

## Some integral inequalities of fractional quantum type

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### Abstract

In this work, some of the most important fractional integral inequalities involving the Riemann Liouville are extended to quantum calculus on the specific time scale  $\mathbb{T}_{t_0} = \{t : t = t_0 q^n, n \text{ a nonnegative integer}\} \cup \{0\}$ , where  $t_0 \in \mathbb{R}$  and  $0 < q < 1$ .

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## 1 Introduction

## 2 Introduction

The literature on fractional inequalities is now vast, and fractional inequalities are important in studying the existence, uniqueness, and other properties of fractional differential equations. Recently many authors have studied integral inequalities on fractional calculus using Riemann-Liouville and Caputo derivative, for more details see [7–13] and references cited therein.

The study of the fractional  $q$ -integral inequalities play a fundamental role in the theory of differential equations and fractional differential equations. In the past several years, integral inequalities have been studied extensively by several researchers in the quantum, for more details we may refer to [1–6] and the references therein.

In this work, we have used some new Riemann-Liouville integral inequalities and we have obtained some new fractional  $q$ -integral inequalities on the specific time scale  $\mathbb{T}_{t_0} = \{t : t = t_0 q^n, n \text{ a nonnegative integer}\} \cup \{0\}$ , where  $t_0 \in \mathbb{R}$  and  $0 < q < 1$ . Our results are extension of [7].

## 3 Preliminaries

In this section, we give some necessary definitions and properties which will be used in the next section of this paper. For more details, we may refer to [1–5].

**Definition 3.1.** [1–3] The specific time scale  $\mathbb{T}_{t_0}$  is defined as

$$\mathbb{T}_{t_0} = \{t : t = t_0 q^n, n \text{ is a nonnegative integer}\} \cup \{0\}, \quad 0 < q < 1. \quad (3.1)$$

If there is no confusion concerning  $t_0$ , we will denote  $\mathbb{T}_{t_0}$  by  $\mathbb{T}$ .

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**Definition 3.2.** The  $q$ -factorial function is defined in the following way

$$(t-s) \binom{(n)}{ } = (t-s)(t-qs)(t-q^2s) \cdots (t-q^{n-1}s), \quad \text{when } n \in \mathbb{N},$$

$$(t-s) \binom{(n)}{ } = t^n \prod_{k=0}^{\infty} \frac{1-(s/t)q^k}{1-(s/t)q^{n+k}}, \quad \text{when } n \notin \mathbb{N}. \tag{3.2}$$

**Definition 3.3.** The  $q$ -derivative of the  $q$ -factorial function with respect to  $t$  is

$$\nabla_q(t-s) \binom{(n)}{ } = \frac{1-q^n}{1-q} (t-s) \binom{(n-1)}{ }, \tag{3.3}$$

and the  $q$ -derivative of the  $q$ -factorial function with respect to  $s$  is

$$\nabla_q(t-s) \binom{(n)}{ } = -\frac{1-q^n}{1-q} (t-qs) \binom{(n-1)}{ }. \tag{3.4}$$

**Definition 3.4.** The  $q$ -exponential function is defined as

$$e_q(t) = \prod_{k=0}^{\infty} (1-q^k t), \quad e_q(0) = 1. \tag{3.5}$$

**Definition 3.5.** The  $q$ -Gamma function is defined by

$$\Gamma_q(v) = \frac{1}{1-q} \int_0^1 \left(\frac{t}{1-q}\right)^{v-1} e_q(qt) \nabla t, \quad v \in \mathbb{R}^+. \tag{3.6}$$

**Remark 3.1.** Observe that

$$\Gamma_q(v+1) = [v]_q \Gamma_q(v), \quad v \in \mathbb{R}^+ \text{ and } [v]_q = \frac{1-q^v}{1-q}.$$

**Definition 3.6.** The fractional  $q$ -integral is defined as

$$\nabla_q^{-v} f(t) = \frac{1}{\Gamma_q(v)} \int_0^t (t-qs)^{\overline{(v-1)}} f(s) \nabla s. \tag{3.7}$$

**Remark 3.2.** For  $f(t) = 1$ , the above definition gives

$$\nabla_q^{-v}(1) = \frac{1}{\Gamma_q(v)} \frac{q-1}{q^v-1} t^{\overline{v}} = \frac{1}{\Gamma_q(v+1)} t^{\overline{v}}.$$

## 4 Main Results

In this section, we will state our main results and give their proofs. We begin with the following lemmas:

**Lemma 4.1.** Let  $f, g$  be two positive functions on  $\mathbb{T}_{t_0}$ . Then for all  $t > 0, v > 0$ , we have

$$\nabla_r^{-v} \left[ \frac{(f(t))^p}{(g(t))^{\frac{p}{q}}} \right] \geq \frac{(\nabla_r^{-v} f(t))^p}{(\nabla_r^{-v} g(t))^{\frac{p}{q}}}, \tag{4.8}$$

where  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ .

*Proof.* Let  $\phi$  and  $\psi$  two functions, then using the fractional Hölder inequality we can write

$$\nabla_r^{-v} |\phi(t)\psi(t)| \leq (\nabla_r^{-v} |\phi(t)|^p)^{\frac{1}{p}} (\nabla_r^{-v} |\psi(t)|^q)^{\frac{1}{q}}, \quad t > 0, \tag{4.9}$$

where  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ .

Putting  $\phi(t) = \frac{f(t)}{(g(t))^{\frac{1}{q}}}$  and  $\psi(t) = (g(t))^{\frac{1}{q}}$  in Eq. (4.9), we get

$$\nabla_r^{-v} f(t) = \nabla_r^{-v} \left( \frac{f(t)}{(g(t))^{\frac{1}{q}}} (g(t))^{\frac{1}{q}} \right) \leq \left[ \nabla_r^{-v} \left( \frac{(f(t))^p}{(g(t))^{\frac{p}{q}}} \right) \right]^{\frac{1}{p}} [\nabla_r^{-v} g(t)]^{\frac{1}{q}}$$

and the proof is complete. □

**Lemma 4.2.** Let  $f, g$  be two positive functions on  $\mathbb{T}_{t_0}$ , such that  $\nabla_r^{-v} f^p(t) < \infty, \nabla_r^{-v} g^q(t) < \infty$ , and  $t > 0$ . If

$$0 < m \leq \frac{f(s)}{g(s)} \leq M < \infty, \quad s \in [0, t]. \tag{4.10}$$

Then for any  $v > 0, p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , we have

$$[\nabla_r^{-v} f(t)]^{\frac{1}{p}} [\nabla_r^{-v} g(t)]^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \nabla_r^{-v} \left[ (f(t))^{\frac{1}{p}} (g(t))^{\frac{1}{q}} \right]. \tag{4.11}$$

*Proof.* Since  $\frac{f(s)}{g(s)} \leq M$  for all  $s \in [0, t], t > 0$ , then we have

$$[g(s)]^{\frac{1}{q}} \geq M^{-\frac{1}{q}} [f(s)]^{\frac{1}{q}}. \tag{4.12}$$

Multiplying both side of (4.12) by  $[f(s)]^{\frac{1}{p}}$ , we have

$$[f(s)]^{\frac{1}{p}} [g(s)]^{\frac{1}{q}} \geq M^{-\frac{1}{q}} f(s). \tag{4.13}$$

Multiplying both side of (4.13) by  $\frac{(t-rs)^{(v-1)}}{\Gamma_r(v)}$ , we have

$$\frac{(t-rs)^{(v-1)}}{\Gamma_r(v)} [f(s)]^{\frac{1}{p}} [g(s)]^{\frac{1}{q}} \geq M^{-\frac{1}{q}} \frac{(t-rs)^{(v-1)}}{\Gamma_r(v)} f(s). \tag{4.14}$$

Integrating both sides of (4.14) with respect to  $s$  on  $(0, t)$ , we obtain

$$\frac{1}{\Gamma_r(v)} \int_0^t (t-rs)^{(v-1)} [f(s)]^{\frac{1}{p}} [g(s)]^{\frac{1}{q}} \nabla s \geq M^{-\frac{1}{q}} \frac{1}{\Gamma_r(v)} \int_0^t (t-rs)^{(v-1)} f(s) \nabla s,$$

or equivalently,

$$\nabla_r^{-v} \left[ (f(t))^{\frac{1}{p}} (g(t))^{\frac{1}{q}} \right] \geq M^{-\frac{1}{q}} \nabla_r^{-v} f(t).$$

This leads to

$$\left( \nabla_r^{-v} \left[ (f(t))^{\frac{1}{p}} (g(t))^{\frac{1}{q}} \right] \right)^{\frac{1}{p}} \geq M^{-\frac{1}{pq}} (\nabla_r^{-v} f(t))^{\frac{1}{p}}. \tag{4.15}$$

But then, since  $m g(s) \leq f(s)$  for all  $s \in [0, t], t > 0$ , so we have

$$[f(s)]^{\frac{1}{p}} \geq m^{\frac{1}{p}} [g(s)]^{\frac{1}{p}}. \tag{4.16}$$

Multiplying both side of (4.16) by  $[g(s)]^{\frac{1}{q}}$ , we have

$$[f(s)]^{\frac{1}{p}} [g(s)]^{\frac{1}{q}} \geq m^{\frac{1}{p}} g(s). \tag{4.17}$$

Multiplying both side of (4.17) by  $\frac{(t-rs)^{(v-1)}}{\Gamma_r(v)}$  and integrating the resulting identity with respect to  $s$  on  $(0, t)$ , we obtain

$$\frac{1}{\Gamma_r(v)} \int_0^t (t-rs)^{(v-1)} [f(s)]^{\frac{1}{p}} [g(s)]^{\frac{1}{q}} \nabla s \geq m^{\frac{1}{p}} \frac{1}{\Gamma_r(v)} \int_0^t (t-rs)^{(v-1)} g(s) \nabla s,$$

or equivalently,

$$\nabla_r^{-v} \left[ (f(t))^{\frac{1}{p}} (g(t))^{\frac{1}{q}} \right] \geq m^{\frac{1}{p}} \nabla_r^{-v} g(t).$$

Hence, we can write

$$\left( \nabla_r^{-v} \left[ (f(t))^{\frac{1}{p}} (g(t))^{\frac{1}{q}} \right] \right)^{\frac{1}{p}} \geq m^{\frac{1}{pq}} (\nabla_r^{-v} g(t))^{\frac{1}{p}}. \tag{4.18}$$

Combining the inequalities (4.15) and (4.18), we obtain the inequality (4.11). □

**Lemma 4.3.** Let  $f, g$  be two positive functions on  $\mathbb{T}_{t_0}$ , such that  $\nabla_r^{-v} f^p(t) < \infty, \nabla_r^{-v} g^q(t) < \infty$ , and  $t > 0$ . If

$$0 < m \leq \frac{f(s)}{g(s)} \leq M < \infty, \quad s \in [0, t].$$

Then for any  $v > 0, p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , we have

$$[\nabla_r^{-v} f^p(t)]^{\frac{1}{p}} [\nabla_r^{-v} g^q(t)]^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \nabla_r^{-v} [f(t)g(t)]. \tag{4.19}$$

*Proof.* Replacing  $f(s)$  and  $g(s)$  by  $(f(s))^p$  and  $(g(s))^q, s \in [0, t], t > 0$ , respectively, in Lemma 4.2, we obtain the required inequality (4.19). □

**Theorem 4.1.** Let  $f$  be a positive function on  $\mathbb{T}_{t_0}$ , such that for all  $t > 0, v > 0$  and  $p > 1$ ,

$$\nabla_r^{-v} f(t) \geq \left(\frac{t^v}{\Gamma_r(v+1)}\right)^{p-1}. \tag{4.20}$$

Then, we have

$$\nabla_r^{-v} [f(t)]^p \geq [\nabla_r^{-v} f(t)]^{p-1}. \tag{4.21}$$

*Proof.* In view of Lemma 4.1, we can write

$$\nabla_r^{-v} [f(t)]^p = \nabla_r^{-v} \left(\frac{[f(t)]^p}{1^{p-1}}\right) \geq \frac{[\nabla_r^{-v} f(t)]^p}{[\nabla_r^{-v} 1]^{p-1}} = \left(\frac{\Gamma_r(v+1)}{t^v}\right)^{p-1} [\nabla_r^{-v} f(t)]^p. \tag{4.22}$$

But from the condition (4.20), we have

$$\left(\frac{\Gamma_r(v+1)}{t^v}\right)^{p-1} \geq [\nabla_r^{-v} f(t)]^{-1}. \tag{4.23}$$

Combining (4.22) and (4.23), we obtain (4.21). □

**Theorem 4.2.** Let  $v > 0, p > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$  and let  $f$  be a positive functions on  $\mathbb{T}_{t_0}$ , such that  $\nabla_r^{-v} f^p(t) < \infty, t > 0$ . If

$$0 < m \leq f^p(s) \leq M < \infty, \quad s \in [0, t]. \tag{4.24}$$

Then, we have

$$[\nabla_r^{-v} f^p(t)]^{\frac{1}{p}} \leq \left(\frac{M}{m}\right)^{\frac{2}{pq}} \left(\frac{t^v}{\Gamma_r(v+1)}\right)^{-\frac{p+1}{q}} \left[\nabla_r^{-v} \left(f^{\frac{1}{p}}(t)\right)\right]^p. \tag{4.25}$$

*Proof.* Putting  $g(s) = 1$  into lemma 4.3, we can write

$$[\nabla_r^{-v} f^p(t)]^{\frac{1}{p}} [\nabla_r^{-v} 1^q(t)]^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \nabla_r^{-v} [f(t) \times 1],$$

or equivalently,

$$[\nabla_r^{-v} f^p(t)]^{\frac{1}{p}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \left(\frac{t^v}{\Gamma_r(v+1)}\right)^{-\frac{1}{q}} \nabla_r^{-v} f(t). \tag{4.26}$$

Now, put  $g(s) = 1$  into lemma 4.2 to write

$$[\nabla_r^{-v} f(t)]^{\frac{1}{p}} [\nabla_r^{-v} (1)]^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \nabla_r^{-v} \left[f^{\frac{1}{p}}(t)\right],$$

or equivalently,

$$[\nabla_r^{-v} f(t)]^{\frac{1}{p}} \leq \left(\frac{M}{m}\right)^{\frac{1}{p^2 q}} \left(\frac{t^v}{\Gamma_r(v+1)}\right)^{-\frac{1}{q}} \nabla_r^{-v} \left[f^{\frac{1}{p}}(t)\right].$$

Obviously,

$$[\nabla_r^{-v} f(t)]^{\frac{1}{p}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \left(\frac{t^v}{\Gamma_r(v+1)}\right)^{-\frac{p}{q}} \left[\nabla_r^{-v} \left(f^{\frac{1}{p}}(t)\right)\right]^p \tag{4.27}$$

Combining (4.26) and (4.27), we obtain the inequality (4.25). □

**Theorem 4.3.** *Let  $f, g$  be two nonnegative functions on  $\mathbb{T}_{t_0}$ , such that  $g$  is non-decreasing. If*

$$\nabla_r^{-v} f(t) \geq \nabla_r^{-v} g(t), \quad t > 0. \tag{4.28}$$

*Then for any  $v > 0, \gamma > 0, \delta > 0$  and  $\gamma - \delta \geq 1$ , we have*

$$\nabla_r^{-v} [f^{\gamma-\delta}(t)] \leq \nabla_r^{-v} [f^\gamma(t)g^{-\delta}(t)]. \tag{4.29}$$

*Proof.* Using the arithmetic-geometric inequality, for  $\gamma > 0, \delta > 0$ , we can write

$$\frac{\gamma}{\gamma-\delta} f^{\gamma-\delta}(s) - \frac{\delta}{\gamma-\delta} g^{\gamma-\delta}(s) \leq f^\gamma(s)g^{-\delta}(s), \quad s \in [0, t], t > 0. \tag{4.30}$$

Multiplying both side of (4.30) by  $\frac{(t-rs)^{(v-1)}}{\Gamma_r(v)}$ , we have

$$\begin{aligned} & \frac{(t-rs)^{(v-1)}}{\Gamma_r(v)} \frac{\gamma}{\gamma-\delta} f^{\gamma-\delta}(s) - \frac{(t-rs)^{(v-1)}}{\Gamma_r(v)} \frac{\delta}{\gamma-\delta} g^{\gamma-\delta}(s) \\ & \leq \frac{(t-rs)^{(v-1)}}{\Gamma_r(v)} f^\gamma(s)g^{-\delta}(s). \end{aligned} \tag{4.31}$$

Integrating both side of (4.31) with respect to  $s$  on  $[0, t]$ , we get

$$\begin{aligned} & \frac{\gamma}{\gamma-\delta} \frac{1}{\Gamma_r(v)} \int_0^t (t-rs)^{(v-1)} f^{\gamma-\delta}(s) \nabla s - \frac{1}{\Gamma_r(v)} \int_0^t (t-rs)^{(v-1)} g^{\gamma-\delta}(s) \nabla s \\ & \leq \frac{1}{\Gamma_r(v)} \int_0^t (t-rs)^{(v-1)} f^\gamma(s)g^{-\delta}(s) \nabla s. \end{aligned}$$

Obviously,

$$\frac{\gamma}{\gamma-\delta} \nabla_r^{-v} [f^{\gamma-\delta}(t)] - \frac{\delta}{\gamma-\delta} \nabla_r^{-v} [g^{\gamma-\delta}(t)] \leq \nabla_r^{-v} [f^\gamma(t)g^{-\delta}(t)].$$

This leads to

$$\frac{\gamma}{\gamma-\delta} \nabla_r^{-v} [f^{\gamma-\delta}(t)] \leq \nabla_r^{-v} [f^\gamma(t)g^{-\delta}(t)] + \frac{\delta}{\gamma-\delta} \nabla_r^{-v} [g^{\gamma-\delta}(t)].$$

This ends the proof. □

**Theorem 4.4.** *Let  $v > 0$  and  $f, g$  be two positive functions on  $\mathbb{T}_{t_0}$ , such that  $f$  is non-decreasing and  $g$  is non-increasing. Then for any  $t > 0, \gamma > 0, \delta > 0$ , we have*

$$\nabla_r^{-v} [f^\gamma(t)g^\delta(t)] \leq \frac{\Gamma_r(v+1)}{t^v} \nabla_r^{-v} [f^\gamma(t)] \nabla_r^{-v} [g^\delta(t)]. \tag{4.32}$$

*Proof.* For any  $t > 0, \gamma > 0, \delta > 0$ , we have

$$(f^\gamma(s) - f^\gamma(\rho)) (g^\delta(s) - g^\delta(\rho)) \geq 0, \quad s, \rho \in [0, t].$$

This may be written as

$$f^\gamma(s)g^\delta(\rho) + f^\gamma(\rho)g^\delta(s) \geq f^\gamma(\rho)g^\delta(\rho) + f^\gamma(s)g^\delta(s).$$

So,

$$\begin{aligned} & \nabla_r^{-v} \left[ f^\gamma(t)g^\delta(t) \right] + \frac{t^v}{\Gamma_r(v+1)} f^\gamma(\rho)g^\delta(\rho) \\ & \leq g^\delta(\rho) \nabla_r^{-v} [f^\gamma(t)] + f^\gamma(\rho) \nabla_r^{-v} [g^\delta(t)]. \end{aligned} \tag{4.33}$$

Multiplying both side of (4.33) by  $\frac{(t-r\rho)^{(v-1)}}{\Gamma_r(v)}$ ,  $\rho \in (0, t)$ , we get

$$\begin{aligned} & \frac{(t-r\rho)^{(v-1)}}{\Gamma_r(v)} \nabla_r^{-v} \left[ f^\gamma(t)g^\delta(t) \right] + \frac{t^v}{\Gamma_r(v+1)} \frac{(t-r\rho)^{(v-1)}}{\Gamma_r(v)} f^\gamma(\rho)g^\delta(\rho) \\ & \leq \frac{(t-r\rho)^{(v-1)}}{\Gamma_r(v)} g^\delta(\rho) \nabla_r^{-v} [f^\gamma(t)] + \frac{(t-r\rho)^{(v-1)}}{\Gamma_r(v)} f^\gamma(\rho) \nabla_r^{-v} [g^\delta(t)]. \end{aligned} \tag{4.34}$$

Integrating both side of (4.34) with respect to  $\rho$  on  $[0, t]$ , we obtain

$$\begin{aligned} & \frac{t^v}{\Gamma_r(v+1)} \nabla_r^{-v} \left[ f^\gamma(t)g^\delta(t) \right] + \frac{t^v}{\Gamma_r(v+1)} \nabla_r^{-v} \left[ f^\gamma(t)g^\delta(t) \right] \\ & \leq \nabla_r^{-v} [g^\delta(t)] \nabla_r^{-v} [f^\gamma(t)] + \nabla_r^{-v} [f^\gamma(t)] \nabla_r^{-v} [g^\delta(t)], \end{aligned}$$

which implies (4.32). □

**Theorem 4.5.** Let  $v > 0$  and  $f, g$  be two positive functions on  $\mathbb{T}_{t_0}$ , such that  $f$  is non-decreasing and  $g$  is non-increasing. Then for any  $t > 0, \gamma > 0, \delta > 0$ , we have

$$\begin{aligned} & \frac{t^v}{\Gamma_r(v+1)} \nabla_r^{-v} \left[ f^\gamma(t)g^\delta(t) \right] + \frac{t^\omega}{\Gamma_r(\omega+1)} \nabla_r^{-\omega} \left[ f^\gamma(t)g^\delta(t) \right] \\ & \leq \nabla_r^{-v} [f^\gamma(t)] \nabla_r^{-\omega} [g^\delta(t)] + \nabla_r^{-\omega} [f^\gamma(t)] \nabla_r^{-v} [g^\delta(t)]. \end{aligned} \tag{4.35}$$

*Proof.* Multiplying both side of (4.33) by  $\frac{(t-r\rho)^{(\omega-1)}}{\Gamma_r(\omega)}$ ,  $\rho \in (0, t)$ , we get

$$\begin{aligned} & \frac{(t-r\rho)^{(\omega-1)}}{\Gamma_r(\omega)} \nabla_r^{-v} \left[ f^\gamma(t)g^\delta(t) \right] + \frac{t^\omega}{\Gamma_r(\omega+1)} \frac{(t-r\rho)^{(\omega-1)}}{\Gamma_r(\omega)} f^\gamma(\rho)g^\delta(\rho) \\ & \leq \frac{(t-r\rho)^{(\omega-1)}}{\Gamma_r(\omega)} g^\delta(\rho) \nabla_r^{-v} [f^\gamma(t)] + \frac{(t-r\rho)^{(\omega-1)}}{\Gamma_r(\omega)} f^\gamma(\rho) \nabla_r^{-v} [g^\delta(t)]. \end{aligned} \tag{4.36}$$

Integrating both side of (4.36) with respect to  $\rho$  on  $[0, t]$ , we obtain

$$\begin{aligned} & \nabla_r^{-v} \left[ f^\gamma(t)g^\delta(t) \right] \int_0^1 \frac{(t-r\rho)^{(\omega-1)}}{\Gamma_r(\omega)} \nabla \rho + \frac{t^\omega}{\Gamma_r(\omega+1)} \int_0^1 \frac{(t-r\rho)^{(\omega-1)}}{\Gamma_r(\omega)} f^\gamma(\rho)g^\delta(\rho) \nabla \rho \\ & \leq \nabla_r^{-v} [f^\gamma(t)] \int_0^1 \frac{(t-r\rho)^{(\omega-1)}}{\Gamma_r(\omega)} g^\delta(\rho) \nabla \rho + \nabla_r^{-v} [g^\delta(t)] \int_0^1 \frac{(t-r\rho)^{(\omega-1)}}{\Gamma_r(\omega)} f^\gamma(\rho) \nabla \rho, \end{aligned}$$

and this ends the proof. □

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