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A Study On Linear and Non linear Schrodinger Equations by Reduced Differential Transform Method

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Abstract

In this paper, reduced differential transform method (RDTM) is used to obtain the exact solution of nonlinear Schrodinger equation. Compared to other existing analytical/numerical methods, RDTM is more efficient and easy to apply.

Keywords: non linear Schrodinger equations, reduced differential transform, reduced differential inverse transform, analytic solution.

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1 Introduction

In this paper we consider the one dimensional linear Schrodinger equation of the form

$$iu_t + u_{xx} = 0, \qquad u(x,0) = f(x), \qquad x \in R, t \geq 0 \tag{1.1}$$

and the nonlinear Schrodinger equation of the form

$$iu_t + u_{xx} + \nu|u|^2 u = 0, \qquad u(x,0) = f(x), \qquad x \in R, t \geq 0, \tag{1.2}$$

where ν is a real constant, $u(x,t)$ is a complex function and $i = \sqrt{-1}$. Such equations has been widely used for studying nonlinear waves in fluid-filled viscoelastic tubes, solitary waves in piezoelectric semiconductors, nonlinear optical waves, hydrodynamics and plasma waves.

Recently, an enormous work has been carried out to the search for efficient analytical / numerical methods for finding the solution of nonlinear Schrodinger equations. For example, Sadhigi and Ganji [6] investigated the linear and nonlinear one dimensional Schrodinger equations using adomian decomposition method and homotopy perturbation method. Wang [4] proposed the finite difference method to obtain the numerical solution of the nonlinear Schrodinger equation. Biazar and Gazvini [5] applied homotopy perturbation method to obtain the solution of cubic Schrodinger equation. Khuri [9] used adomian decomposition method to solve the nonlinear Schrodinger equation. Wazwaz [2] presented the exact solution of the linear and nonlinear one dimensional Schrodinger equation obtained by means of variation iteration method. Borhanifar and Reza Abazari [7] proposed differential transform method to obtain the exact solution of Schrodinger equation.

The aim of this work is to propose employ reduced differential transform method (RDTM) on nonlinear Schrodinger equations. The reduced DTM was first envisioned by Keskin [13] and successfully applied to many nonlinear partial differential equations. Also, Keskin and Oturnac [11, 12] applied this method to obtain the analytical solutions of generalized KdV equations. RDTM has been widely used by many researchers [10] and [14–17] successfully for different nonlinear physical systems such as higher dimensional Burger

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equations, Burgers-Huxley equations, Newell-Whitehead-Segel equation and generalized Hirota-Satsuma coupled KdV equation. By comparing with adomian decomposition method, reduced diferential transform can apply directly to solve the problem without using adomian polynomials. The solution procedure is very simple and easy in RDTM and the computational time is also less than that of traditional differential transform method. Another advantage of RDTM is that it does not require auxiliary parameter to control the convergence region, which was used in homotopy analysis method.

This paper is sketched as follows: Section 2 describes to show, how to use the reduced differential transform method (RDTM) will be presented and we show how to use the RDTM to approximate the solution. In section 3, we apply this method to some linear and nonlinear Schrodinger equations. The results of numerical experiments are also presented in this section.

2 Basic Idea of Reduced Differential Transform

With reference to the articles [10–17], the basic definitions of reduced differential transform are introduced as follows:

Consider a function $u(x,t)$ of two variables and assume that it can be represented as a product of two single variable functions, i.e., $u(x,t) = f(x)g(t)$. On the basis of the properties of the one dimensional differential transform, the function $u(x,t)$ can be represented as

$$u(x,t) = \sum_{h=0}^{\infty} F(h)x^h \sum_{k=0}^{\infty} G(k)t^k = \sum_{h=0}^{\infty}\sum_{k=0}^{\infty} U(h,k)x^h t^k$$

where $U(h,k) = F(h)G(k)$ is called the spectrum of $u(x,t)$.

The basic definitions and properties of reduced differential transform method(RDTM) are introduced below.

The reduced differential transform of $u(x,t)$ at $t = 0$ is defined as

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=0} \tag{2.3}$$

Where $u(x,t)$ is the given function and $U_k(x)$ is the transformed function.

The reduced differential inverse transform of $U_k(x)$ is defined as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k \tag{2.4}$$

and from (1.1) and (1.2), we have

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=0} t^k \tag{2.5}$$

In this work, the lower case $u(x,t)$ represents the original function while upper case $U_k(x)$ represents the transformed function. On the basis of the definions (2.3) and (2.4), we have the following results:

Theorem 2.1. *If* $w(x,t) = u(x,t) + v(x,t)$ *then* $W_k(x) = U_k(x) + V_k(x)$

Theorem 2.2. *If* $w(x,t) = \alpha u(x,t)$ *then* $W_k(x) = \alpha U_k(x)$

Theorem 2.3. *If* $w(x,t) = \alpha \frac{\partial^n u(x,t)}{\partial t^n}$ *then* $W_k(x) = \alpha \frac{(k+n)!}{k!} U_{k+n}(x)$

Theorem 2.4. *If* $w(x,t) = x^m t^n$ *then* $W_k(x) = x^m \delta(k-n) = \begin{cases} x^m & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$

Theorem 2.5. *If* $w(x,t) = \alpha \frac{\partial^n u(x,t)}{\partial x^n}$ *then* $W_k(x) = \alpha \frac{\partial^n U_k(x)}{\partial x^n}$

Theorem 2.6. *If* $w(x,t) = u(x,t)v(x,t)$ *then* $W_k(x) = \sum_{k_1=0}^{k} U_{k_1}(x)V_{k-k_1}(x)$

Theorem 2.7. *If* $w(x,t) = x^m t^n u(x,t)$ *then* $W_k(x) = x^m U_{k-n}(x)$

Theorem 2.8. *If* $w(x,t) = t^n u(x,t)$ *then* $W_k(x) = U_{k-n}(x)$

3 Numerical Examples

Example 3.1. *We first consider the one dimensional linear Schrodinger equation*

$$iu_t + u_{xx} = 0, \qquad x \in R, t \geq 0 \tag{3.6}$$

Subject to the initial condition

$$u(x,0) = 1 + 2\cosh ax, \quad \textit{where a is constant} \tag{3.7}$$

The transformed version of eqn. (3.6) is

$$i(k+1)U_{k+1}(x) = -\frac{\partial^2}{\partial x^2}U_k(x) \tag{3.8}$$

The transformed version of eqn. (3.7) is

$$U_0(x) = 1 + 2\cosh ax \tag{3.9}$$

Using the recurrence equation (3.8), with transformed initial condition (3.9), for $k = 0, 1, 2, 3, 4$, the first few components of $U_k(x)$ are obtained as follows:

$$\begin{aligned}
U_1(x) &= -ia^2 coshax \\
U_2(x) &= -\frac{a^4}{2}\cosh ax \\
U_3(x) &= i\frac{a^6}{6}\cosh ax \\
U_4(x) &= \frac{a^8}{24}\cosh ax \\
U_5(x) &= -i\frac{a^{10}}{120}\cosh ax
\end{aligned}$$

and so on. Finally substituting all values of $U_k(x)$ in to Eq. (2.4), we obtain the series solution as follows:

$$u(x,t) = 1 + \cosh ax\left(1 - \frac{(ia^2t)}{1!} + \frac{(ia^2t)^2}{2!} - \frac{(ia^2t)^3}{3!} + \frac{(ia^2t)^4}{4!} - \frac{(ia^2t)^5}{5!} + \cdots\right)$$

Consequently the series in the closed form

$$u(x,t) = 1 + \cosh axe^{-a^2it} \tag{3.10}$$

which is exactly the same as the results obtained by DTM [3], HPM [1, 6], HAM [8] and ADM [6] by setting $a = 2$ in the eqn. (3.10).

Example 3.2. *Consider the one dimensional nonlinear Schrodinger equation*

$$iu_t + u_{xx} + m|u|^2u = 0, \quad x \in R, t \geq 0 \tag{3.11}$$

Subject to the initial condition

$$u(x,0) = e^{inx} \tag{3.12}$$

Where m is constant.

The transformed version of eqn. (3.11) is

$$i(k+1)U_{k+1}(x) = -\frac{\partial^2}{\partial x^2}U_k(x) - m\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\overline{U}_{k_1}(x)U_{k_2-k_1}(x)U_{k-k_2}(x) \tag{3.13}$$

The transformed version of initial condition (3.12) is

$$U_0(x) = e^{inx} \tag{3.14}$$

Similar to the previous problem, the first few components of $U_k(x)$ are obtained as follows:

$$U_1(x) = i(m-n^2)e^{inx}$$
$$U_2(x) = -\frac{1}{2}(m-n^2)^2e^{inx}$$
$$U_3(x) = -\frac{i}{6}(m-n^2)^3e^{inx}$$
$$U_4(x) = \frac{1}{24}(m-n^2)^4e^{inx}$$
$$U_5(x) = \frac{i}{120}(m-n^2)^5e^{inx}$$

and so on. Substituting all these values in Eq. (2.4) yields

$$u(x,t) = e^{inx} + i(m-n^2)e^{inx}t + \frac{i^2}{2!}(m-n^2)^2e^{inx}t^2 + \frac{i^3}{3!}(m-n^2)^3e^{inx}t^3 + \frac{i^4}{4!}(m-n^2)^4t^4 + \cdots$$

Consequently the solution in closed form is

$$u(x,t) = e^{i(nx+(m-n^2)t)} \tag{3.15}$$

which is exactly the same as the result obtained by HPM [6], VIM [2], DTM [3] with $m = 2$, $n = 1$ and ADM [6] with $m = -2$, $n = 1$ in the eqn. (3.15).

Example 3.3. *Finally we consider the following non-linear Schrodinger equation*

$$iu_t = mu_{xx} + u\cos^2 x + |u|^2u \quad x \in R, t \geq 0 \tag{3.16}$$

Subject to the initial condition

$$u(x,0) = \sin ax \tag{3.17}$$

where m and a are constants.

The transformed version of (3.16) gives the following recursive formula:

$$i(k+1)U_{k+1}(x) = m\frac{\partial^2}{\partial x^2}U_k(x) + U_k(x)\cos^2 x + \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\overline{U}_{k_1}(x)U_{k_2-k_1}(x)U_{k-k_2}(x) \tag{3.18}$$

Where $|u| = \overline{u}u$ and $\overline{u}$ is the conjugate of u. The transformed version of initial condition (3.17) is

$$U_0(x) = \sin ax \tag{3.19}$$

Substituting the eqn. (3.19) in to the eqn. (3.18), we obtained the first few components of $U_k(x)$ as

$$U_1(x) = i(ma^2-1)\sin ax$$
$$U_2(x) = -\frac{1}{2}(ma^2-1)^2\sin ax$$
$$U_3(x) = -\frac{i}{6}(ma^2-1)^3\sin ax$$
$$U_4(x) = \frac{1}{24}(ma^2-1)^4\sin ax$$
$$U_5(x) = \frac{i}{120}(ma^2-1)^5\sin ax$$

and so on and finally substituting all these values in the eqn. (2.4) yields

$$u(x,t) = \sin ax + i(ma^2-1)t\sin ax + \frac{i^2}{2!}(ma^2-1)^2t^2\sin ax + \frac{i^3}{3!}(ma^2-1)^3t^3 sinax + \cdots$$

Consequently, the solution in closed form is

$$u(x,t) = \sin axe^{i(ma^2-1)t} \tag{3.20}$$

which is exactly the same as the results obtained by DTM [3, 7], HPM [5], HAM [8] and FDM [4] by setting $m = -1/2$ and $a = 1$ in the eqn. (3.20).

4 Conclusion

In this paper, the reduced differential transform method was applied to obtain the exact solution of linear and nonlinear Schrodinger equations. The results of test examples confirmed that, RDTM is efficient and powerful tool in finding the analytical solution of linear and nonlinear partial differential equations. This method is better than numerical methods since it is free from round of error and does not require large computer memory. Comparing with other existing methods it does not require linearization, perturbation or discretization.

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