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Inverse Fourier Transform for Bi-Complex Variables

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Abstract

In this paper we examine the existence of bicomplexified inverse Fourier transform as an extension of its complexified inverse version within the region of convergence of bicomplex Fourier transform. In this paper we use the idempotent representation of bicomplex-valued functions as projections on the auxiliary complex spaces of the components of bicomplex numbers along two orthogonal,idempotent hyperbolic directions.

Keywords: Bicomplex numbers, Fourier transform, Inverse Fourier transform.

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1 Introduction

In 1892, in search for special algebras, Corrado Segre [11] published a paper in which he treated an infinite family of algebras whose elements are commutative generalization of complex numbers called bicomplex numbers, tricomplex numbers,.....etc. Segre [11] defined a bicomplex number ξ as follows:

$$\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4,$$

where x_1, x_2, x_3, x_4 are real numbers, $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. The set of bicomplex numbers, complex numbers and real numbers are respectively denoted by $\mathbb{C}_2$, $\mathbb{C}_1$ and $\mathbb{C}_0$. Thus

$$\mathbb{C}_2 = \{\xi : \xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 : \ x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$$

$$\text{i.e., } \mathbb{C}_2 = \{\xi = z_1 + i_2 z_2 : \ z_1 (= x_1 + i_1 x_2), z_2 (= x_3 + i_1 x_4) \in \mathbb{C}_1\}.$$

There are two non trivial elements $e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$ in $\mathbb{C}_2$ with the properties $e_1^2 = e_1, e_2^2 = e_2, e_1 \cdot e_2 = e_2 \cdot e_1 = 0$ and $e_1 + e_2 = 1$ which means that e_1 and e_2 are idempotents alternatively called orthogonal idempotents. By the help of the idempotent elements e_1 and e_2, any bicomplex number

$$\xi = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 = (a_0 + i_1 a_1) + i_2 (a_2 + i_1 a_3) = z_1 + i_2 z_2$$

where $a_0, a_1, a_2, a_3 \in \mathbb{C}_0,$, $z_1 (= a_0 + i_1 a_1)$ and $z_2 (= a_2 + i_1 a_3) \in \mathbb{C}_1$ can be expressed as

$$\xi = z_1 + i_2 z_2 = \xi_1 e_1 + \xi_2 e_2$$

where $\xi_1 (= z_1 - i_1 z_2) \in \mathbb{C}_1$ and $\xi_2 (= z_1 + i_1 z_2) \in \mathbb{C}_1$.

2 Fourier Transform

Let $f(t)$ be a real valued continuous function in $(-\infty, \infty)$ which satisfies the estimates

$$|f(t)| \leq C_1 \exp(-\alpha t), t \geq 0, \alpha > 0$$
$$\text{and } |f(t)| \leq C_2 \exp(-\beta t), t \leq 0, \beta > 0. \tag{1}$$

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Then the bicomplex Fourier transform [2] of $f(t)$ can be defined as

$$\widehat{f}(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} \exp(i_1\omega t) f(t) dt, \omega \in \mathbb{C}_2.$$

The Fourier transform $\widehat{f}(\omega)$ exists and holomorphic for all $\omega \in \Omega$ where

$$\Omega = \{\omega = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 \in \mathbb{C}_2 : -\infty < a_0, a_3 < \infty,$$

$$-\alpha + |a_2| < a_1 < \beta - |a_2| \; and \; 0 \leq |a_2| < \frac{\alpha+\beta}{2}\}$$

is the region of absolute convergence of $\widehat{f}(\omega)$ (see the figure 1 in the appendix).

2.1 Complex version of Fourier inverse transform.

We start with the complex version of Fourier inverse transform and in this connection we consider a continuous function $f(t)$ for $-\infty < t < \infty$ satisfying the estimates (1) possessing the Fourier transform $\widehat{f_1}$ in complex variable ω_1=$x_1 + i_1 x_2$ i.e.,

$$\begin{aligned} \widehat{f}_1(\omega_1) &= \int_{-\infty}^{\infty} \exp(i_1\omega_1 t) f(t) dt \\ &= \int_{-\infty}^{\infty} \exp(i_1 x_1 t)\{\exp(-x_2 t) f(t)\} dt = \phi(x_1, x_2). \end{aligned}$$

In fact, one may identify $\phi(x_1, x_2)$ as the Fourier transform of $g(t) = \exp(-x_2 t) f(t)$ in usual complex exponential form [1, 6].

Towards this end, we assume that $f(t)$ is continuous and $f'(t)$ is piecewise continuous on the whole real line. Then $\widehat{f}_1(\omega_1)$ converges absolutely for $-\alpha < x_2 < \beta$ and

$$|\widehat{f}_1(\omega_1)\,| < \infty$$

which implies that

$$\begin{aligned} &\int_{-\infty}^{\infty} |\exp(i_1\omega_1 t) f(t)| dt \\ &= \int_{-\infty}^{\infty} |\exp(i_1 x_1) g(t)| dt \\ &= \int_{-\infty}^{\infty} |g(t)| dt < \infty. \end{aligned}$$

The later condition shows $g(t)$ is absolutely integrable. Then by the Fourier inverse transform in complex exponential form [1, 6],

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i_1 x_1 t) \phi(x_1, x_2) dx_1$$

which implies that

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(x_2 t) \exp(-i_1 x_1 t) \phi(x_1, x_2) dx_1.$$

Now if we integrate along a horizontal line then x_2 is constant and so for complex variable ω_1=$x_1 + i_1 x_2$ (which implies $d\omega_1 = dx_1$), the above inversion formula can be extended upto complex Fourier inverse transform

$$\begin{aligned} f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{-i_1(x_1 + i_1 x_2)t\} \phi(x_1, x_2) dx_1 \\ &= \frac{1}{2\pi} \int_{-\infty + i_1 x_2}^{\infty + i_1 x_2} \exp(-i_1\omega_1 t) \widehat{f}_1(\omega_1) d\omega_1 \\ &= \frac{1}{2\pi} \lim_{x_1 \longrightarrow \infty} \int_{-x_1 + i_1 x_2}^{x_1 + i_1 x_2} \exp(-i_1\omega_1 t) \widehat{f}_1(\omega_1) d\omega_1. \end{aligned} \quad (2)$$

Here the integration is to be performed along a horizontal line in complex ω_1-plane employing contour integration method.

We first consider the case $Im(\omega_1) = x_2 \geq 0$.We observe that $\widehat{f_1}(\omega_1)$ is continuous for $x_2 \geq 0$ and in particular it is holomorphic (and so it has no singularities) for $0 \leq x_2 < \beta$. We now introduce a contour Γ_R consisting of the segment $[-R, R]$ and a semicircle C_R of radius $|\omega_1| = R > \beta$ with centre at the origin. Then all possible singularities (if exists) of $\widehat{f_1}(\omega_1)$ must lie in the region above the horizontal line $x_2 = \beta$. At this stage we now consider the following two cases:

CaseI: We assume that $\widehat{f_1}(\omega_1)$ is holomorphic in $x_2 > \beta$ except for having a finite number of poles $\omega_1^{(k)}$ for $k = 1, 2, ...n$ therein (See Figure 2 in Appendix). By taking $R \to \infty$, we can guarantee that all these poles lie inside the contour Γ_R. Since $exp(-i_1\omega_1 t)$ never vanishes then the status of these poles $\omega_1^{(k)}$ of $\widehat{f_1}(\omega_1)$ is not affected by multiplication of it with $exp(-i_1\omega_1 t)$.Then by Cauchy's residue theorem,

$$\begin{aligned}
&\lim_{R\longrightarrow\infty}\int_{\Gamma_R} \exp(-i_1\omega_1 t)\widehat{f_1}(\omega_1)d\omega_1 \\
&= 2\pi i_1 \sum_{Im(\omega_1^{(k)})>0} \text{Res}\,\{\exp(-i_1\omega_1 t)\widehat{f_1}(\omega_1) : \omega_1 = \omega_1^{(k)}\}.
\end{aligned} \tag{3}$$

Furthermore as $x_2 \geq 0$, we can get $|\exp(-i_1\omega_1 t)| \leq 1$ for $\omega_1 \in C_R$ only when $t \leq 0$. In particular for $t < 0$,

$$\begin{aligned}
M(R) &= \max_{\omega_1\in C_R} |\widehat{f_1}(\omega_1)| = \max_{\omega_1\in C_R} |\int_{-\infty}^{0} \exp(i_1\omega_1 t)f(t)dt| \\
&\leq C_2 \max_{\omega_1\in C_R} |\int_{-\infty}^{0} \exp\{(\beta + i_1\omega_1)t\}dt| = C_2 \max_{\omega_1\in C_R} |\frac{1}{\beta + i_1\omega_1}| \\
&\leq C_2 \max_{\omega_1\in C_R} \frac{1}{\beta + |i_1||\omega_1|}
\end{aligned}$$

where we use the estimate 1. Now for $\omega_1| = R \to \infty$, we obtain that $M(R) \to 0$. Thus the conditions for Jordan's lemma [10] are met and so employing it we get that

$$\lim_{R\longrightarrow\infty}\int_{C_R} \exp(-i_1\omega_1 t)\widehat{f_1}(\omega_1)d\omega_1 = 0. \tag{4}$$

Finally as,

$$\begin{aligned}
&\lim_{R\longrightarrow\infty}\int_{\Gamma_R} \exp(-i_1\omega_1 t)\widehat{f_1}(\omega_1)d\omega_1 \\
&= \int_{C_R} \exp(-i_1\omega_1 t)\widehat{f_1}(\omega_1)d\omega_1 + \int_{-R+i_1x_2}^{R+i_1x_2} \exp(-i_1\omega_1 t)\widehat{f_1}(\omega_1)d\omega_1
\end{aligned}$$

then for $R \to \infty$, on using (3) and (4) we obtain that

$$\begin{aligned}
&\int_{-\infty+i_1x_2}^{\infty+i_1x_2} \exp(-i_1\omega_1 t)\widehat{f_1}(\omega_1)d\omega_1 \\
&= 2\pi i_1 \sum_{Im(\omega_1^{(k)})>0} \text{Res}\,\{\exp(-i_1\omega_1 t)\widehat{f_1}(\omega_1) : \omega_1 = \omega_1^{(k)}\} \text{ for } t < 0
\end{aligned}$$

and so

$$f(t) = i_1 \sum_{Im(\omega_1^{(k)})>0} \text{Res}\,\{\exp(-i_1\omega_1 t)\widehat{f_1}(\omega_1) : \omega_1 = \omega_1^{(k)}\} \text{ for } t < 0.$$

Case II: Suppose $\widehat{f_1}(\omega_1)$ has infinitely many poles $\omega_1^{(k)}$ for $k = 1, 2, ...n$ in $x_2 > \beta$ and we arrange them in such a way that $\omega_1^{(1)} \leq |\omega_1^{(2)}| \leq |\omega_1^{(3)}|$..... where $\omega_1^{(k)}| \to \infty$ as $k \to \infty$. We then consider a sequence of contours Γ_k consisting of the segments $[-\, x_1^{(k)} + i_1x_2, x_1^{(k)} + i_1x_2]$ and the semicircles C_k of radii $r_k = \omega_1^{(k)}| > \beta$ enclosing the first k poles $\omega_1^{(1)}, \omega_1^{(2)}, \omega_1^{(3)}, ...\omega_1^{(k)}$ (See Figure 3 in Appendix).Then by Cauchy's residue theorem we get that

$$
\begin{aligned}
2\pi i_1 \sum_{Im(\omega_1^{(k)})>0} & \text{Res}\,\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = \omega_1^{(k)}\} \\
&= \int_{\Gamma_R} \exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1)d\omega_1 \\
&= \int_{C_R} \exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1)d\omega_1 \\
&+ \int_{-x_1^{(k)}+i_1x_2}^{x_1^{(k)}+i_1x_2} \exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1)d\omega_1.
\end{aligned} \tag{5}
$$

Now for $t < 0$, in the procedure similar to Case I, employing Jordan lemma here also we may deduce that

$$\lim_{|\omega_1^{(k)}|\longrightarrow\infty} \int_{C_R} \exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1)d\omega_1 = 0.$$

Hence in the limit $|\omega_1^{(k)}| \longrightarrow \infty$ (which implies that $|x_1^{(k)}| \longrightarrow \infty$) , (5) leads to

$$
\begin{aligned}
&\int_{-\infty+i_1x_2}^{\infty+i_1x_2} \exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1)d\omega_1 \\
&= 2\pi i_1 \sum_{Im(\omega_1^{(k)})>0} \text{Res}\,\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = \omega_1^{(k)}\} \text{ for } t < 0
\end{aligned}
$$

and as its consequence

$$f(t) = i_1 \sum_{Im(\omega_1^{(k)})>0} \text{Res}\,\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = \omega_1^{(k)}\} \text{ for } t < 0.$$

Thus for $x_2 \geq 0$, whatever the number of poles is finite or infinite, from the above two cases we obtain the complex version of Fourier inverse transform as

$$f(t) = i_1 \sum_{Im(\omega_1^{(k)})>0} \text{Res}\,\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = \omega_1^{(k)}\} \text{ for } t < 0. \tag{6}$$

We now consider the Case Im **(**ω_1**) =** $x_2 \leq 0$. The complex valued function $\widehat{f}_1(\omega_1)$ is continuous for $x_2 \leq 0$ and holomorphic in $-\alpha < x_2 \leq 0$. Introducing a contour $\Gamma'_{R'}$ consisting of the segment $[-R', R']$ and a semicircle $C'_{R'}$ of radius $\omega_1| = R' > \alpha$ with centre at the origin, we see that all possible singularities (if exists) of $\widehat{f}_1(\omega_1)$ must lie in the region below the horizontal line $x_2 = -\alpha$. If $\overline{\omega}_1^{(k)}$ for $k = 1, 2...$ are the poles in $x_2 < \alpha$, whatever the number of poles are finite or not for $R' \to \infty$, in similar to the previous consideration for $x_2 \geq 0$ we see that for $t > 0$ the conditions for Jordan lemma are met and so

$$f(t) = -i_1 \sum_{Im(\omega_1^{(k)})<0} \text{Res}\,\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = \overline{\omega}_1^{(k)}\} \text{ for } t > 0. \tag{7}$$

We then assign the value of $f(t)$ at $t = 0$ fulfilling the requirement of continuity of it in $-\infty < t < \infty$. This completes our procedure in complex ω_1 plane.

Similarly in $\omega_2(= y_1 + i_1y_2)$ plane the complex version of Fourier inverse transform of $\widehat{f}_2(\omega_2)$ will be

$$f(t) = \frac{1}{2\pi} \lim_{y_1\longrightarrow\infty} \int_{-y_1+i_1y_2}^{y_1+i_1y_2} \exp(-i_1\omega_2 t)\widehat{f}_2(\omega_2)d\omega_2 \tag{8}$$

where the integration is to be performed along the horizontal line in ω_2 plane. Employing the contour integration method, we can obtain that

$$
\begin{aligned}
f(t) &= i_1 \sum_{Im(\omega_2^{(k)})>0} \text{Res}\,\{\exp(-i_1\omega_2 t)\widehat{f}_2(\omega_2) : \omega_2 = \omega_2^{(k)}\} \text{ for } t < 0 \\
&= -i_1 \sum_{Im(\omega_2^{(k)})<0} \text{Res}\,\{\exp(-i_1\omega_2 t)\widehat{f}_2(\omega_2) : \omega_2 = \omega_2^{(k)}\} \text{ for } t > 0
\end{aligned} \tag{9}
$$

and the value of $f(t)$ at $t = 0$ can be assigned fulfilling the requirement of continuity of it in $-\infty < t < \infty$.

2.2 Bicomplex version of Fourier inverse transform.

Suppose $\widehat{f}(\omega)$ is the bicomplex Fourier transform of the real valued continuous function $f(t)$ for $-\infty < t < \infty$ where $\omega = \omega_1 e_1 + \omega_2 e_2$ and $\widehat{f}(\omega) = \widehat{f_1}(\omega_1)e_1 + \widehat{f_2}(\omega_2)e_2$ in their idempotent representations. Here the symbols $\omega_1, \omega_2, \widehat{f_1}$ and $\widehat{f_2}$ have their same representation as defined in section 2.1. Then $\widehat{f}(\omega)$ is holomorphic in

$$\begin{aligned} \Omega = \{\omega = (x_1 + i_1 x_2)e_1 + (y_1 + i_1 y_2)e_2 \in \mathbb{C}_2 \\ : -\alpha < x_2, y_2 < \beta, -\infty < x_1, y_1 < \infty\}. \end{aligned} \tag{10}$$

Now using complex inversions 2 and 8, we obtain that

$$\begin{aligned} f(t) &= f(t)e_1 + f(t)e_2 \\ &= [\frac{1}{2\pi}\int_{D_1} \exp(-i_1\omega_1 t)\widehat{f_1}(\omega_1)d\omega_1]e_1 + [\frac{1}{2\pi}\int_{D_2} \exp(-i_1\omega_2 t)\widehat{f_2}(\omega_2)d\omega_2]e_2 \\ &= \frac{1}{2\pi}\int_D \exp\{-i_1(\omega_1 e_1 + \omega_2 e_2)t\}\{\widehat{f_1}(\omega_1)e_1 + \widehat{f_2}(\omega_2)e_2\}d(\omega_1 e_1 + \omega_2 e_2) \\ &= \frac{1}{2\pi}\int_D \exp\{-i_1(\omega t)\widehat{f}(\omega)d\omega \end{aligned} \tag{11}$$

where

$$D_1 = \{\omega = x_1 + i_1 x_2 \in \mathbb{C}(i_1) : -\infty < x_1 < \infty, -\alpha < x_2 < \beta\},$$
$$D_2 = \{\omega = y_1 + i_1 y_2 \in \mathbb{C}(i_1) : -\infty < y_1 < \infty, -\alpha < y_2 < \beta\}$$

and D be such that $D_1 = P_1(D), D_2 = P_2(D)$. The integration in D_1 and D_2 are to be performed along the lines parallel to x_1 -axis in ω_1 plane and y_1-axis in ω_2 plane respectively inside the respective strips $-\alpha < x_2 < \beta$ and $-\alpha < y_2 < \beta$. As a result,

$$D = \{\omega \in \mathbb{C}_2 : \omega = \omega_1 e_1 + \omega_2 e_2 = (x_1 + i_1 x_2)e_1 + (y_1 + i_1 y_2)e_2\} \tag{12}$$

where $-\infty < x_1, y_1 < \infty, -\alpha < x_2, y_2 < \beta$. In four-component form D can be alternatively expressed as

$$\begin{aligned} D = \{\omega \in \mathbb{C}_2 : \frac{x_1 + y_1}{2} + i_1\frac{x_2 + y_2}{2} + i_2\frac{y_2 - x_2}{2} + i_1 i_2\frac{x_1 - y_1}{2}, \\ -\infty < x_1, y_1 < \infty, -\alpha < x_2, y_2 < \beta\}. \end{aligned}$$

Conversely, if the integration in D is performed then the integrations in mutually complementary projections of D namely D_1 and D_2 are to be performed along the lines parallel to x_1-axis in ω_1 plane and y_1-axis in ω_2 plane respectively inside the strips $-\alpha < x_2, y_2 < \beta$ by using the contour integration technique. So using 2 and 8, we obtain that

$$\begin{aligned} &\frac{1}{2\pi}\int_D \exp\{-i_1(\omega t)\widehat{f}(\omega)d\omega \\ &= \frac{1}{2\pi}\int_D \exp\{-i_1(\omega_1 e_1 + \omega_2 e_2)t\}\{\widehat{f_1}(\omega_1)e_1 + \widehat{f_2}(\omega_2)e_2\}d(\omega_1 e_1 + \omega_2 e_2) \\ &= [\frac{1}{2\pi}\int_{D_1} \exp(-i_1\omega_1 t)\widehat{f_1}(\omega_1)d\omega_1]e_1 + [\frac{1}{2\pi}\int_{D_2} \exp(-i_1\omega_2 t)\widehat{f_2}(\omega_2)d\omega_2]e_2 \\ &= [\frac{1}{2\pi}\int_{-\infty+i_1x_2}^{\infty+i_1x_2} \exp(-i_1\omega_1 t)\widehat{f_1}(\omega_1)d\omega_1]e_1 + [\frac{1}{2\pi}\int_{-\infty+i_1y_2}^{\infty+i_1y_2} \exp(-i_1\omega_2 t)\widehat{f_2}(\omega_2)d\omega_2]e_2 \\ &= f(t)e_1 + f(t)e_2 = f(t) \end{aligned}$$

which guarantees the existence of Fourier inverse transform for bicomplex-valued functions.

In the following, we define the bicomplex version of Fourier inverse transform method.

Definition 1. *Let $\widehat{f}(\omega)$ be the bicomplex Fourier transform of a real valued continuous function $f(t)$ for $-\infty < t < \infty$ which is holomorphic in 12.The Fourier inverse transform of $\widehat{f}(\omega)$ is defined as*

$$f(t) = \frac{1}{2\pi}\int_D \exp\{-i_1(\omega t)\widehat{f}(\omega)d\omega$$

where D is given by 12. *On using 6,7 and 9 this inversion method amounts to*

$$f(t) = i_1 e_1 \sum_{Im(\omega_2^{(k)})>0} \text{Res}\,\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = \omega_1^{(k)}\}$$
$$+ i_1 e_2 \sum_{Im(\omega_2^{(k)})>0} \text{Res}\,\{\exp(-i_1\omega_2 t)\widehat{f}_2(\omega_2) : \omega_2 = \omega_2^{(k)}\} \text{ for } t < 0 \tag{13}$$

and

$$f(t) = -i_1 e_1 \sum_{Im(\omega_1^{(k)})<0} \text{Res}\,\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = \overline{\omega}_1^{(k)}\}$$
$$- i_1 e_2 \sum_{Im(\omega_2^{(k)})<0} \text{Res}\,\{\exp(-i_1\omega_2 t)\widehat{f}_2(\omega_2) : \omega_2 = \omega_2^{(k)}\} \text{ for } t > 0. \tag{14}$$

We assign the value of $f(t)$ at $t = 0$ fulfilling the requirement of continuity of it in the whole real line $(-\infty < t < \infty)$.

The following examples will make our notion clear:

Example 2. *1. If* $\widehat{f}(\omega) = \frac{2a}{a^2+\omega^2}$ *for* $a>0$ *then*

$$\widehat{f}_1(\omega_1) = \frac{2a}{a^2+\omega_1^2},$$

$$\widehat{f}_2(\omega_2) = \frac{2a}{a^2+\omega_2^2}$$

and in each of ω_1 and ω_2 planes the poles are simple at $i_1 a$ and $i_1 a$. Now employing 13 and 14, for $t < 0$ we obtain that

$$f(t) = i_1 e_1 \text{ Res}\,\{\exp(-i_1\omega_1 t)\frac{2a}{a^2+\omega_1^2} : \omega_1 = i_1 a\}$$
$$+ i_1 e_2 \text{ Res}\,\{\exp(-i_1\omega_2 t)\frac{2a}{a^2+\omega_2^2} : \omega_2 = i_1 a\}$$
$$= i_1 e_1\{-i_1 \exp(at)\} + i_1 e_2\{-i_1 \exp(at)\} = \exp(-a|t|)$$

and for $t > 0$,

$$f(t) = -i_1 e_1 \text{ Res}\,\{\exp(-i_1\omega_1 t)\frac{2a}{a^2+\omega_1^2} : \omega_1 = i_1 a\}$$
$$- i_1 e_2 \text{ Res}\,\{\exp(-i_1\omega_2 t)\frac{2a}{a^2+\omega_2^2} : \omega_2 = i_1 a\}$$
$$= -i_1 e_1\{i_1 \exp(-at)\} - i_1 e_2\{i_1 \exp(at)\} = \exp(-a|t|).$$

Now for the continuity of t in the real line, we find $f(0) = 1$. Thus the Fourier inverse transform of $\widehat{f}(\omega)$ is $f(t) = \exp(-a|t|)$.

Example 3. *2. If*

$$\widehat{f}(\omega) = \frac{1}{2}[\frac{1}{\omega+\omega_0+\frac{i_1}{T}} - \frac{1}{\omega-\omega_0+\frac{i_1}{T}}] \text{ for } T, \omega_0 > 0$$

then in each of ω_1 and ω_2 plane the poles are at $(\omega_0 - \frac{i_1}{T})$ and $(-\omega_0 - \frac{i_1}{T})$. For both the poles the imaginary components are negative and so the poles are in lower half of both the planes. In otherwords, no poles exist in upper half of ω_1 or ω_2 planes and as its consequence $f(t) = 0$ for $t < 0$. Now at $t > 0$,

$$f(t) = -i_1 e_1 \text{ Res}\,\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = -\omega_0 - \frac{i_1}{T}\}$$
$$- i_1 e_1 \text{ Res}\,\{\exp(-i_1\omega_1 t)\widehat{f}_1(\omega_1) : \omega_1 = \omega_0 - \frac{i_1}{T}\}$$
$$- i_1 e_2 \text{ Res}\,\{\exp(-i_1\omega_2 t)\widehat{f}_2(\omega_2) : \omega_2 = -\omega_0 - \frac{i_1}{T}\}$$
$$- i_1 e_2 \text{ Res}\,\{\exp(-i_1\omega_2 t)\widehat{f}_2(\omega_2) : \omega_2 = \omega_0 - \frac{i_1}{T}\}$$

$$= -i_1 e_1 \frac{1}{2} \exp(-\frac{t}{T}) \exp(i_1 \omega_0 t) + i_1 e_1 \frac{1}{2} \exp(-\frac{t}{T}) \exp(-i_1 \omega_0 t)$$
$$- i_1 e_2 \frac{1}{2} \exp(-\frac{t}{T}) \exp(i_1 \omega_0 t) + i_1 e_2 \frac{1}{2} \exp(-\frac{t}{T}) \exp(-i_1 \omega_0 t)$$
$$= -i_1 \frac{1}{2} \exp(-\frac{t}{T}) \exp(i_1 \omega_0 t) + i_1 \frac{1}{2} \exp(-\frac{t}{T}) \exp(-i_1 \omega_0 t)$$
$$= \exp(-\frac{t}{T}) \sin(\omega_0 t).$$

Finally, the continuity of $f(t)$ in the whole real line implies that $f(0) = 0$.

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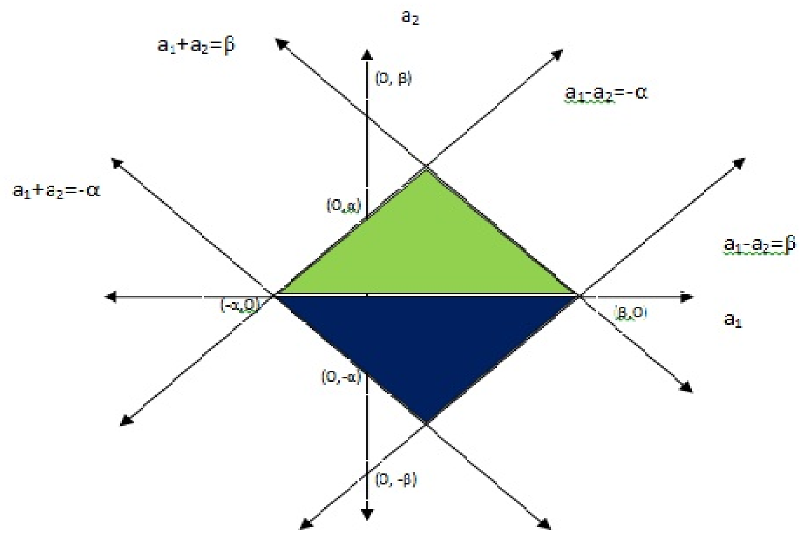
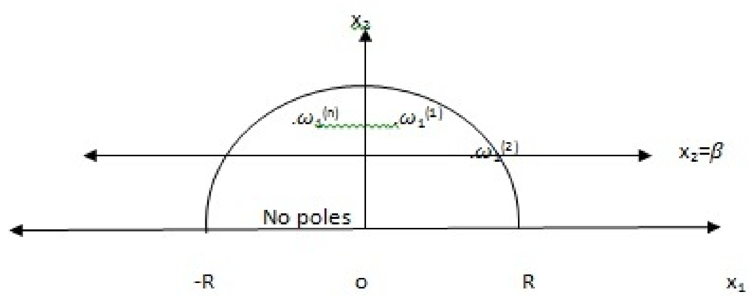
Appendix



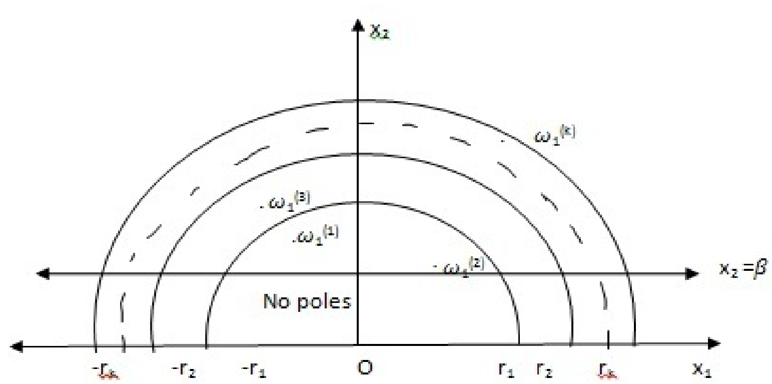
$a_1 - a_2$ plane in the concerned region of convergence

Fig-1



Upper half of ω_1 plane

Fig-1



Upper half of ω_1 plane

Fig-2