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Hermite-Hadamard Type Inequalities for $(n, m, h_1, h_2, \varphi)$ −Convex Functions Via Fractional Integrals

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Abstract

In this paper, we obtain new generalizations for Hermite-Hadamard inequality by using Riemann-Liouville fractional integral and new type convex functions.

Keywords: Integral inequalities, Riemann-Liouville Fractional integral, Hermite-Hadamard Inequality, $(n, m, h_1, h_2, \varphi)$ −Convex Functions

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1 Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

is known in the literature as Hermite-Hadamard inequality for convex mappings. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f.

It is well known that the Hermite-Hadamard's inequality plays an important role in nonlinear analysis. Over the last decade, this classical inequality has been improved and generalized in a number of ways; there have been a large number of research papers written on this subject, (see, [3, 5, 12, 13, 15, 16, 18, 20]) and the references there in.

Definition 1.1. *([9]) A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on I if inequality*

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b), \tag{1.2}$$

holds for all $a, b \in I$ and $t \in [0, 1]$.

It is remarkable that Sarikaya et al. [11] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

with $\alpha > 0$.

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Definition 1.2. *([7]) Let $s \in (0,1]$. A function $f : I \subseteq \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ is said to be $s-$convex in the second sense if*

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b), \tag{1.4}$$

holds for all $a, b \in I$ and $t \in [0,1]$. This class of $s-$convex functions is usually denoted by K_s^2.

Definition 1.3. *([10]) Let $(0,1) \subseteq J \subseteq \mathbb{R}$, $I \subseteq \mathbb{R}$ be an interval, and $h : I \to \mathbb{R}_0$ is said to be $h-$convex if the inequality*

$$f(ta + (1-t)b) \leq h(t) f(a) + h(1-t) f(b). \tag{1.5}$$

Definition 1.4. *([1, 8, 17]) Let $f \in L_1[a,b]$. The Riemann-Liouville fractional integral $J_{a+}^{\alpha} f(x)$ and $J_{b-}^{\alpha} f(x)$ of order $\alpha \geq 0$ are defined by*

$$J_{a+}^{\alpha}[f(x)] = \tfrac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt \quad x > a \tag{1.6}$$

and

$$J_{b-}^{\alpha}[f(x)] = \tfrac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt \quad x < b \tag{1.7}$$

respectively. Where $\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du$ is Gamma function and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

We give the following properties:

$$J^{\alpha} J^{\beta}[f(t)] = J^{\alpha+\beta}[f(t)], \quad \alpha \geq 0, \beta \geq 0, \tag{1.8}$$

$$J^{\alpha} J^{\beta}[f(t)] = J^{\beta} J^{\alpha}[f(t)], \quad \alpha \geq 0, \beta \geq 0. \tag{1.9}$$

Definition 1.5. *([2]) A function f is said to be in the $L_p(a,b)$ space if*

$$L_p(a,b) = \left\{ f : \|f\|_{L_p} = \left(\int_a^b |f(t)|^p\, dt \right)^{\frac{1}{p}} < \infty,\ 1 \leq p < \infty \right\}, \tag{1.10}$$

and for the case $p = \infty$

$$||f||_{\infty} = ess \sup_{a \leq t \leq b} |f(t)|. \tag{1.11}$$

Our goal in this paper is to state and prove the Hermite-Hadamard type inequality for convex functions. In order to achieve our goal, we give an important identity and then we prove some integral inequalities by using this identity.

In order to established main results, we first give following generalized definition.

In paper ([6]), (α, β, a, b) $-$convex functions are defined as solutions f of the functional inequality

$$f(\alpha(t) x + \beta(t) y) \leq a(t) f(x) + b(t) f(y)$$

where $0 \neq T \subseteq [0,1]$ and $\alpha, \beta, a, b : T \to \mathbb{R}$ are given functions. We introduce a definition of $(n, m, h_1, h_2, \varphi)$ $-$convex functions.

Definition 1.6. *Let $\varphi : [a,b] \subset \mathbb{R} \to [a,b]$. A function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$, $h_1, h_2 : [0,1] \to \mathbb{R}_0$, $m, n \in (0,1]$. Then f is said to be $(n, m, h_1, h_2, \varphi)$ $-$convex if the inequality*

$$f(nt\varphi(a) + m(1-t)\varphi(b)) \leq nh_1(t) f(\varphi(a)) + mh_2(t) f(\varphi(b)). \tag{1.12}$$

holds for all $a, b \in I$ and $t \in [0,1]$. If the inequality (1.12) reverses, then f is said to be $(n, m, h_1, h_2, \varphi)$ $-$concave on I.

Taking $\varphi(x) = x$, $h_1(t) = t$, $h_2(t) = 1 - t$ and $m = n = 1$ in Definition 1.6, we obtain Definition 1.1,

$$f(ta + (1-t)b) \leq tf(a) + (1-t) f(b).$$

Taking $\varphi(x) = x$, $h_1(t) = t$ and $h_2(t) = 1 - t$ in Definition 1.6, we obtain $(n, m)-$convex functions in ([19]),

$$f(nta + m(1-t)b) \leq ntf(a) + m(1-t) f(b).$$

Taking $\varphi(x) = x$, $h_1(t) = t^{\beta}$ and $h_2(t) = 1 - t^{\alpha}$ in Definition 1.6, we obtain $(\beta, \alpha, n, m)-$convex functions in ([4]),

$$f(nta + m(1-t)b) \leq nt^{\beta} f(a) + m(1-t^{\alpha}) f(b).$$

The following Lemma will be used to established our main results:

Lemma 1.1. *([14]) Let $f : I \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L_1[\varphi(a),\varphi(b)]$ for $\varphi(a),\varphi(b) \in I$, then the following equality for fractional integrals holds:*

$$\begin{aligned} &\frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha}\left[J^\alpha_{\varphi(a)+}f(\varphi(b)) + J^\alpha_{\varphi(b)-}f(\varphi(a))\right] \\ &= \frac{\varphi(b)-\varphi(a)}{2}\int_0^1 \left[(1-t)^\alpha - t^\alpha\right] f'(t\varphi(a)+(1-t)\varphi(b))dt. \end{aligned} \tag{1.13}$$

Proof. It suffices to note that

$$\begin{aligned} I &= \int_0^1 \left[(1-t)^\alpha - t^\alpha\right] f'(t\varphi(a)+(1-t)\varphi(b))dt \\ &= \int_0^1 (1-t)^\alpha f'(t\varphi(a)+(1-t)\varphi(b))dt + \int_0^1 (-t^\alpha) f'(t\varphi(a)+(1-t)\varphi(b))dt \\ &= I_1 + I_2 \end{aligned}$$

By integration by parts, we get

$$\begin{aligned} I_1 &= \int_0^1 (1-t)^\alpha f'(t\varphi(a)+(1-t)\varphi(b))dt \\ &= (1-t)^\alpha \frac{f(t\varphi(a)+(1-t)\varphi(b))}{\varphi(a)-\varphi(b)}\Bigg|_0^1 + \frac{\alpha}{\varphi(a)-\varphi(b)}\int_0^1 (1-t)^{\alpha-1} f(t\varphi(a)+(1-t)\varphi(b))dt \\ &= \frac{f(\varphi(b))}{\varphi(b)-\varphi(a)} - \frac{\alpha}{\varphi(b)-\varphi(a)}\int_0^1 (1-t)^{\alpha-1} f(t\varphi(a)+(1-t)\varphi(b))dt \\ &= \frac{f(\varphi(b))}{\varphi(b)-\varphi(a)} - \frac{\alpha}{(\varphi(b)-\varphi(a))^{\alpha+1}}\int_{\varphi(a)}^{\varphi(b)} (u-\varphi(a))^{\alpha-1} f(u)du \\ &= \frac{f(\varphi(b))}{\varphi(b)-\varphi(a)} - \frac{\Gamma(\alpha+1)}{(\varphi(b)-\varphi(a))^{\alpha+1}} J^\alpha_{\varphi(b)-}f(\varphi(a)), \end{aligned}$$

and similarly,

$$\begin{aligned} I_2 &= \int_0^1 (-t^\alpha) f'(t\varphi(a)+(1-t)\varphi(b))dt \\ &= (-t^\alpha)\frac{f(t\varphi(a)+(1-t)\varphi(b))}{\varphi(a)-\varphi(b)}\Bigg|_0^1 + \frac{\alpha}{\varphi(a)-\varphi(b)}\int_0^1 t^{\alpha-1} f(t\varphi(a)+(1-t)\varphi(b))dt \\ &= \frac{f(\varphi(a))}{\varphi(b)-\varphi(a)} - \frac{\alpha}{\varphi(b)-\varphi(a)}\int_0^1 t^{\alpha-1} f(t\varphi(a)+(1-t)\varphi(b))dt \\ &= \frac{f(\varphi(a))}{\varphi(b)-\varphi(a)} - \frac{\alpha}{(\varphi(b)-\varphi(a))^{\alpha+1}}\int_{\varphi(a)}^{\varphi(b)} (\varphi(a)-u)^{\alpha-1} f(u)du \\ &= \frac{f(\varphi(a))}{\varphi(b)-\varphi(a)} - \frac{\Gamma(\alpha+1)}{(\varphi(b)-\varphi(a))^{\alpha+1}} J^\alpha_{\varphi(a)+}f(\varphi(b)). \end{aligned}$$

Thus can write,

$$I = I_1 + I_2 = \frac{f(\varphi(a))+f(\varphi(b))}{\varphi(b)-\varphi(a)} - \frac{\Gamma(\alpha+1)}{(\varphi(b)-\varphi(a))^{\alpha+1}}\left[J^\alpha_{\varphi(a)+}f(\varphi(b)) + J^\alpha_{\varphi(b)-}f(\varphi(a))\right]$$

Multiplying the both sides by $\frac{\varphi(b)-\varphi(a)}{2}$, we obtain lemma which completes the proof. □

2 Main results

Theorem 2.1. *Let I be an interval $a,b \in I$ with $0 \leq a < b$ and $\varphi : I \to \mathbb{R}$ a continuous increasing function. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°. If $f' \in L_1([\varphi(a),\varphi(b)])$ for $\varphi(a),\varphi(b) \in I$, $n,m \in (0,1]$ and $h_1,h_2 : [0,1] \to \mathbb{R}_0$, then*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)+}f(\varphi(b))+J^{\alpha}_{\varphi(b)-}f(\varphi(a))\right]\right| \leq \frac{\varphi(b)-\varphi(a)}{2}\frac{2(1-2^{-\alpha})}{\alpha+1}\left(n\left|f'\left(\frac{\varphi(a)}{n}\right)\right|\,||h_1||_{\infty}+m\left|f'\left(\frac{\varphi(b)}{m}\right)\right|\,||h_2||_{\infty}\right). \tag{2.14}$$

Proof. From Lemma 1.1 and (n,m,h_1,h_2,φ) −convexity of $|f'|$, we obtain

$$\begin{aligned}
&\left|\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)+}f(\varphi(b))+J^{\alpha}_{\varphi(b)-}f(\varphi(a))\right]\right| \\
&\leq \frac{\varphi(b)-\varphi(a)}{2}\int_0^1 \left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'(t\varphi(a)+(1-t)\varphi(b))\right|dt \\
&= \frac{\varphi(b)-\varphi(a)}{2}\int_0^1 \left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'(nt\frac{\varphi(a)}{n}+m(1-t)\frac{\varphi(b)}{m})\right|dt \\
&\leq \frac{\varphi(b)-\varphi(a)}{2}\int_0^1 \left|(1-t)^{\alpha}-t^{\alpha}\right|\left(n\left|f'\left(\frac{\varphi(a)}{n}\right)\right|\,||h_1||_{\infty}+m\left|f'\left(\frac{\varphi(b)}{m}\right)\right|\,||h_2||_{\infty}\right)dt \\
&= \frac{\varphi(b)-\varphi(a)}{2}\left\{\int_0^{1/2}\left[(1-t)^{\alpha}-t^{\alpha}\right]\left(n\left|f'\left(\frac{\varphi(a)}{n}\right)\right|\,||h_1||_{\infty}+m\left|f'\left(\frac{\varphi(b)}{m}\right)\right|\,||h_2||_{\infty}\right)dt\right. \\
&\left.+\int_{1/2}^1\left[t^{\alpha}-(1-t)^{\alpha}\right]\left(n\left|f'\left(\frac{\varphi(a)}{n}\right)\right|\,||h_1||_{\infty}+m\left|f'\left(\frac{\varphi(b)}{m}\right)\right|\,||h_2||_{\infty}\right)dt\right\},
\end{aligned}$$

where

$$\int_0^{1/2}\left[(1-t)^{\alpha}-t^{\alpha}\right]dt = \int_{1/2}^1\left[t^{\alpha}-(1-t)^{\alpha}\right]dt = \frac{1-2^{-\alpha}}{\alpha+1},$$

which completes the proof. □

Corollary 2.1. *Under the assumptions of Theorem 2.1 with $h_1(t) = h(t)$, $h_2(t) = h(1-t)$, , then the following inequality holds*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)+}f(\varphi(b))+J^{\alpha}_{\varphi(b)-}f(\varphi(a))\right]\right| \leq \frac{\varphi(b)-\varphi(a)}{2}\frac{2-2^{1-\alpha}}{\alpha+1}\,||h_1||_{\infty}\left(n\left|f'\left(\frac{\varphi(a)}{n}\right)\right|+m\left|f'\left(\frac{\varphi(b)}{m}\right)\right|\right).$$

Furthermore, if $n = m = 1$, then

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)+}f(\varphi(b))+J^{\alpha}_{\varphi(b)-}f(\varphi(a))\right]\right| \leq \frac{\varphi(b)-\varphi(a)}{2}\frac{2-2^{1-\alpha}}{\alpha+1}\,||h_1||_{\infty}\left(|f'(\varphi(a))|+|f'(\varphi(b))|\right).$$

Corollary 2.2. *Under the assumptions of Corollary 2.1 with $h_1(t) = h(t) = t^s$, $n = m = 1$, then the following inequality holds*

$$\left|\frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha}\left[J^\alpha_{\varphi(a)+}f(\varphi(b)) + J^\alpha_{\varphi(b)-}f(\varphi(a))\right]\right|$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\frac{2-2^{1-\alpha}}{\alpha+1}\frac{1}{s+1}\left(|f'(\varphi(a))| + |f'(\varphi(b))|\right).$$

Specially, $\alpha = s = n = m = 1$, then

$$\left|\frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b)-\varphi(a))}\int\limits_{\varphi(a)}^{\varphi(b)} f(x)dx\right|$$

$$\leq \frac{\varphi(b)-\varphi(a)}{4}\left(|f'(\varphi(a))| + |f'(\varphi(b))|\right)$$

Theorem 2.2. *Let I be an interval $a, b \in I$ with $0 \leq a < b$ and $\varphi : I \to \mathbb{R}$ a continuous increasing function. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°. If $f' \in L_1([\varphi(a), \varphi(b)])$ for $\varphi(a), \varphi(b) \in I$, $n, m \in (0, 1]$ and $h_1, h_2 : [0, 1] \to \mathbb{R}_0$, then*

$$\left|\frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha}\left[J^\alpha_{\varphi(a)+}f(\varphi(b)) + J^\alpha_{\varphi(b)-}f(\varphi(a))\right]\right|$$
$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left\{\left\{\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right\}^{1/p}\left[n\,||h_1||_q\left|f'(\frac{\varphi(a)}{n})\right| + m\,||h_2||_q\left|f'(\frac{\varphi(b)}{m})\right|\right]\right\}. \tag{2.15}$$

Proof. From Lemma 1.1, Hölder inequality, and the $(n, m, h_1, h_2, \varphi)$ −convexity of $|f'|^q$, we obtain

$$\left|\frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha}\left[J^\alpha_{\varphi(a)+}f(\varphi(b)) + J^\alpha_{\varphi(b)-}f(\varphi(a))\right]\right|$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\int\limits_0^1 |(1-t)^\alpha - t^\alpha|\,|f'(t\varphi(a) + (1-t)\varphi(b))|\,dt$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\int\limits_0^1 |(1-t)^\alpha - t^\alpha|\left|f'(nt\frac{\varphi(a)}{n} + m(1-t)\frac{\varphi(b)}{m})\right|dt$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\int\limits_0^1 |(1-t)^\alpha - t^\alpha|\left[n\,|h_1(t)|\left|f'(\frac{\varphi(a)}{n})\right| + m\,|h_2(t)|\left|f'(\frac{\varphi(b)}{m})\right|\right]dt$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left\{\left(\int\limits_0^1 |(1-t)^\alpha - t^\alpha|^p\,dt\right)^{1/p}\left(\int\limits_0^1 n^q\,|h_1(t)|^q\left|f'(\frac{\varphi(a)}{n})\right|^q dt\right)^{1/q}\right.$$

$$\left. + \left(\int\limits_0^1 |(1-t)^\alpha - t^\alpha|^p\,dt\right)^{1/p}\left(\int\limits_0^1 m^q\,|h_2(t)|^q\left|f'(\frac{\varphi(b)}{m})\right|^q dt\right)^{1/q}\right\}$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left\{\left(\int_0^1 \left|(1-t)^\alpha - t^\alpha\right|^p dt\right)^{1/p}\left[n\,||h_1||_q\left|f'(\frac{\varphi(a)}{n})\right| + m\,||h_2||_q\left|f'(\frac{\varphi(b)}{m})\right|\right]\right\}$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left\{\left(\int_0^{1/2}\left[(1-t)^\alpha - t^\alpha\right]^p dt + \int_{1/2}^1\left[t^\alpha-(1-t)^\alpha\right]^p dt\right)^{1/p}\right.$$

$$\left.\times\left[n\,||h_1||_q\left|f'(\frac{\varphi(a)}{n})\right| + m\,||h_2||_q\left|f'(\frac{\varphi(b)}{m})\right|\right]\right\}$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left\{\left(\int_0^{1/2}\left[(1-t)^{\alpha p} - t^{\alpha p}\right] dt + \int_{1/2}^1\left[t^{\alpha p}-(1-t)^{\alpha p}\right] dt\right)^{1/p}\right.$$

$$\left.\times\left[n\,||h_1||_q\left|f'(\frac{\varphi(a)}{n})\right| + m\,||h_2||_q\left|f'(\frac{\varphi(b)}{m})\right|\right]\right\}$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left\{\left\{\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right\}^{1/p}\left[n\,||h_1||_q\left|f'(\frac{\varphi(a)}{n})\right| + m\,||h_2||_q\left|f'(\frac{\varphi(b)}{m})\right|\right]\right\}$$

where

$$\int_0^{1/2}\left[(1-t)^{\alpha p} - t^{\alpha p}\right] dt = \int_{1/2}^1\left[t^{\alpha p}-(1-t)^{\alpha p}\right] dt = \frac{1}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right),$$

which completes the proof. □

Corollary 2.3. *Under the assumptions of Theorem* 2.2 *with* $h_1(t) = h(t)$, $h_2(t) = h(1-t)$, *, then the following inequality holds*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha}\left[J^\alpha_{\varphi(a)+}f(\varphi(b)) + J^\alpha_{\varphi(b)-}f(\varphi(a))\right]\right|$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\,||h_1||_q\left\{\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right\}^{1/p}\left(n\left|f'(\frac{\varphi(a)}{n})\right| + m\left|f'(\frac{\varphi(b)}{m})\right|\right).$$

Furthermore, if $n = m = 1$, *then*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha}\left[J^\alpha_{\varphi(a)+}f(\varphi(b)) + J^\alpha_{\varphi(b)-}f(\varphi(a))\right]\right|$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\,||h_1||_q\left\{\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right\}^{1/p}\left(|f'(\varphi(a))| + |f'(\varphi(b))|\right).$$

Corollary 2.4. *Under the assumptions of Corollary* 2.3 *with* $h_1(t) = h(t) = t^s$, $n = m = 1$ *, then the following inequality holds*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha}\left[J^\alpha_{\varphi(a)+}f(\varphi(b)) + J^\alpha_{\varphi(b)-}f(\varphi(a))\right]\right|$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left(\frac{1}{sq+1}\right)^{1/q}\left\{\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right\}^{1/p}\left(|f'(\varphi(a))| + |f'(\varphi(b))|\right);$$

Specially, $\alpha = s = n = m = 1$, *then*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{1}{(\varphi(b)-\varphi(a))}\int_{\varphi(a)}^{\varphi(b)} f(x)dx\right|$$

$$\leq \frac{(\varphi(b)-\varphi(a))}{2}\left(\frac{1}{q+1}\right)^{1/q}\left\{\frac{2}{p+1}\left(1-\frac{1}{2^{p}}\right)\right\}^{1/p}\left(|f'(\varphi(a))| + |f'(\varphi(b))|\right).$$

Corollary 2.5. *Under the assumptions of Corollary 2.4 with $h_1(t) = h(t) = t$, $n = m = 1$ and $\varphi(x) = x$, , then the following inequality holds*

$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a) \right] \right|$$

$$\leq \frac{b-a}{2} \left(\frac{1}{s+1} \right)^{1/q} \left\{ \frac{2}{\alpha p+1} \left(1 - \frac{1}{2^{\alpha p}} \right) \right\}^{1/p} \left(|f'(a)| + |f'(b)| \right);$$

Specially, $\alpha = s = n = m = 1$, then

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{(b-a)} \int_a^b f(x)dx \right|$$

$$\leq \frac{(b-a)}{2} \left(\frac{1}{q+1} \right)^{1/q} \left\{ \frac{2}{p+1} \left(1 - \frac{1}{2^{p}} \right) \right\}^{1/p} \left(|f'(a)| + |f'(b)| \right).$$

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