpha)} \left(\frac{\delta^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}} \right)^{1-\alpha_2}.$$

It is obvious that the right-hand side of the above inequality tends to zero as $\delta \to 0$. Therefore, F is continuous on J, that is, $F(x) \in C(J, X)$. Moreover, for $x \in B_r$ and all $t \in J$, by using (4.9), we have

$$\| (F(x))(t) \| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \| f(s, x(s), (Sx)(s)) \| ds$$
$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \| f(s, x(s), (Sx)(s)) \| ds + \frac{|c|}{|a+b|}$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds$$
$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s) ds + \frac{|c|}{|a+b|}$$
$$\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_0^t (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2}$$
$$+ \frac{|b|}{(|a+b|)\Gamma(\alpha)} \left(\int_0^T (T-s)^{\frac{\alpha-1}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_0^T (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2} + \frac{|c|}{|a+b|}$$

$$\leq \frac{HT^{\alpha-\alpha_2}}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{|b|}{|a+b|} \times \frac{HT^{\alpha-\alpha_2}}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{|c|}{|a+b|}$$

$$\leq r.$$

Thus, $\|F(x)\|_\infty \leq r$ and we conclude that for all $x \in B_r$, $F(x) \in B_r$, that is, $F : B_r \to B_r$.

Step 2. F is contraction mapping on B_r.
For $x, y \in B_r$ and any $t \in J$, by using (4.7), (H2) and Hölder inequality, we have

$$\|(F(x))(t) - (F(y))(t)\|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x(s), (Sx)(s)) - f(s, y(s), (Sy)(s))\| \, ds$$

$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|f(s, x(s), (Sx)(s)) - f(s, y(s), (Sy)(s))\| \, ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m_1(s) \big(\|x(s) - y(s)\| + \|(Sx)(s) - (Sy)(s)\|\big) ds$$

$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} m_1(s) \big(\|x(s) - y(s)\| + \|(Sx)(s) - (Sy)(s)\|\big) ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m_1(s)$$

$$\times \left(\|x(s) - y(s)\| + \int_0^s \|k(s, \tau, x(\tau)) - k(s, \tau, y(\tau))\| \, d\tau \right) ds$$

$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} m_1(s)$$

$$\times \left(\|x(s) - y(s)\| + \int_0^s \|k(s, \tau, x(\tau)) - k(s, \tau, y(\tau))\| \, d\tau \right) ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m_1(s)$$

$$\times \left(\|x(s) - y(s)\| + \int_0^s m_2(s) \|x(\tau) - y(\tau)\| \, d\tau \right) ds$$

$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} m_1(s)$$

$$\times \left(\|x(s) - y(s)\| + \int_0^s m_2(s) \|x(\tau) - y(\tau)\| \, d\tau \right) ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m_1(s) \big(\|x - y\|_\infty + m_2(s)T\|x - y\|_\infty\big) ds$$

$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} m_1(s) \big(\|x - y\|_\infty + m_2(s)T\|x - y\|_\infty\big) ds$$

$$\leq \frac{\|x - y\|_\infty}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}} ds \right)^{1-\alpha_1} \left(\int_0^t (m_1(s) + m_1(s)m_2(s)T)^{\frac{1}{\alpha_1}} ds \right)^{\alpha_1}$$

$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \left(\int_0^T (T-s)^{\frac{\alpha-1}{1-\alpha_1}} ds \right)^{1-\alpha_1}$$

$$\times \left(\int_0^T (m_1(s) + m_1(s)m_2(s)T)^{\frac{1}{\alpha_1}} ds \right)^{\alpha_1}$$

$$\leq \frac{M\|x - y\|_\infty}{\Gamma(\alpha)} \left(\frac{t^{\frac{\alpha-\alpha_1}{1-\alpha_1}}}{\frac{\alpha-\alpha_1}{1-\alpha_1}} \right)^{1-\alpha_1} + \frac{M|b|\|x - y\|_\infty}{|a+b|\Gamma(\alpha)} \left(\frac{T^{\frac{\alpha-\alpha_1}{1-\alpha_1}}}{\frac{\alpha-\alpha_1}{1-\alpha_1}} \right)^{1-\alpha_1}$$

$$\leq \left[\frac{M}{\Gamma(\alpha)} \frac{T^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} \left(1 + \frac{|b|}{|a+b|} \right) \right] \|x - y\|_\infty.$$

$$= \Omega_{\alpha,T} \|x - y\|_\infty.$$

Thus, we have

$$\|F(x) - F(y)\|_\infty \le \Omega_{\alpha,T} \|x - y\|_\infty.$$

Since $\Omega_{\alpha,T} < 1$, F is contraction. By Banach contraction principle, we can deduce that F has a unique fixed point which is the unique solution of the fractional BVP (1.1). □

Our second main result is based on the well known Schaefer's fixed point theorem.

Theorem 4.2. *Assume that(H1), (H4) and (H5) hold. Then the fractional BVP* (1.1) *has at least one solution on J .*

Proof. Transform the fractional BVP (1.1) into a fixed point problem. Consider the operator $F : C(J, X) \to C(J, X)$ defined as (4.10). It is obvious that F is well defined due to (H1), Hölder inequality and the lemma 2.5.

For the sake of convenience, we subdivide the proof into several steps.
Step 1. F is continuous operator.
Let $\{x_n\}$ be a sequence such that $x_n \to x$ in C(J,X). Then for each $t \in J$, we have

$$\|(F(x_n))(t) - (F(x))(t)\|$$
$$\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| f(s, x_n(s), (Sx_n)(s)) - f(s, x(s), (Sx)(s)) \right\| ds$$
$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left\| f(s, x_n(s), (Sx_n)(s)) - f(s, x(s), (Sx)(s)) \right\| ds$$
$$\le \left\| f(., x_n(.), (Sx_n)(.)) - f(., x(.), (Sx)(.)) \right\|_\infty \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds$$
$$+ \left\| f(., x_n(.), (Sx_n)(.)) - f(., x(.), (Sx)(.)) \right\|_\infty \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds$$
$$\le \left\| f(., x_n(.), (Sx_n)(.)) - f(., x(.), (Sx)(.)) \right\|_\infty \frac{t^\alpha}{\alpha\Gamma(\alpha)}$$
$$+ \left\| f(., x_n(.), (Sx_n)(.)) - f(., x(.), (Sx)(.)) \right\|_\infty \frac{|b|}{|a+b|\Gamma(\alpha)} \frac{T^\alpha}{\alpha}$$
$$\le \frac{T^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|} \right) \left\| f(., x_n(.), (Sx_n)(.)) - f(., x(.), (Sx)(.)) \right\|_\infty.$$

Taking supremum, we get

$$\|Fx_n - Fx\|_\infty$$
$$\le \frac{T^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|} \right) \left\| f(., x_n(.), (Sx_n)(.)) - f(., x(.), (Sx)(.)) \right\|_\infty,$$

since f is continuous, we have

$$\|Fx_n - Fx\|_\infty \to 0 \text{ as } n \to \infty.$$

Therefore, F is continuous operator.

Step 2. F maps bounded sets into bounded sets in $C(J, X)$.
Indeed, it is enough to show that for any $\eta^* > 0$, there exists a $l > 0$ such that for each

$x \in B_{\eta^*} = \{x \in C(J, X) : \|x\|_\infty \leq \eta^*\}$, we have $\|Fx\|_\infty \leq l$.
For each $t \in J$, by (H4), we get

$$\|(F(x))(t)\| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x(s), (Sx)(s))\| ds$$

$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|f(s, x(s), (Sx)(s))\| ds + \frac{|c|}{|a+b|}$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N_f (1 + \|x(s)\|^\lambda + \|(Sx)(s)\|) ds$$

$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} N_f (1 + \|x(s)\|^\lambda + \|(Sx)(s)\|) ds + \frac{|c|}{|a+b|}$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N_f \left(1 + \|x(s)\|^\lambda + \int_0^s \|k(s, \tau, x(\tau))\| d\tau\right) ds$$

$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}$$

$$\times N_f \left(1 + \|x(s)\|^\lambda + \int_0^s \|k(s, \tau, x(\tau))\| d\tau\right) ds + \frac{|c|}{|a+b|}$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N_f \left(1 + \|x(s)\|^\lambda + \int_0^s N_k(1 + \|x(\tau)\|^\lambda) d\tau\right) ds$$

$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}$$

$$\times N_f \left(1 + \|x(s)\|^\lambda + \int_0^s N_k(1 + \|x(\tau)\|^\lambda) d\tau\right) ds + \frac{|c|}{|a+b|}$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N_f (1 + \|x\|_\infty^\lambda + N_k(1 + \|x\|_\infty^\lambda)T) ds$$

$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}$$

$$\times N_f (1 + \|x\|_\infty^\lambda + N_k(1 + \|x\|_\infty^\lambda)T) ds + \frac{|c|}{|a+b|}$$

$$\leq \frac{N_f(1 + (\eta^*)^\lambda)(1 + N_k T)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds$$

$$+ \frac{N_f(1 + (\eta^*)^\lambda)(1 + N_k T)|b|}{\Gamma(\alpha)|a+b|} \int_0^T (T-s)^{\alpha-1} ds + \frac{|c|}{|a+b|}$$

$$\leq \frac{N_f(1 + (\eta^*)^\lambda)(1 + N_k T)}{\Gamma(\alpha)} \frac{t^\alpha}{\alpha}$$

$$+ \frac{N_f(1 + (\eta^*)^\lambda)(1 + N_k T)|b|}{\Gamma(\alpha)|a+b|} \frac{T^\alpha}{\alpha} + \frac{|c|}{|a+b|}$$

$$\leq \left(1 + \frac{|b|}{|a+b|}\right) \frac{T^\alpha}{\Gamma(\alpha+1)} N_f(1 + (\eta^*)^\lambda)(1 + N_k T) + \frac{|c|}{|a+b|}$$

$$\leq l,$$

where

$$l := \left(1 + \frac{|b|}{|a+b|}\right) \frac{T^\alpha}{\Gamma(\alpha+1)} N_f(1 + (\eta^*)^\lambda)(1 + N_k T) + \frac{|c|}{|a+b|}.$$

Thus, we have
$\|(F(x))(t)\| \leq l$ and hence $\|Fx\|_\infty \leq l$.

Step 3. F maps bounded sets into equicontinuous sets of $C(J, X)$.
Let $0 \leq t_1 \leq t_2 \leq T, x \in B_{\eta^*}$. Using (H4), again we have

$$\|(F(x))(t_2) - (F(x))(t_1)\|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] \|f(s, x(s), (Sx)(s))\| \, ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \|f(s, x(s), (Sx)(s))\| \, ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] N_f (1 + \|x(s)\|^\lambda + \|(Sx)(s)\|) ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} N_f (1 + \|x(s)\|^\lambda + \|(Sx)(s)\|) ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] N_f (1 + \|x\|_\infty^\lambda + N_k (1 + \|x\|_\infty^\lambda) T) ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} N_f (1 + \|x\|_\infty^\lambda + N_k (1 + \|x\|_\infty^\lambda) T) ds$$

$$\leq \frac{N_f (1 + (\eta^*)^\lambda)(1 + N_k T)}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] ds$$

$$+ \frac{N_f (1 + (\eta^*)^\lambda)(1 + N_k T)}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds$$

$$\leq \frac{N_f (1 + (\eta^*)^\lambda)(1 + N_k T)}{\Gamma(\alpha + 1)} (t_2^\alpha - t_1^\alpha)$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero and since x is an arbitrary in B_{η^*}, F is equicontinuous.

Now, let $\{x_n\}, n = 1, 2, \ldots$ be a sequence on B_{η^*}, and

$$(Fx_n)(t) = (F_1 x_n)(t) + (F_2 x_n)(T), t \in J,$$

where

$$(F_1 x_n)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, x_n(s), (Sx_n)(s)) ds, t \in J,$$

$$(F_2 x_n)(T) = -\frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1} f(s, x_n(s), (Sx_n)(s)) ds + \frac{c}{a+b}.$$

In view of hypothesis (H5) and lemma 2.6, the set $\overline{conv} K_1$ is compact. For any $t^* \in J$,

$$(F_1 x_n)(t^*) = \frac{1}{\Gamma(\alpha)} \int_0^{t^*} (t^* - s)^{\alpha-1} f(s, x_n(s), (Sx_n)(s)) ds$$

$$= \frac{1}{\Gamma(\alpha)} \lim_{k \to \infty} \sum_{i=1}^k \frac{t^*}{k} \left(t^* - \frac{it^*}{k} \right)^{\alpha-1} f\left(\frac{it^*}{k}, x_n(\frac{it^*}{k}), (Sx_n)(\frac{it^*}{k}) \right)$$

$$= \frac{t^*}{\Gamma(\alpha)} \zeta_n,$$

where

$$\zeta_n = \lim_{k \to \infty} \sum_{i=1}^k \frac{1}{k} \left(t^* - \frac{it^*}{k} \right)^{\alpha-1} f\left(\frac{it^*}{k}, x_n(\frac{it^*}{k}), (Sx_n)(\frac{it^*}{k}) \right).$$

Now, we have $\{(F_1 x_n)(t)\}$ is a function family of continuous mappings $F_1 x_n : J \to X$, which is uniformly bounded and equicontinuous. As $\overline{conv} K_1$ is convex and compact, we know $\zeta_n \in \overline{conv} K_1$. Hence, for any

$t^* \in J = [0, T]$, the set $\{(F_1 x_n)(t^*)\}$, is relatively compact. Therefore by lemma 2.7, every $\{(F_1 x_n)(t)\}$ contains a uniformly convergent subsequence $\{(F_1 x_{n_k})(t)\}, k = 1, 2, \ldots$, on J. Thus, $\{F_1 x : x \in B_{\eta^*}\}$ is relatively compact. Similarly, one can obtain $\{(F_2 x_n)(t)\}$ contains a uniformly convergent subsequence $\{(F_2 x_{n_k})(t)\}, k = 1, 2, \ldots$, on J. Thus, $\{F_2 x : x \in B_{\eta^*}\}$ is relatively compact. As a result, the set $\{Fx : x \in B_{\eta^*}\}$ is relatively compact.

As a consequence of steps 1-3, we can conclude that F is continuous and completely continuous.

Step 4. A priori bounds.
Now it remains to show that the set

$$E(F) = \{x \in C(J, X) : x = \eta Fx \text{ for some } \eta \in [0, 1]\},$$

is bounded
Let $x \in E(F)$, then $x = \eta Fx$ for some $\eta \in [0, 1]$. Thus, for each $t \in J$, we have

$$x(t) = \eta \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), (Sx)(s)) ds \right.$$
$$\left. - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x(s), (Sx)(s)) ds + \frac{c}{a+b} \right).$$

Using (H4), for each $t \in J$, we have

$$\|x(t)\| \leq \|(F(x))(t)\|$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N_f \left(1 + \|x(s)\|^\lambda + \int_0^s N_k (1 + \|x(\tau)\|^\lambda) d\tau \right) ds$$
$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}$$
$$\times N_f \left(1 + \|x(s)\|^\lambda + \int_0^s N_k (1 + \|x(\tau)\|^\lambda) d\tau \right) ds + \frac{|c|}{|a+b|}$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N_f (1 + \|x(s)\|^\lambda + N_k (1 + \|x_s\|_B^\lambda) T) ds$$
$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}$$
$$\times N_f (1 + \|x(s)\|^\lambda + N_k (1 + \|x_s\|_B^\lambda) T) ds + \frac{|c|}{|a+b|}$$
$$\leq \frac{N_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{N_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x(s)\|^\lambda ds$$
$$+ \frac{N_f N_k T}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{N_f N_k T}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_s\|_B^\lambda ds$$
$$+ \frac{|b| N_f}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds$$
$$+ \frac{|b| N_f}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|x(s)\|^\lambda ds$$
$$+ \frac{|b| N_f N_k T}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds$$
$$+ \frac{|b| N_f N_k T}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|x_s\|_B^\lambda ds + \frac{|c|}{|a+b|}$$
$$\leq \frac{N_f T^\alpha}{\Gamma(\alpha+1)} + \frac{N_f N_k T^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{|b| N_f T^\alpha}{|a+b|\Gamma(\alpha+1)} + \frac{|b| N_f N_k T^{\alpha+1}}{|a+b|\Gamma(\alpha+1)} + \frac{|c|}{|a+b|}$$
$$+ \frac{N_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x(s)\|^\lambda ds + \frac{|b| N_f}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|x(s)\|^\lambda ds$$

$$+ \frac{N_f N_k T}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_s\|_B^\lambda ds + \frac{|b| N_f N_k T}{|a+b| \Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|x_s\|_B^\lambda ds.$$

By lemma 3.10, there exists a $N > 0$ such that $\|x(t)\| \leq N, t \in J$.

Thus for every $t \in J$, we have $\|x\|_\infty \leq N$. This show that the set $E(F)$ is bounded.

As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point that is solution of fractional BVP (1.1). $\qquad\square$

5 Examples

In this section, we give one example to illustrate the usefulness of our main results.

Example 5.1.

$$\begin{cases} {}^c D^{\frac{1}{2}} x(t) = \frac{e^{-\sigma t}}{1+e^t} \left(\frac{|x(t)|}{1+|x(t)|} + \int_0^t \frac{(s+|x(s)|)}{(2+t)^2(1+|x(s)|)} ds \right), t \in J_1, \alpha \in (0,1], \\ x(0) + x(T) = 0, \end{cases}$$

(5.11)

where $\sigma > 0$ is constant.

Take $X_1 = [0, \infty)$, $J_1 = [0,1]$ and so $T = 1$.

Set

$$f_1\big(t, x(t), (Sx)(t)\big) = \frac{e^{-\sigma t}}{1+e^t} \left(\frac{|x(t)|}{1+|x(t)|} + (Sx)(t) \right), \quad k_1(t,s,x(s)) = \frac{s+|x(s)|}{(2+t)^2(1+|x(s)|)}.$$

Let $x_1, x_2 \in C(J_1, X_1)$ and $t \in [0,1]$, we have

$$\begin{aligned} |k_1(t,s,x_1(s)) - k_1(t,s,x_2(s))| &\leq \frac{1}{(2+t)^2} \left| \frac{(1-s)(|x_1(s)| - |x_2(s)|)}{(1+|x_1(s)|)(1+|x_2(s)|)} \right| \\ &\leq \frac{1}{(2+t)^2} |x_1(s) - x_2(s)| \\ &\leq \frac{1}{4} |x_1(s) - x_2(s)|, \end{aligned}$$

and

$$\begin{aligned} &\left| f_1\big(t, x_1(t), (Sx_1)(t)\big) - f_1\big(t, x_2(t), (Sx_2)(t)\big) \right| \\ &\leq \frac{e^{-\sigma t}}{1+e^t} \left(\left| \frac{|x_1(t)|}{1+|x_1(t)|} - \frac{|x_2(t)|}{1+|x_2(t)|} \right| + \left| (Sx_1)(t) - (Sx_2)(t) \right| \right) \\ &\leq \frac{e^{-\sigma t}}{2} \left(\left| \frac{|x_1(t)| - |x_2(t)|}{(1+|x_1(t)|)(1+|x_2(t)|)} \right| + \left| (Sx_1)(t) - (Sx_2)(t) \right| \right) \\ &\leq \frac{e^{-\sigma t}}{2} \left(\left| x_1(t) - x_2(t) \right| + \left| (Sx_1)(t) - (Sx_2)(t) \right| \right). \end{aligned}$$

Also, for all $x \in C(J_1, X_1)$ and each $t \in J_1$, we have

$$\begin{aligned} |f_1\big(t, x(t), (Sx)(t)\big)| &\leq \frac{e^{-\sigma t}}{1+e^t} \left(\left| \frac{|x(t)|}{1+|x(t)|} \right| + \int_0^t \left| \frac{s+|x(s)|}{(2+t)^2(1+|x(s)|)} \right| ds \right) \\ &\leq \frac{e^{-\sigma t}}{2} \left(1 + \frac{1}{4} \right) \leq \left(\frac{5}{4} \right) \frac{e^{-\sigma t}}{2}. \end{aligned}$$

For $t \in J_1, \beta \in (0, \frac{1}{2})$, we have

$m_1(t) = \frac{e^{-\sigma t}}{2} \in L^{\frac{1}{\beta}}(J_1, \mathbb{R}), m_2(t) = \frac{1}{4} \in L^{\frac{1}{\beta}}(J_1, \mathbb{R}), h(t) = \left(\frac{5}{4}\right)\frac{e^{-\sigma t}}{2} \in L^{\frac{1}{\beta}}(J_1, \mathbb{R})$ and $M = \| \left(\frac{5}{4}\right)\frac{e^{-\sigma t}}{2} \|_{L^{\frac{1}{\beta}}(J_1, \mathbb{R})}$.

Choosing some $\sigma > 0$ large enough and $\beta = \frac{1}{4} \in (0, \frac{1}{2})$, one can arrive at the following inequality

$$\Omega_{\frac{1}{2}, \frac{1}{4}} = \frac{M}{\Gamma(\frac{1}{2})} \frac{1}{(\frac{\frac{1}{2}-\frac{1}{4}}{1-\frac{1}{4}})^{1-\frac{1}{4}}} \left(1 + \frac{1}{2}\right) < 1.$$

All the assumptions in Theorem 4.1 are satisfied, and therefore, the fractional BVP 5.11 has a unique solution on J_1.

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