Malaya J. Mat. 4(1)(2016) 25-36

Malaya Journal of Matematik

www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

On Optional Deterministic Server Vacations in a Batch Arrival Queueing System with a Single Server Providing First Essential Service Followed by One of the Two Types of Additional Optional Service

Kailash C. Madan

Ahlia University, PO Box 10878, GOSI Complex, Kingdom of Bahrain, kmadan@ahlia.edu.bh

Abstract

We analyze a batch arrival queue with a single server providing first essential service (FES) followed by one of the two types of additional optional service (AOS). After completion of the FES, a customer has the option to leave the system or to choose one of the two types of AOS and as soon as a customer leaves (either after the FES or after completing one of the chosen AOS, the server may take a vacation or may continue staying in the system. The vacation times are assumed to be deterministic and the server vacations are based on Bernoulli schedules under a single vacation policy. We obtain explicit queue size distribution at a random epoch under the steady state. In addition, some important performance measures such as the steady state expected queue size and the expected waiting time of a customer at a random epoch are obtained. Further, some interesting particular cases are also discussed.

Keywords: Batch arrivals, compound Poisson process, first essential service (FES), additional optional service (AOS), deterministic server vacation, queue size distribution.

AMS subject classification: Primary 60 K 25, Secondary 60J15, 90B10.

1 Introduction

Server vacations are a common phenomenon in many real life queueing situations. In recent years, research on queueing systems with server vacations has acquired great importance. Queueing systems with a wide variety of vacation policies have been studied by a large number of authors. Many researchers including Borthakur and Choudhury [1], Choudhury [2, 3], Madan and Choudhury [15], Gaver [5], Kielson and Servi [6], Lee and Srinivan [8], Rosenberg and Yechialli [18] and Tegham [22] have studied queues with Bernoulli schedule vacations or modified Bernoulli schedule vacations. Among the authors who studied queueing systems with vacation policies other than the Bernoulli type vacations, we mention Shanthikumar [19] who studied generalized vacations, Takagi [20] and, Madan and Abu-Rub [12–14] who studied vacations based exhaustive service and Madan [9] considered a priority queueing system with exhaustive service in which the server cannot take a vacation till all priority units present in the system are served. Recently, Krishnamoorthy and Sreeniwan [7] and Tao et al [21] studied queueing system with working vacations wherein they assume that the server is on vacation but keeps working in the system at a lower arte. Further, most of the above-mentined authors assumed that the server takes a single vacation and some, e.g. Choudhury and Madan [4] studied a queueing system in which server may take multiple vacations. Further, majority of authors who studied vacation queues assume that the server takes a vacation of random length. However, in many real life situations, the server may take a break or a vacation of fixed lenghth as it happens in a factory, a bank, a railway station and a post office etc.. In order to minimize uncertainty of availability of a server, a fixed length vacation is more realistic in many queueing situations. Madan [10, 16] studied queueing systems with deterministic vacations. Further, Madan [11] introduced the idea of a second optional service in a single server queue without server vacations. Very recently Madan [17] cosidered a queueing system which provides two stages of general service followed by a third stage optional service with deterministic server vacations. In the present paper, we study a queueing sytem with a single server providing the first essentail service followed by one of the two types of additional optional service. This system allows deterministic server vacations. We generalize results obtained by Madan [10], Madan [11] and Madan [16].

2 The Mathematical Model

Customers (units) arrive at the system in batches of variable size in a compound Poisson process. Let $\lambda c_i\, dt$ $(i = 1, 2, 3, \cdots)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t,\ t + dt]$, where $0 \leq c_i \leq 1$ and $\sum\limits_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches. There is a single server which provides first essential service (FES) to every customer. We assume that

the first essential service (FES) time random variable S_E follows a general probability law with the distribution function $B(S_E)$ and the probability density function $b(S_E)$ with the k-th moment $E(S_E^k)$, $(k = 1, 2, 3, \ldots)$.

Let $h(x)$ be the conditional probability of completion of FES during the interval $(x,\ x + dx]$, given that the elapsed service time is x, so that

$$h(x) = \frac{h(x)}{1 - H(x)}\,, \tag{2.1}$$

and therefore,

$$h(S_E) = h(S_E) \exp\left[-\int_0^s h(x)\, dx\right]. \tag{2.2}$$

After completion of his FES, a customer may opt to take one of the two kinds of additional optional service (AOS) with probability α or else may leave the system with probability $1 - \alpha$. Each customer opting for the AOS has the option to choose type 1, AOS (1) with probability θ_1 or type 2, AOS (2) with probability θ_2, where $\theta_1 + \theta_2 = 1$. We assume that the service time random variable S_j of type j, AOS (j) follows a general probability law with $B_j(S_j)$ as the distribution function, $b_j(S_j)$ as the probability density function and $E(S_j^k)$ as the k-th moment $(k = 1, 2, 3, \cdots)$ of the service time, $j = 1, 2$.

Let $\mu_j(x)$ be the conditional probability of completion of AOS (j), $j = 1, 2$, during the interval $(x,\ x + dx]$, given that the elapsed service time is x, so that

$$\mu_j(x) = \frac{b_j(x)}{1 - B_j(x)}\,,\ \ j = 1, 2, \tag{2.3}$$

and therefore,

$$b_j(s_j) = \mu_j(s_j)\left[-\int_0^s \mu_j(x)\, dx\right],\ \ j = 1, 2, \tag{2.4}$$

Next, we assume that as soon as the number of services required by a customer i.e. either FES alone or FES followed by one of the two types of AOS are complete, the server may decide to take a vacation for a constant duration d with probability δ or else with probability $1 - \delta$ may continue to be in the system, either providing service to the next customer, if any or else remains idle and waits for the next batch of customers to arrive.

We further assume that whenever the server takes a vacation, it is always a single vacation. In other words, on completion of a vacation, the server must be back to the system even if there is no customer present in the system.

Finally, it is assumed that the inter-arrival times of the customers, the service times of each kind of service and vacation times of the server, all these stochastic processes involved in the system are independent of each other.

3 Definitions and Notations

Assuming that the steady state exists, let $P_n^E(x)$ denote the steady state probability that there are n (≥ 0) customers in the queue excluding one customer in FES and the elapsed service time of this customer is x. Accordingly, $P_n^E = \int_0^\infty P_n^E(x)\, dx$ denotes the corresponding steady state probability irrespective of the elapsed service time x. Next, we define $P_{n,j}(x)$ to be the steady state probability that that there are n (≥ 0) customers in the queue excluding one customer in AOS (j) with elapsed service time x. Accordingly, $P_{n,j} = \int_0^\infty P_{n,j}(x)\, dx$ is the steady state probability that that there are n (≥ 0) customers in the queue excluding one customer in AOS (j), irrespective of the elapsed service time x. Next, we define V_n as a steady state probability that there are n (≥ 0) customers in the queue and the server is on vacation. Finally, let Q denote the steady state probability that the system is empty, i.e., there is no customer either in queue or in service and the server is idle but available in the system. We further assume that K_r is the probability of r arrivals during the vacation period and therefore, ,

$$K_r = \frac{-\exp(\lambda d)(\lambda d)^r}{r!}\, , r = 0, 1, 2, \dots . \tag{3.1}$$

In addition, we define the following probability generating functions (PGFs):

$$P^E(x,z) = \sum_{n=0}^{\infty} P_n^E(x) z^n\, , \;\; P^E(z) = \sum_{n=0}^{\infty} P^E z^n\, , \tag{3.2}$$

$$P_j(x,z) = \sum_{n=0}^{\infty} P_{n,j}(x) z^n\, , \;\; P_j(z) = \sum_{n=0}^{\infty} P_{n,j} z^n\, , j = 1, 2\, , \tag{3.3}$$

$$K(z) = \sum_{n=0}^{\infty} K_n z^n = \sum_{n=0}^{\infty} \frac{\exp(-\lambda d)(\lambda d)^n}{n!} z^n = \exp\left[-\lambda d(1-z)\right]\, , |z| \leq 1\, , \tag{3.4}$$

$$C(z) = \sum_{n=0}^{\infty} c_n z^n\, , |z| \leq 1\, . \tag{3.5}$$

Following the usual probability arguments, we obtain the following steady equations for our model.

4 Steady State Equations Governing the System

$$\frac{d}{dx} P_n^E(x) + (\lambda + h(x)) P_n^E(x) = \lambda \sum_{i=1}^{n} P_n^E(x)\, , n \geq 1\, , \tag{4.1}$$

$$\frac{d}{dx} P_0^E(x) + (\lambda + h(x)) P_0^E(x) = 0\, , \tag{4.2}$$

$$\frac{d}{dx} P_{n,1}(x) + (\lambda + \mu_1(x)) P_{n,1}(x) = \lambda \sum_{i=1}^{n} P_{n-i,1}(x)\, , n \geq 1\, , \tag{4.3}$$

$$\frac{d}{dx} P_{0,1}(x) + (\lambda + \mu_1(x)) P_{0,1}(x) = 0\, , \tag{4.4}$$

$$\frac{d}{dx} P_{n,2}(x) + (\lambda + \mu_2(x)) P_{n,2}(x) = \lambda \sum_{i=1}^{n} P_{n-i,2}(x)\, , n \geq 1\, , \tag{4.5}$$

$$\frac{d}{dx} P_{0,2}(x) + (\lambda + \mu_2(x)) P_{0,2}(x) = 0\, , \tag{4.6}$$

$$V_n = \delta(1-\alpha) \int_0^\infty P_n^E(x) h(x)\, dx + \delta \sum_{j=1}^{2} \int_0^\infty P_{n,j}(x) \mu_j(x)\, dx\, , n \geq 0\, , \tag{4.7}$$

$$\lambda Q = (1-\delta)(1-\alpha) \int_0^\infty P_0^E(x) h(x)\, dx + (1-\delta) \sum_{j=1}^{2} \int_0^\infty P_{0,j}(x) \mu_j(x)\, dx + V_0 K_0\, . \tag{4.8}$$

We will solve the above equations subject to the following boundary conditions:

$$P_n^E(0) = (1-\delta)(1-\alpha)\int_0^\infty P_{n+1}^E(x)h(x)\,dx + (1-\delta)\sum_{j=1}^{2}\int_0^\infty P_{n+1,j}(x)\mu_j(x)\,dx + V_0K_{n+1} \tag{4.9}$$

$$+V_1K_n + \cdots + V_{n+1}K_0 + \lambda c_{n+1}Q\,,\, n \geq 1\,, \tag{4.10}$$

$$P_0^E(0) = (1-\delta)\sum_{j=1}^{2}\int_0^\infty P_{1,j}(x)\mu_j(x)\,dx + V_0K_1 + V_1K_0 + \lambda c_1 Q\,,n \tag{4.11}$$

$$P_{n,1}(0) = \alpha\theta_1\int_0^\infty P_n^E(x)h(x)\,dx\,,\, n \geq 0\,, \tag{4.12}$$

$$P_{n,2}(0) = \alpha\theta_2\int_0^\infty P_n^E(x)h(x)\,dx\,,\, n \geq 0\,. \tag{4.13}$$

5 Steady State Solution in Terms of Probability Generating Functions

We multiply both sides of equation (4.1) by suitable powers of z, add equation (4.2) in the result and use (3.2) and on simplifying we obtain,

$$\frac{d}{dz}P^E(x,z) + (\lambda - \lambda C(z) + h(x))P^E(x,z) = 0\,. \tag{5.1}$$

Similar operations on equations (4.3) and (4.4); (4.5) and (4.6); and (4.7) yield

$$\frac{d}{dz}P_1(x,z) + (\lambda - \lambda C(z) + \mu_1(x))P_1(x,z) = 0\,, \tag{5.2}$$

$$\frac{d}{dz}P_2(x,z) + (\lambda - \lambda C(z) + \mu_2(x))P_2(x,z) = 0\,, \tag{5.3}$$

$$V(z) = \delta(1-\alpha)\int_0^\infty P^E(x,z)h(x)\,dx + \delta\sum_{j=1}^{2}\int_0^\infty P_j(x,z)\mu_j(x)\,dx\,. \tag{5.4}$$

(5.5)

Yet again we use a similar operation on (4.9) and (4.10), use (4.8) and simplify. Thus we obtain

$$zP^E(0,z) = (1-\delta)(1-\alpha)\int_0^\infty P^E(x,z)h(x)\,dx + (1-\delta)\sum_{j=1}^{2}\int_0^\infty P_j(x,z)\mu_j(x)\,dx \tag{5.6}$$

$$+V(z)K(z) + \lambda(C(z)-1)Q\,. \tag{5.7}$$

Finally, with the similar operations of (4.11) and (4.12), we obtain,

$$P_1(0,z) = \alpha\theta_1\int_0^\infty P^E(x,z)h(x)\,dx\,, \tag{5.8}$$

$$P_2(0,z) = \alpha\theta_2\int_0^\infty P^E(x,z)h(x)\,dx\,. \tag{5.9}$$

Next, we integrate equations (5.1), (5.2) and (5.3) between the limits 0 and x and obtain

$$P^E(x,z) = P^E(0,z)\exp\left[-(\lambda-\lambda C(z))x - \int_0^\infty h(t)\,dt\right], \tag{5.10}$$

$$P_1(x,z) = P_1(0,z)\exp\left[-(\lambda-\lambda C(z))x - \int_0^\infty \mu_1(t)\,dt\right], \tag{5.11}$$

$$P_2(x,z) = P_2(0,z)\exp\left[-(\lambda-\lambda C(z))x - \int_0^\infty \mu_2(t)\,dt\right]. \tag{5.12}$$

where $P^E(0,z)$, $P_1(0,z)$ and $P_2(0,z)$ have been obtained above in (5.5), (5.6) and (5.7) respectively.

We again integrate (5.8), (5.9) and (5.10) with respect to x and obtain

$$\begin{aligned} P^E(z) &= P^E(0,z)\frac{1-\overline{H}(\lambda-\lambda C(z))}{\lambda-\lambda C(z)}, && (5.13)\\ P_1(z) &= P_1(0,z)\frac{1-\overline{B_1}(\lambda-\lambda C(z))}{\lambda-\lambda C(z)}, && (5.14)\\ P_2(z) &= P_1(0,z)\frac{1-\overline{B_2}(\lambda-\lambda C(z))}{\lambda-\lambda C(z)}, && (5.15)\end{aligned}$$

where $\overline{H}(\lambda-\lambda C(z)) = \int\limits_0^\infty \exp\left[-(\lambda-\lambda C(z))x\right]\, dH(x)$ is the Laplace-Stieltjes transform of S_E, the service time of FES and $\overline{B_j}(\lambda-\lambda C(z)) = \int\limits_0^\infty \exp\left[-(\lambda-\lambda C(z))x\right]\, dB_j(x)$ is the Laplace-Stieltjes transform of S_j, the service time of AOS (j), $j=1,2$.

Now, we multiply equations (5.11), (5.12) and (5.13) by $h(x)$, $\mu_1(x)$ and $\mu_2(x)$ respectively and integrate them with respect to x and use (2.2) and (2.4). Thus we obtain

$$\begin{aligned} \int_0^\infty P^E(x,z)h(x)\,dx &= P^E(0,z)\overline{H}(\lambda-\lambda C(z)), && (5.16)\\ \int_0^\infty P_1(x,z)\mu_1(x)\,dx &= P_1(0,z)\overline{B_1}(\lambda-\lambda C(z)), && (5.17)\\ \int_0^\infty P_2(x,z)\mu_2(x)\,dx &= P_2(0,z)\overline{B_2}(\lambda-\lambda C(z)). && (5.18)\end{aligned}$$

We further use (5.14), (5.15) and (5.16) into equations (5.4), (5.5), (5.6) and (5.7), simplify and get

$$\begin{aligned} V(z) &= \delta(1-\alpha)P^E(0,z)\overline{H}(\lambda-\lambda C(z)) + P_1(0,z)\overline{B_1}(\lambda-\lambda C(z)) + P_2(0,z)\overline{B_2}(\lambda-\lambda C(z)), && (5.19)\\ zP^E(0,z) &= (1-\delta)(1-\alpha)P^E(0,z)\overline{H}(\lambda-\lambda C(z)) + (1-\delta)P_1(0,z)\overline{B_1}(\lambda-\lambda C(z)) && (5.20)\\ &\quad +(1-\delta)P_2(0,z)\overline{B_2}(\lambda-\lambda C(z)) + V(z)K(z) + \lambda(C(z)-1)Q, && (5.21)\\ P_1(0,z) &= \alpha\theta_1 P^E(0,z)\overline{H}(\lambda-\lambda C(z)), && (5.22)\\ P_2(0,z) &= \alpha\theta_2 P^E(0,z)\overline{H}(\lambda-\lambda C(z)), && (5.23)\end{aligned}$$

Now, solving (5.17), (5.18), (5.19) and (5.20), utilizing (5.11), (5.12) and (5.13) and simplifying, we obtain

$$\begin{aligned} V(z) &= \frac{\delta\left[(1-\alpha)+\alpha\theta_1\overline{B_1}(\lambda-\lambda C(z))+\alpha\theta_2\overline{B_2}(\lambda-\lambda C(z))\right]\lambda\left[C(z)-1\right]\overline{H}(\lambda-\lambda C(z))Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, && (5.24)\\ P^E(z) &= \frac{\left(\overline{H}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, && (5.25)\\ P_1(z) &= \frac{\alpha\theta_1\overline{H}(\lambda-\lambda C(z))\left(\overline{B_1}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, && (5.26)\\ P_2(z) &= \frac{\alpha\theta_2\overline{H}(\lambda-\lambda C(z))\left(\overline{B_2}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, && (5.27)\end{aligned}$$

where

$$\begin{aligned} \Psi(z) &= (1-\alpha)(1-\delta)+(1-\delta)\alpha\theta_1\overline{B_1}(\lambda-\lambda C(z))+(1-\delta)\alpha\theta_2\overline{B_2}(\lambda-\lambda C(z))\\ &\quad +\delta\left[(1-\alpha)+\alpha\theta_1\overline{B_1}(\lambda-\lambda C(z))+\alpha\theta_2\overline{B_2}(\lambda-\lambda C(z))\right]K(z).\end{aligned}$$

Now, in order to determine the only unknown Q, we proceed as follows:

$$\begin{aligned} V(1) &= \lim_{z\to 1} V(z)\\ &= \lim_{z\to 1}\frac{\delta\left[(1-\alpha)+\alpha\theta_1\overline{B_1}(\lambda-\lambda C(z))+\alpha\theta_2\overline{B_2}(\lambda-\lambda C(z))\right]\lambda\left[C(z)-1\right]\overline{H}(\lambda-\lambda C(z))Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}\\ &= \frac{\delta\lambda E(I)Q}{1-\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1E(S_1)+\alpha\theta_2E(S_2)\right\}+\delta\lambda d\right]}, && (5.28)\end{aligned}$$

where $E(I)$ is the average batch size, $E(S_E)$, $E(S_1)$ and $E(S_2)$ are the average service time of FES, AOS (1) and AOS(2), respectively.

$$\begin{aligned} P^E(1) &= \lim_{z\to 1} P^E(z) = \lim_{z\to 1} \frac{\left(\overline{H}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))} \\ &= \frac{\lambda E(I)E(S_E)Q}{1-[\,\lambda E(I)\,\{E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\}+\delta\lambda d\,]}, \end{aligned} \tag{5.29}$$

$$\begin{aligned} P_1(1) &= \lim_{z\to 1} P_1(z) = \lim_{z\to 1} \frac{\alpha\theta_1\overline{H}(\lambda-\lambda C(z))\left(\overline{B_1}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))} \\ &= \frac{\alpha\theta_1\lambda E(I)E(S_1)Q}{1-[\,\lambda E(I)\,\{E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\}+\delta\lambda d\,]}, \end{aligned} \tag{5.30}$$

$$\begin{aligned} P_2(1) &= \lim_{z\to 1} P_2(z) = \lim_{z\to 1} \frac{\alpha\theta_2\overline{H}(\lambda-\lambda C(z))\left(\overline{B_2}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))} \\ &= \frac{\alpha\theta_2\lambda E(I)E(S_2)Q}{1-[\,\lambda E(I)\,\{E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\}+\delta\lambda d\,]}, \end{aligned} \tag{5.31}$$

Next, we use the results found in (4.37), (4.38), (4.39) and (4.40) in the normalizing condition:

$$Q+V(1)+P^E(1)+P_1(1)+P_2(1)=1\,. \tag{5.32}$$

On simplifying, (4.41) yields

$$Q=\frac{1-[\,\lambda E(I)\,\{E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\}+\delta\lambda d\,]}{1+\delta\lambda E(I)-\delta\lambda d}\,. \tag{5.33}$$

The result (4.42) gives the probability that the server is idle and the stability condition which emerges from this equation is given by

$$[\,\lambda E(I)\,\{E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\}+\delta\lambda d\,]<1\,. \tag{5.34}$$

Now, we define ρ, the utilization factor of the system as the proportion of time the server is providing any kind of service and using results (4.38), (4.39) and (4.40) and simplifying, we get

$$\rho=P^E(1)+P_1(1)+P_2(1)=\frac{\lambda E(I)\,[\,E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\,]}{1+\delta\lambda E(I)-\delta\lambda d}\,. \tag{5.35}$$

6 Steady State Average Queue Length and Average Waiting Time

Let $P_Q(z)$ be the steady state probability generating function for the number of customers in the queue so that adding (4.37), (4.38), (4.39) and (4.40) we get

$$P_Q(z)=V(z)+P^E(z)+P_1(z)+P_2(z)=\frac{N(z)}{D(z)}\,. \tag{6.1}$$

Next, we define L_q to be the steady state average number of customers in the queue. Then $L_Q=\frac{d}{dz}P_Q(z)\,|_{z=1}$. However, since $P_Q(z)=0/0$ at $z=1$, we use double differentiation and obtain

$$L_Q=\lim_{z\to 1}\frac{d}{dz}P_Q(z)=\lim_{z\to 1}\frac{D'(z)N''(z)-N'(z)D''(z)}{2\,(D'(z))^2}=\frac{D'(1)N''(1)-N'(1)D''(1)}{2\,(D'(1))^2}, \tag{6.2}$$

where primes mean derivatives with respect to z and after a lot of algebra and simplification, we obtain

$$\begin{aligned} L_Q &= \frac{(\lambda E(I))^2\,[\,E(S_E^2)+\alpha\theta_1 E(S_1^2)+\alpha\theta_2 E(S_2^2)+2(E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2))\,]}{2\,\{1-[\,\lambda E(I)\,\{E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\}+\delta\lambda d\,]\}} \\ &\quad+\frac{2\delta\lambda^2 dE(I)\,[\,E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\,]+\delta\lambda^2 d}{2\,\{1-[\,\lambda E(I)\,\{E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\}+\delta\lambda d\,]\}} \\ &\quad+\frac{[\,\lambda E(I)\,\{E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\}+\delta\lambda d\,]\,E(I(I-1))}{2\,\{1-[\,\lambda E(I)\,\{E(S_E)+\alpha\theta_1 E(S_1)+\alpha\theta_2 E(S_2)\}+\delta\lambda d\,]\}}, \end{aligned} \tag{6.3}$$

where $E(S_E^2)$, $E(S_1^2)$ and $E(S_2^2)$ are the second moments of the FES, AOS (1) and AOS (2) service times respectively and $E(I(I-1))$ is the second factorial moment of the batch size.

Note that using L_q obtained in (6.3) into Littles formulae, we can obtain the following:
The steady state average number of customers in the system is

$$L = L_q + \rho\,. \tag{6.4}$$

where ρ is given by (5.32). Further, the steady state average waiting time in the queue is

$$W_q = \frac{L_q}{\lambda}\,. \tag{6.5}$$

The steady state average waiting time in the system is

$$W = \frac{L}{\lambda}\,. \tag{6.6}$$

7 Particular Cases

Case 1: We assume single Poisson arrivals with FES, AOS (1) and AOS (2) all having exponential distribution

In this case we have $E(I) = 1$, $E(S_E) = 1/h$, $E(S_E^2) = 2/h^2$, $E(S_1) = 1/\mu_1$, $E(S_1^2) = 2/\mu_1^2$, $E(S_2) = 1/\mu_2$, $E(S_2^2) = 2/\mu_2^2$, and $E(I(I-1)) = 0$. Furthermore,

$$\overline{H}\,[\,\lambda - \lambda C(z)\,] = \frac{h}{h+\lambda-\lambda C(z)}\,,\ \overline{B_1}\,[\,\lambda - \lambda C(z)\,] = \frac{\mu_1}{\mu_1+\lambda-\lambda C(z)} \text{ and } \overline{B_2}\,[\,\lambda - \lambda C(z)\,] = \frac{\mu_2}{\mu_2+\lambda-\lambda C(z)}\,.$$

Consequently,

$$\frac{1-\overline{H}\,[\,\lambda-\lambda C(z)\,]}{\lambda-\lambda C(z)} = \frac{1}{h+\lambda-\lambda C(z)}\,,\quad \frac{1-\overline{B_1}\,[\,\lambda-\lambda C(z)\,]}{\lambda-\lambda C(z)} = \frac{1}{\mu_1+\lambda-\lambda C(z)}\,,$$

and

$$\frac{1-\overline{B_2}\,[\,\lambda-\lambda C(z)\,]}{\lambda-\lambda C(z)} = \frac{1}{\mu_2+\lambda-\lambda C(z)}\,.$$

Substituting these values in the main results, we obtain:

$$V(z) = \frac{\delta\left[\,(1-\alpha)+\alpha\theta_1\left(\frac{\mu_1}{\mu_1+\lambda-\lambda C(z)}\right)+\alpha\theta_2\left(\frac{\mu_2}{\mu_2+\lambda-\lambda C(z)}\right)\right]\lambda\,[\,C(z)-1\,]\left(\frac{h}{h+\lambda-\lambda C(z)}\right)Q}{z-\Psi(z)\left(\frac{h}{h+\lambda-\lambda C(z)}\right)}\,, \tag{7.1}$$

$$P^E(z) = \frac{\left(\frac{h}{h+\lambda-\lambda C(z)}-1\right)Q}{z-\Psi(z)\left(\frac{h}{h+\lambda-\lambda C(z)}\right)}\,, \tag{7.2}$$

$$P_1(z) = \frac{\alpha\theta_1\left(\frac{h}{h+\lambda-\lambda C(z)}\right)\left(\frac{\mu_1}{\mu_1+\lambda-\lambda C(z)}-1\right)Q}{z-\Psi(z)\left(\frac{h}{h+\lambda-\lambda C(z)}\right)}\,, \tag{7.3}$$

$$P_2(z) = \frac{\alpha\theta_2\left(\frac{h}{h+\lambda-\lambda C(z)}\right)\left(\frac{\mu_2}{\mu_2+\lambda-\lambda C(z)}-1\right)Q}{z-\Psi(z)\left(\frac{h}{h+\lambda-\lambda C(z)}\right)}\,, \tag{7.4}$$

where, in this case,

$$\begin{aligned}\Psi(z) &= (1-\alpha)(1-\delta)+(1-\delta)\alpha\theta_1\left(\frac{\mu_1}{\mu_1+\lambda-\lambda C(z)}\right)+(1-\delta)\alpha\theta_2\left(\frac{\mu_2}{\mu_2+\lambda-\lambda C(z)}\right)\\ &\quad+\delta\left[\,(1-\alpha)+\alpha\theta_1\left(\frac{\mu_1}{\mu_1+\lambda-\lambda C(z)}\right)+\alpha\theta_2\left(\frac{\mu_2}{\mu_2+\lambda-\lambda C(z)}\right)\right]K(z)\,.\end{aligned}$$

and

$$Q = \frac{1-\lambda\left[\frac{1}{h}+\frac{\alpha\theta_1}{\mu_1}+\frac{\alpha\theta_2}{\mu_2}+\delta d\right]}{1+\delta\lambda-\delta\lambda d}. \tag{7.5}$$

Further,

$$\begin{aligned} V(1) &= \frac{\delta\lambda}{1+\delta-\delta\lambda d}, & (7.6)\\ P^E(1) &= \frac{\lambda}{h(1+\delta-\delta\lambda d)}, & (7.7)\\ P_1(1) &= \frac{\alpha\theta_1\lambda}{\mu_1(1+\delta-\delta\lambda d)}, & (7.8)\\ P_2(1) &= \frac{\alpha\theta_2\lambda}{\mu_2(1+\delta-\delta\lambda d)}, & (7.9)\end{aligned}$$

$$\begin{aligned} L_Q &= \frac{\lambda^2\left[\frac{2}{h^2}+\frac{2\alpha\theta_1}{\mu_1^2}+\frac{2\alpha\theta_2}{\mu_2^2}+2\left(\frac{1}{h}+\frac{\alpha\theta_1}{\mu_1}+\frac{\alpha\theta_2}{\mu_2}\right)\right]}{2-2\lambda\left[\frac{1}{h}+\frac{\alpha\theta_1}{\mu_1}+\frac{\alpha\theta_2}{\mu_2}+\delta d\right]}+\frac{2\delta\lambda^2 d\left[\frac{1}{h}+\frac{\alpha\theta_1}{\mu_1}+\frac{\alpha\theta_2}{\mu_2}\right]+\delta\lambda^2 d}{2-2\lambda\left[\frac{1}{h}+\frac{\alpha\theta_1}{\mu_1}+\frac{\alpha\theta_2}{\mu_2}+\delta d\right]}\\ &+\frac{\lambda\left[\frac{1}{h}+\frac{\alpha\theta_1}{\mu_1}+\frac{\alpha\theta_2}{\mu_2}\right]}{2-2\lambda\left[\frac{1}{h}+\frac{\alpha\theta_1}{\mu_1}+\frac{\alpha\theta_2}{\mu_2}+\delta d\right]}. \end{aligned} \tag{7.10}$$

Next, the steady state average number of customers in the system,

$$L = L_q + \rho, \tag{7.11}$$

where

$$\rho = \frac{\lambda\left[\frac{1}{h}+\frac{\alpha\theta_1}{\mu_1}+\frac{\alpha\theta_2}{\mu_2}\right]}{1+\delta\lambda-\delta\lambda d}. \tag{7.12}$$

Further, the steady state average waiting time in the queue is

$$W_q = \frac{L_q}{\lambda}. \tag{7.13}$$

The steady state average waiting time in the system is

$$W = \frac{L}{\lambda}. \tag{7.14}$$

Case 2: The first essential service is compulsorily followed by one of AOS (1) or AOS (2)

The results corresponding to this particular case can be obtained by putting $\alpha = 1$ in the main results.

$$\begin{aligned} V(z) &= \frac{\delta\left[\theta_1\overline{B_1}(\lambda-\lambda C(z))+\theta_2\overline{B_2}(\lambda-\lambda C(z))\right]\lambda\left[C(z)-1\right]\overline{H}(\lambda-\lambda C(z))Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, & (7.15)\\ P^E(z) &= \frac{\left(\overline{H}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, & (7.16)\\ P_1(z) &= \frac{\theta_1\overline{H}(\lambda-\lambda C(z))\left(\overline{B_1}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, & (7.17)\\ P_2(z) &= \frac{\theta_2\overline{H}(\lambda-\lambda C(z))\left(\overline{B_2}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, & (7.18)\end{aligned}$$

where

$$\begin{aligned}\Psi(z) &= (1-\delta)\theta_1\overline{B_1}(\lambda-\lambda C(z)) + (1-\delta)\theta_2\overline{B_2}(\lambda-\lambda C(z)) \\ &\quad +\delta\left[\theta_1\overline{B_1}(\lambda-\lambda C(z)) + \theta_2\overline{B_2}(\lambda-\lambda C(z))\right]K(z),\end{aligned}$$

$$Q = \frac{1-[\lambda E(I)\{E(S_E)+\theta_1 E(S_1)+\theta_2 E(S_2)\}+\delta\lambda d]}{1+\delta\lambda E(I)-\delta\lambda d}, \tag{7.19}$$

$$\rho = \frac{\lambda E(I)[E(S_E)+\theta_1 E(S_1)+\theta_2 E(S_2)]}{1+\delta\lambda E(I)-\delta\lambda d}, \tag{7.20}$$

$$\begin{aligned}L_Q &= \frac{(\lambda E(I))^2\left[E(S_E^2)+\theta_1 E(S_1^2)+\theta_2 E(S_2^2)+2(E(S_E)+\theta_1 E(S_1)+\theta_2 E(S_2))\right]}{2\{1-[\lambda E(I)\{E(S_E)+\theta_1 E(S_1)+\theta_2 E(S_2)\}+\delta\lambda d]\}} \\ &\quad +\frac{2\delta\lambda^2 dE(I)[E(S_E)+\theta_1 E(S_1)+\theta_2 E(S_2)]+\delta\lambda^2 d}{2\{1-[\lambda E(I)\{E(S_E)+\theta_1 E(S_1)+\theta_2 E(S_2)\}+\delta\lambda d]\}} \\ &\quad +\frac{[\lambda E(I)\{E(S_E)+\theta_1 E(S_1)+\theta_2 E(S_2)\}+\delta\lambda d]E(I(I-1))}{2\{1-[\lambda E(I)\{E(S_E)+\theta_1 E(S_1)+\theta_2 E(S_2)\}+\delta\lambda d]\}},\end{aligned} \tag{7.21}$$

Case 3: No AOS(1): $M^X/(G_E,G_2)/D/1$ **Queue**

In this case we put $\theta_1=0$ in the main results (5.20) to (5.23), (5.30), (5.31) and (6.2) to obtain

$$V(z) = \frac{\delta\left[(1-\alpha)+\alpha\theta_2\overline{B_2}(\lambda-\lambda C(z))\right]\lambda[C(z)-1]\overline{H}(\lambda-\lambda C(z))Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, \tag{7.22}$$

$$P^E(z) = \frac{\left(\overline{H}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, \tag{7.23}$$

$$P_1(z) = 0, \tag{7.24}$$

$$P_2(z) = \frac{\alpha\theta_2\overline{H}(\lambda-\lambda C(z))\left(\overline{B_2}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, \tag{7.25}$$

where

$$\Psi(z) = (1-\alpha)(1-\delta)+(1-\delta)\alpha\theta_2\overline{B_2}(\lambda-\lambda C(z))+\delta\left[(1-\alpha)+\alpha\theta_2\overline{B_2}(\lambda-\lambda C(z))\right]K(z),$$

$$Q = \frac{1-[\lambda E(I)\{E(S_E)+\alpha\theta_2 E(S_2)\}+\delta\lambda d]}{1+\delta\lambda E(I)-\delta\lambda d}, \quad \lambda E(I)\{E(S_E)+\alpha\theta_2 E(S_2)\}+\delta\lambda d<1, \tag{7.26}$$

$$\begin{aligned}L_Q &= \frac{(\lambda E(I))^2\left[E(S_E^2)+\alpha\theta_2 E(S_2^2)+2(E(S_E)+\alpha\theta_2 E(S_2))\right]}{2\{1-[\lambda E(I)\{E(S_E)+\alpha\theta_2 E(S_2)\}+\delta\lambda d]\}} \\ &\quad +\frac{2\delta\lambda^2 dE(I)[E(S_E)+\alpha\theta_2 E(S_2)]+\delta\lambda^2 d}{2\{1-[\lambda E(I)\{E(S_E)+\alpha\theta_2 E(S_2)\}+\delta\lambda d]\}} \\ &\quad +\frac{[\lambda E(I)\{E(S_E)+\alpha\theta_2 E(S_2)\}+\delta\lambda d]E(I(I-1))}{2\{1-[\lambda E(I)\{E(S_E)+\alpha\theta_2 E(S_2)\}+\delta\lambda d]\}}.\end{aligned} \tag{7.27}$$

Case 4: No AOS (2): $M^X/(G_E,G_1)/D/1$ **Queue**

In this case we put $\theta_2 = 0$ in the main results (5.20) to (5.23), (5.30), (5.31) and (6.2) to obtain

$$V(z) = \frac{\delta\left[(1-\alpha)+\alpha\theta_1\overline{B_1}(\lambda-\lambda C(z))\right]\lambda\left[C(z)-1\right]\overline{H}(\lambda-\lambda C(z))Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, \tag{7.28}$$

$$P^E(z) = \frac{\left(\overline{H}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, \tag{7.29}$$

$$P_1(z) = \frac{\alpha\theta_1\overline{H}(\lambda-\lambda C(z))\left(\overline{B_1}(\lambda-\lambda C(z))-1\right)Q}{z-\Psi(z)\overline{H}(\lambda-\lambda C(z))}, \tag{7.30}$$

$$P_2(z) = 0, \tag{7.31}$$

where

$$\Psi(z) = (1-\alpha)(1-\delta)+(1-\delta)\alpha\theta_1\overline{B_1}(\lambda-\lambda C(z))+\delta\left[(1-\alpha)+\alpha\theta_1\overline{B_1}(\lambda-\lambda C(z))\right]K(z),$$

$$Q = \frac{1-\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1E(S_1)\right\}+\delta\lambda d\right]}{1+\delta\lambda E(I)-\delta\lambda d}, \quad \lambda E(I)\left\{E(S_E)+\alpha\theta_1E(S_1)\right\}+\delta\lambda d<1, \tag{7.32}$$

$$\begin{aligned} L_Q = {} & \frac{(\lambda E(I))^2\left[E(S_E^2)+\alpha\theta_1E(S_1^2)+2(E(S_E)+\alpha\theta_1E(S_1))\right]}{2\left\{1-\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1E(S_1)\right\}+\delta\lambda d\right]\right\}} \\ & + \frac{2\delta\lambda^2dE(I)\left[E(S_E)+\alpha\theta_2E(S_1)\right]+\delta\lambda^2d}{2\left\{1-\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1E(S_1)\right\}+\delta\lambda d\right]\right\}} \\ & + \frac{\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1E(S_1)\right\}+\delta\lambda d\right]E(I(I-1))}{2\left\{1-\left[\lambda E(I)\left\{E(S_E)+\alpha\theta_1E(S_1)\right\}+\delta\lambda d\right]\right\}}. \end{aligned} \tag{7.33}$$

Case 5: None of the AOS: $M^X/G_E/D/1$ **Queue**

In this case we put $\alpha = \theta_1 = \theta_2 = 0$ in the main results (5.20) to (5.23), (5.30), (5.31) and (6.2) to obtain

$$V(z) = \frac{\delta\lambda\left[C(z)-1\right]\overline{H}(\lambda-\lambda C(z))Q}{z-\left\{1-\delta+\delta K(z)\right\}\overline{H}(\lambda-\lambda C(z))}, \tag{7.34}$$

$$P^E(z) = \frac{\left(\overline{H}(\lambda-\lambda C(z))-1\right)Q}{z-\left\{1-\delta+\delta K(z)\right\}\overline{H}(\lambda-\lambda C(z))}, \tag{7.35}$$

$$P_1(z) = 0, \tag{7.36}$$

$$P_2(z) = 0, \tag{7.37}$$

where

$$Q = \frac{1-\lambda E(I)E(S_E)-\delta\lambda d}{1+\delta\lambda E(I)-\delta\lambda d}, \quad \lambda E(I)E(S_E)+\delta\lambda d<1, \tag{7.38}$$

$$\begin{aligned} L_Q = {} & \frac{(\lambda E(I))^2\left[E(S_E^2)+2E(S_E)\right]}{2\left\{1-\left[\lambda E(I)E(S_E)+\delta\lambda d\right]\right\}} + \frac{2\delta\lambda^2dE(I)E(S_E)+\delta\lambda^2d}{2\left\{1-\left[\lambda E(I)E(S_E)+\delta\lambda d\right]\right\}} \\ & + \frac{\left[\lambda E(I)E(S_E)+\delta\lambda d\right]E(I(I-1))}{2\left\{1-\left[\lambda E(I)E(S_E)+\delta\lambda d\right]\right\}}. \end{aligned} \tag{7.39}$$

Case 6: No Server Vacations: $M^X/G_E/1$ Queue

In this case we put $\delta = 0$ and $d = 0$. Consequently $K(z) = 1$ in the results of case 4 and obtain

$$V(z) = 0\,, \tag{7.40}$$

$$P^E(z) = \frac{\left(1 - \overline{H}(\lambda - \lambda C(z))\right) Q}{z - \overline{H}(\lambda - \lambda C(z))}\,, \tag{7.41}$$

$$P_1(z) = 0\,, \tag{7.42}$$

$$P_2(z) = 0\,, \tag{7.43}$$

$$Q = 1 - \lambda E(I)E(S_E)\,, \quad \lambda E(I)E(S_E) < 1\,, \tag{7.44}$$

$$L_Q = \frac{(\lambda E(I))^2\left[E(S_E^2) + 2E(S_E)\right] + \lambda E(I)E(S_E)E(I(I-1))}{2[1 - \lambda E(I)E(S_E)]}\,. \tag{7.45}$$

The results of this case are known results of the ordinary $M^X/G/1$ queue.

References

[1] Borthakur, A. and Chaudhury, G. (1997): On a batch arrival Poisson queue with generalized vacation. Sankhya Ser. B, 59, 369-383.

[2] Choudhury, G. (2000): An $M^x/G/1$ queueing system with a setup period and a vacation period. Queueing Systems, 36 , 23 - 38 .

[3] Choudhury, G. (2002): A batch arrival queue with a vacation time under single vacation policy. Computers and Operations Research , 29, 1941-1955.

[4] Choudhury G. and Madan, Kailash C. (2006): A batch arrival queue with Bernoulli vacation schedule under multiple vacation policy. International Journal of Management Sciences, Volume 12, Number 2, pp 1-18 .

[5] Gaver, D.P. (1962): A Waiting Line with Interrupted Service Including Priorities. Journal of Royal Statistical Society, Ser. - B, 24 (1962), 73-90.

[6] J. Keilson, J. and Servi, L.D. (1986): Oscillating random walk models for $GI/G/1$ vacation systems with Bernoulli schedules. Journal of Applied Probability, 23, 790-802.

[7] Krishnamoorthy A. and Sreeniwasaan, C. (2012): An $M/M/2$ queueing system with heterogeneous servers including one working vacation. International Journal of Stochastic Analysis, Article ID 145867, 16 pages.

[8] Lee, H.S. and Srinivasan, M.M. (1989): Control policies for the $M^x/G/1$ queueing systems. Management Sciences, 35, 708-721.

[9] Madan, Kailash C. (2011) : 'A non-preemptive priority queueing system with a single server serving two queues $M/G/1$ and $M/D/1$ with Optional server vacations based on exhaustive service of the priority units. Applied Mathematics, Vol. 3, No. 1, pp 03-08.

[10] Madan, Kailash C. (1999): An $M/G/1$ Queue with optional deterministic server vacations. Metron, LVII No. 3-4 (1999) 83-95.

[11] Madan, Kailash C. (2000): An $M/G/1$ queue with second optional service. Queueing Systems 34, 37-46.

[12] Madan, Kailash C. and Abu-Rub, Adel (2004): On a single server queue with optional phase type server vacations based on exhaustive deterministic service with a single vacation policy, Applied Math. And Computation, 149, 723-734.

[13] Madan, Kailash C. and Abu-Rub, Adel (2004): Transient and steady state solution of a single server queue with modified Bernoulli server vacation based on exhaustive service and a single vacation policy. Investigacion Operacional, 25, 158-165.

[14] Madan, Kailash C. and Abu-Rub, Adel (2005): On a Single Server Queue with Phase Type Service and Phase Type Server Vacations Based on Exhaustive Deterministic Service Under a Single Vacation Policy, Journal of Applied Statistical Science, 15(1), 49-66.

[15] Madan, Kailash C. and Choudhury, G. (2004): An $M(X)/G/1$ queue with a Bernoulli vacation schedule under restricted admissibility policy. Sankhya, Vol. 66, Part 1, pp 175-193.

[16] Madan, Kailash C. (2001): On a single server queue with two-stage heterogeneous service and deterministic server vacations. International J. of Systems Science 32, 7, 837-844.

[17] Madan, Kailash C. (2015): On a $M(X)/(G1, G2)/1$ queue with third stage optional service and deterministic server vacations. Journal of Mathematical and Computational Science, Vol. 5, No. 2, pp. 195-206.

[18] Rosenberg, E. and Yechiali, U. (1993): The queue with single and multiple vacations under LIFO service regime. Open. Res. Lett., 14, 171-179.

[19] Shanthikumar, J.G. (1988): On stochastic decomposition in The $M/G/1$ type queues with generalized vacations. Operations Research, 36, 566-569.

[20] Takagi, H. (1992): Time dependent process of $M/G/1$ vacation models with exhaustive service. J. Appl. Prob., 29, 418-429.

[21] Tao, Li, Zhang Liyuan and Gao, Shan. (2014): $M/M/1$ retrial queue with working vacation interruption and feedback under N policy. Journal of Applied mathematics, pages 1-11, Article Id 414739.

[22] Tegham, L.J. (1990): On a decomposition result for a class of vacation queueing systems. Journal of Applied Probability 27, 227-231.

Received: June 26, 2015; *Accepted*: August 23, 2015

UNIVERSITY PRESS

Website: http://www.malayajournal.org/