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The Natural Lift of the Fixed Centrode of a Non-null Curve in Minkowski 3-Space

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Abstract

In this study, we dealt with the natural lift curves of the fixed centrode of a non-null curve.Furthermore, some interesting result about the original curve were obtained, depending on the assumption that the natural lift curves should be the integral curve of the geodesic spray on the tangent bundle $T\left(S_1^2\right)$ and $T\left(H_0^2\right)$.

Keywords: Natural lift, geodesic spray, Darboux vector.

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1 Introduction

Thorpe gave the concepts of the natural lift curve and geodesic spray in [12]. Thorpe provied the natural lift $\overline{\alpha}$ of the curve α is an integral curve of the geodesic spray iff α is an geodesic on M. Çalışkan at al. studied the natural lift curves of the spherical indicatries of tangent, principal normal, binormal vectors and fixed centrode of a curve in [11]. They gave some interesting results about the original curve, depending on the assumption that the natural lift curve should be the integral curve of the geodesic spray on the tangent bundle $T\left(S^2\right)$. Some properties of $\overline{M}$-vector field Z defined on a hypersurface M of $\overline{M}$ were studied by Agashe in [1]. $\overline{M}$-integral curve of Z and $\overline{M}$-geodesic spray are defined by Çalışkan and Sivridağ. They gave the main theorem: The natural lift $\overline{\alpha}$ of the curve α (in $\overline{M}$) is an $\overline{M}$-integral curve of the geodesic spray Z iff α is an $\overline{M}$-geodesic in [5]. Bilici et al. have proposed the natural lift curves and the geodesic sprays for the spherical indicatrices of the the involute evolute curve couple in Euclidean 3-space. They gave some interesting results about the evolute curve, depending on the assumption that the natural lift curve of the spherical indicatrices of the involute should be the integral curve on the tangent bundle $T\left(S^2\right)$ in [3]. Then Bilici applied this problem to involutes of a timelike curve in Minkowski 3-space (see [4]). Ergün and Çalışkan defined the concepts of the natural lift curve and geodesic spray in Minkowski 3-space in [7]. The anologue of the theorem of Thorpe was given in Minkowski 3-space by Ergün and Çalışkan in [7]. Çalışkan and Ergün defined $\overline{M}$-vector field Z, $\overline{M}$-geodesic spray, $\overline{M}$-integral curve of Z, $\overline{M}$-geodesic in [6].The anologue of the theorem of Sivridağ and Çalışkan was given in Minkowski 3-space by Ergün and Çalışkan in [5]. Walrave characterized the curve with constant curvature in Minkowski 3-space in [12]. In differential geometry, especially the theory of space curve, the Darboux vector is the areal velocity vector of the Frenet frame of a spacere curve. It is named after Gaston Darboux who discovered it. In term of the Frenet-Serret apparatus, the darboux vector W can be expressed as $W = \tau T + \kappa B$, details are given in Lambert et al. in [8].

In this study,we studied the fixed centrode curve of a curve and characterized the curve if the natural lift of the fixed centrode curve is an integral curve of the geodesic sprays.

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Let Minkowski 3-space $\mathbb{R}^3_1$ be the vector space $\mathbb{R}^3$ equipped with the Lorentzian inner product g given by

$$g(X,X) = -x_1^2 + x_2^2 + x_3^2$$

where $X = (x_1, x_2, x_3) \in \mathbb{R}^3$. A vector $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ is said to be timelike if $g(X,X) < 0$, spacelike if $g(X,X) > 0$ and lightlike (or null) if $g(X,X) = 0$. Similarly, an arbitrary curve $\alpha = \alpha(t)$ in $\mathbb{R}^3_1$ where t is a pseudo-arclength parameter, can locally be timelike, spacelike or null (lightlike), if all of its velocity vectors $\dot{\alpha}(t)$ are respectively timelike, spacelike or null (lightlike), for every $t \in I \subset \mathbb{R}$. A lightlike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ $(resp. x_1 < 0)$ and a timelike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ $(resp.\ x_1 < 0)$. The norm of a vector X is defined by $\|X\|_{IL} = \sqrt{|g(X,X)|}$, [9].

The Lorentzian sphere and hyperbolic sphere of radius 1 in $\mathbb{R}^3_1$ are given by

$$S_1^2 = \left\{ X = (x_1, x_2, x_3) \in \mathbb{R}^3_1 \,:\, g(X,X) = 1 \right\}$$

and

$$H_0^2 = \left\{ X = (x_1, x_2, x_3) \in \mathbb{R}^3_1 \,:\, g(X,X) = -1 \right\}$$

respectively,[8].The vectors $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in \mathbb{R}^3_1$ are orthogonal if and only if $g(X,X) = 0$, [9].

Now let X and Y be two vectors in $\mathbb{R}^3_1$, then the Lorentzian cross product is given by

$$X \times Y = (x_3 y_2 - x_2 y_3,\ x_1 y_3 - x_3 y_1,\ x_1 y_2 - x_2 y_1),\ [2].$$

We denote by $\{T(t), N(t), B(t)\}$ the moving Frenet frame along the curve α. Then T, N and B are the tangent, the principal normal and the binormal vector of the curve α, respectively.

Let α be a unit speed timelike space curve with curvature κ and torsion τ. Let Frenet vector fields of α be $\{T, N, B\}$. In this trihedron, T is timelike vector field, N and B are spacelike vector fields.For this vectors, we can write

$$T \times N = B, \quad N \times B = -T, \quad B \times T = N,$$

where $\times$ is the Lorentzian cross product, [2]. in space $\mathbb{R}^3_1$ Then, Frenet formulas are given by

$$\dot{T} = \kappa N,\ \dot{N} = \kappa T + \tau B,\ \dot{B} = -\tau N,\ [13].$$

The Frenet instantaneous rotation vector for the timelike curve is given by $W = \tau T + \kappa B$.

Let α be a unit speed spacelike space curve with a spacelike binormal. In this trihedron, we assume that T and B are spacelike vector fields and N is a timelike vector field In this situation,.

$$T \times N = B, \quad N \times B = T, \quad B \times T = -N,$$

Then, Frenet formulas are given by

$$\dot{T} = \kappa N,\ \dot{N} = \kappa T + \tau B,\ \dot{B} = \tau N,\ [13].$$

The Frenet instantaneous rotation vector for the spacelike space curve with a spacelike binormal is given by $W = \tau T - \kappa B$.

Lemma 1.1. *Let Xand Y be nonzero Lorentz orthogonal vectors in $\mathbb{R}^3_1$. If X is timelike, then Y is spacelike,* [10].

Lemma 1.2. *Let X and Y be pozitive (negative) timelike vectors in $\mathbb{R}^3_1$. Then*

$$g(X,Y) \leq \|X\| \, \|Y\|$$

whit equality if and only if X and Y are linearly dependent, [10].

Lemma 1.3. *i) Let X and Y be pozitive (negative) timelike vectors in* $\mathbb{R}^3_1$. *By the Lemma 2, there is unique nonnegative real number* $\varphi(X,Y)$ *such that*

$$g(X,Y) = \|X\| \|Y\| \cosh \varphi(X,Y)$$

the Lorentzian timelike angle between X and Y is defined to be $\varphi(X,Y)$.
ii) Let X and Y be spacelike vektors in $\mathbb{R}^3_1$ *that span a spacelike vector subspace. Then we have*

$$|g(X,Y)| \leq \|X\| \|Y\|.$$

Hence, there is a unique real number $\varphi(X,Y)$ *between* 0 *and* π *such that*

$$g(X,Y) = \|X\| \|Y\| \cos \varphi(X,Y)$$

the Lorentzian spacelike angle between X and Y is defined to be $\varphi(X,Y)$.
iii) Let X and Y be spacelike vectors in $\mathbb{R}^3_1$ *that span a timelike vector subspace. Then we have*

$$g(X,Y) > \|X\| \|Y\|.$$

Hence, there is a unique pozitive real number $\varphi(X,Y)$ *between* 0 *and* π *such that*

$$|g(X,Y)| = \|X\| \|Y\| \cosh \varphi(X,Y)$$

the Lorentzian timelike angle between X and Y is defined to be $\varphi(X,Y)$.
iv) Let X be a spacelike vector and Y be a pozitive timelike vector in $\mathbb{R}^3_1$. *Then there is a unique nonnegative reel number* $\varphi(X,Y)$ *such that*

$$|g(X,Y)| = \|X\| \|Y\| \sinh \varphi(X,Y)$$

the Lorentzian timelike angle between X and Y is defined to be $\varphi(X,Y)$, [10].

Theorem 1.1. *Let* α *be a unit speed timelike space curve.Then we have*

1. $\kappa = 0$ *if and only if* α *is a part of a timelike straight line;*
2. $\tau = 0$ *if and only if* α *is a planar timelike curve;*
3. $\tau = 0$ *and* $\kappa =$*constant* > 0 *if and only if* α *is a part of a orthogonal hyperbola;*
4. $\kappa =$*constant* > 0, $\tau =$*constant* $\neq 0$ *and* $|\tau| > \kappa$ *if and only if* α *is a part of a timelike circular helix,*

   $$\alpha(s) = \frac{1}{K}\left(\sqrt{\tau^2 K}s, \kappa\cos\left(\sqrt{K}s\right), \kappa\sin\left(\sqrt{K}s\right)\right)$$

   with $K = \tau^2 - \kappa^2$;
5. $\kappa =$*constant* > 0, $\tau =$*constant* $\neq 0$ *and* $|\tau| < \kappa$ *if and only if* α *is a timelike hyperbolic helix,*

   $$\alpha(s) = \frac{1}{K}\left(\kappa\sinh\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s, \kappa\cosh\left(\sqrt{K}s\right)\right)$$

   with $K = \kappa^2 - \tau^2$;
6. $\kappa =$*constant* > 0, $\tau =$*constant* $\neq 0$ *and* $|\tau| = \kappa$ *if and only if* α *can be parameterized by*

   $$\alpha(s) = \frac{1}{6}\left(\kappa^2 s^3 + 6s, 3\kappa s^2, \kappa\tau s^3\right)$$

   [13].

Theorem 1.2. *Let* α *be a unit speed spacelike space curve with a spacelike binormal. Then we have*

1. $\tau = 0$ *and* $\kappa =$*constant* > 0 *if and only if* α *is a part of a orthogonal hyperbola;*

2. *κ =constant > 0, τ =constant $\neq 0$ if and only if α is a part of a spacelike hyperbolic helix,*

$$\alpha(s) = \frac{1}{K}\left(, \kappa \cosh\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s, \kappa \sinh\left(\sqrt{K}s\right)\right)$$

with $K = \kappa^2 + \tau^2$, [13].

Theorem 1.3. *Let α be a unit speed spacelike space curve with a timelike binormal. Then we have*

1. *$\tau = 0$ and κ =constant > 0 if and only if α is a part of a circle;*
2. *κ =constant > 0, τ =constant $\neq 0$ and $|\tau| > \kappa$ if and only if α is a part of a spacelike hyperbolic helix,*

$$\alpha(s) = \frac{1}{K}\left(\kappa \sinh\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s, \kappa \cosh\left(\sqrt{K}s\right)\right)$$

with $K = \tau^2 - \kappa^2$;

3. *κ =constant > 0, τ =constant $\neq 0$ and $|\tau| < \kappa$ if and only if α is a part of a spacelike circular helix,*

$$\alpha(s) = \frac{1}{K}\left(\sqrt{\tau^2 K}s, \kappa \cos\left(\sqrt{K}s\right), \kappa \sin\left(\sqrt{K}s\right)\right)$$

with $K = \kappa^2 - \tau^2$;

4. *κ =constant > 0, τ =constant $\neq 0$ and $|\tau| = \kappa$ if and only if α can be parameterized by*

$$\alpha(s) = \frac{1}{6}\left(\kappa\tau s^3, -\kappa^2 s^3 + 6s, 3\kappa s^2\right)$$

[13].

2 The Natural Lift of the Fixed Centrode of a Non-null Curve in Minkowski 3-Space

Definition 2.1. *Let M be a hypersurface in $\mathbb{R}^3_1$ and let $\alpha : I \longrightarrow M$ be a parametrized curve. α is called an integral curve of X if*

$$\frac{d}{dt}(\alpha(t)) = X(\alpha(t)) \ \ (\text{for all } t \in I)$$

where X is a smooth tangent vector field on M, [9]. *We have*

$$TM = \underset{P\in M}{} T_P M = \chi(M)$$

where $T_P M$ is the tangent space of M at P and $\chi(M)$ is the space of vector fields of M.

Definition 2.2. *For any parametrized curve $\alpha : I \longrightarrow M$, $\overline{\alpha} : I \longrightarrow TM$ given by*

$$\overline{\alpha}(t) = \left(\alpha(t), \dot{\alpha}(t)\right) = \dot{\alpha}(t)\,|_{\alpha(t)}$$

is called the natural lift of α on TM.Thus, we can write

$$\frac{d\overline{\alpha}}{dt} = \frac{d}{dt}\left(\dot{\alpha}(t)\,|_{\alpha(t)}\right) = D_{\dot{\alpha}(t)}\dot{\alpha}(t)$$

where D is the Levi-Civita connection on $\mathbb{R}^3_1$, [7].

Definition 2.3. *A $X \in \chi(TM)$ is called a geodesic spray if for $V \in TM$ $X(V) = +\varepsilon g(S(V), V)N$,where $\varepsilon = g(N, N)$,*[7].

Theorem 2.1. *The natural lift* $\overline{\alpha}$ *of the curve* α *is an integral curve of geodesic spray* X *if and only if* α *is a geodesic on* M,[7].

Definition 2.4. *(Unit Vector C of Direction W for Non-null Curves):*

1. *For the curve* α *with a timelike tanget,* θ *being a Lorentzian timelike angle between the spacelike binormal unit* $-B$ *and the Frenet instantaneous rotation vector* W.

   (i)If $|\kappa| > |\tau|$, *then* W *is a spacelike vector. In this situation, from Lemma 1.3 iii) we can write*

$$\begin{aligned} \kappa &= \|W\| \cosh\theta \\ \tau &= \|W\| \sinh\theta \end{aligned}$$

   $\|W\|^2 = g(W,W) = \kappa^2 - \tau^2$ *and* $C = \frac{W}{\|W\|} = \sinh\theta T + \cosh\theta B$, *where* C *is unit vector of direction* W.
   (ii)If $|\kappa| < |\tau|$, *then* W *is a timelike vector. In this situation, from Lemma 1.3 iv) we can write*

$$\begin{aligned} \kappa &= \|W\| \sinh\theta \\ \tau &= \|W\| \cosh\theta \end{aligned}$$

   $\|W\|^2 = -g(W,W) = -\left(\kappa^2 - \tau^2\right)$ *and* $C = \cosh\theta T + \sinh\theta B$.

2. *For the curve* α *with a timelike principal normal,* θ *being an angle between the* B *and the* W, *if* B *and* W *spacelike vectors that span a spacelike vektor subspace then by the Lemma 3 ii) we can write*

$$\begin{aligned} \kappa &= \|W\| \cos\theta \\ \tau &= \|W\| \sin\theta \end{aligned}$$

   $\|W\|^2 = g(W,W) = \kappa^2 + \tau^2$ *and* $C = \sin\theta T - \cos\theta B$.

3. *For the curve* α *with a timelike binormal,* θ *being a Lorentzian timelike angle between the* $-B$ *and the* W.
   (i)If $|\kappa| < |\tau|$, *then* W *is a spacelike vector. In this situation, from Lemma 3 iv) we can write*

$$\begin{aligned} \kappa &= \|W\| \sinh\theta \\ \tau &= \|W\| \cosh\theta \end{aligned}$$

   $\|W\|^2 = g(W,W) = \tau^2 - \kappa^2$ *and* $C = -\cosh\theta T + \sinh\theta B$.
   (ii)If $|\kappa| > |\tau|$, *then* W *is a timelike vector. In this situation, from Lemma 3 i) we have*

$$\begin{aligned} \kappa &= \|W\| \cosh\theta \\ \tau &= \|W\| \sinh\theta \end{aligned}$$

   $\|W\|^2 = -g(W,W) = -\left(\tau^2 - \kappa^2\right)$ *and* $C = -\sinh\theta T + \cosh\theta B$.

Let D, $\overline{D}$ and $\overline{\overline{D}}$ be connections in $\mathbb{R}^3_1$, S^2_1 and H^2_0 respectively and ξ be a unit normal vector field of S^2_1 and H^2_0. Then Gauss Equations are given by the followings

$$\begin{aligned} D_X Y &= \overline{D}_X Y + \varepsilon g\left(S\left(X\right), Y\right)\xi, \\ D_X Y &= \overline{\overline{D}}_X Y + \varepsilon g\left(S\left(X\right), Y\right)\xi, \end{aligned}$$

where $\varepsilon = g(\xi,\xi)$ and S is the shape operator of S^2_1 and H^2_0 .

Let α_C be the fixed centrode of the motion described by the curve α. Then the curve is given by $\alpha_C = C(s)$ and $C = \frac{W}{\|W\|}$,where W being the Darboux vector.

We have investigate how α must be curve satifying the condition that $\overline{\alpha}_C$ is an integral curve of the geodesic spray, where $\overline{\alpha}_C$ is the natural lift of the curve α_C.

(i) **Let α be a unit speed timelike space curve.**

(a) Let W is a spacelike vector. If $\overline{\alpha}_C$ is an integral curve of the geodesic spray, then by means of Theorem 2.1

$$\bar{D}_{\dot{\alpha_C}}\dot{\alpha_C} = 0$$

that is

$$D_{\dot{\alpha_C}}\dot{\alpha_C} = \bar{D}_{\dot{\alpha_C}}\dot{\alpha_C} + \varepsilon g\left(S\left(\dot{\alpha_C}\right), \dot{\alpha_C}\right)\xi$$

$$D_{\dot{\alpha_C}}\dot{\alpha_C} = \varepsilon g\left(S\left(\dot{\alpha_C}\right), \dot{\alpha_C}\right)C$$

where $\epsilon = g(\xi, \xi)$ and $\xi = C$. Since T, N, B are linearly independent, we have $\dot{\theta} = 0$ or $\tau = \kappa = 0$.

Corollary 2.1. *If the natural lift $\overline{\alpha}_C$ of α_C is an integral curve of the geodesic spray on the tangent bundle $T\left(S_1^2\right)$ then α is a part of a timelike hyperbolic helix,*

$$\alpha(s) = \frac{1}{K}\left(\kappa\sinh\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s, \kappa\cosh\left(\sqrt{K}s\right)\right)$$

with $K = \kappa^2 - \tau^2$.

(b) Let W is a timelike vector. If $\overline{\alpha}_C$ is an integral curve of the geodesic spray, then by means of Theorem 2.1

$$\bar{\bar{D}}_{\dot{\alpha_C}}\dot{\alpha_C} = 0$$

that is

$$D_{\dot{\alpha}_{CF}}\dot{\alpha_C} = \bar{\bar{D}}_{\dot{\alpha_C}}\dot{\alpha_C} + \varepsilon g\left(S\left(\dot{\alpha_C}\right), \dot{\alpha_C}\right)\xi$$

$$D_{\dot{\alpha_C}}\dot{\alpha_C} = \varepsilon g\left(S\left(\dot{\alpha_C}\right), \dot{\alpha_C}\right)C$$

where $\varepsilon = g(\xi, \xi)$ and $\xi = C$. Since T, N, B are linearly independent, we have $\dot{\theta} = 0$ or $\tau = \kappa = 0$.

Corollary 2.2. *If the natural lift $\overline{\alpha}_C$ of α_C is an integral curve of the geodesic spray on the tangent bundle $T\left(H_0^2\right)$ then α is a part of a timelike circular helix,*

$$\alpha(s) = \frac{1}{K}\left(\sqrt{\tau^2 K}s, \kappa\cos\left(\sqrt{K}s\right), \kappa\sin\left(\sqrt{K}s\right)\right)$$

with $K = \tau^2 - \kappa^2$.

(ii) **Let α be a unit speed spacelike space curve with a spacelike binormal.**

W is a spacelike vector. If $\overline{\alpha}_C$ is an integral curve of the geodesic spray, then by means of Theorem 2.1

$$\bar{D}_{\dot{\alpha_C}}\dot{\alpha_C} = 0$$

that is

$$D_{\dot{\alpha}_{CF}}\dot{\alpha_C} = \bar{D}_{\dot{\alpha_C}}\dot{\alpha_C} + \varepsilon g\left(S\left(\dot{\alpha_C}\right), \dot{\alpha_C}\right)\xi$$

$$D_{\dot{\alpha_C}}\dot{\alpha_C} = \varepsilon g\left(S\left(\dot{\alpha_C}\right), \dot{\alpha_C}\right)C$$

where $\varepsilon = g(\xi, \xi)$ and $\xi = C$. Because T, N, B are linearly independent, we have $\dot{\theta} = 0$ or $\tau = \kappa = 0$.

Corollary 2.3. *If the natural lift $\overline{\alpha}_C$ of α_C is an integral curve of the geodesic spray on the tangent bundle $T\left(S_1^2\right)$ then α is a part of a spacelike hyperbolic helix,*

$$\alpha\left(s\right)=\frac{1}{K}\left(\kappa\cosh\left(\sqrt{K}s\right),\sqrt{\tau^2K}s,\kappa\sinh\left(\sqrt{K}s\right)\right)$$

with $K=\kappa^2+\tau^2$.

(iii) **Let α be a unit speed spacelike space curve with a timelike binormal.**

(a) Let W is a spacelike vector. If $\overline{\alpha}_C$ is an integral curve of the geodesic spray, then by means of Theorem 2.1

$$\bar{D}_{\dot{\alpha_C}}\dot{\alpha_C}=0$$

that is

$$D_{\dot{\alpha_{CF}}}\dot{\alpha_C}=\bar{D}_{\dot{\alpha_C}}\dot{\alpha_C}+\varepsilon g\left(S\left(\dot{\alpha_C}\right),\dot{\alpha_C}\right)\xi$$

$$D_{\dot{\alpha_C}}\dot{\alpha_C}=\varepsilon g\left(S\left(\dot{\alpha_C}\right),\dot{\alpha_C}\right)C$$

where $\varepsilon=g\left(\xi,\xi\right)$ and $\xi=C$. Because T, N, B are linearly independent, we have $\dot{\theta}=0$ or $\tau=\kappa=0$.

Corollary 2.4. *If the natural lift $\overline{\alpha}_C$ of α_C is an integral curve of the geodesic spray on the tangent bundle $T\left(S_1^2\right)$ then α is a part of a spacelike hyperbolic helix,*

$$\alpha\left(s\right)=\frac{1}{K}\left(\kappa\sinh\left(\sqrt{K}s\right),\sqrt{\tau^2K}s,\kappa\cosh\left(\sqrt{K}s\right)\right)$$

with $K=\tau^2-\kappa^2$.

(b) Let W is a timelike vector.If $\overline{\alpha}_C$ is an integral curve of the geodesic spray, then by means of Theorem 2.1

$$\bar{\bar{D}}_{\dot{\alpha_C}}\dot{\alpha_C}=0$$

that is

$$D_{\dot{\alpha_{CF}}}\dot{\alpha_C}=\bar{\bar{D}}_{\dot{\alpha_C}}\dot{\alpha_C}+\varepsilon g\left(S\left(\dot{\alpha_C}\right),\dot{\alpha_C}\right)\xi$$

$$D_{\dot{\alpha_C}}\dot{\alpha_C}=\varepsilon g\left(S\left(\dot{\alpha_C}\right),\dot{\alpha_C}\right)C$$

where $\varepsilon=g\left(\xi,\xi\right)$ and $\xi=C$. Since T, N, B are linearly independent, we have $\dot{\theta}=0$ or $\tau=\kappa=0$.

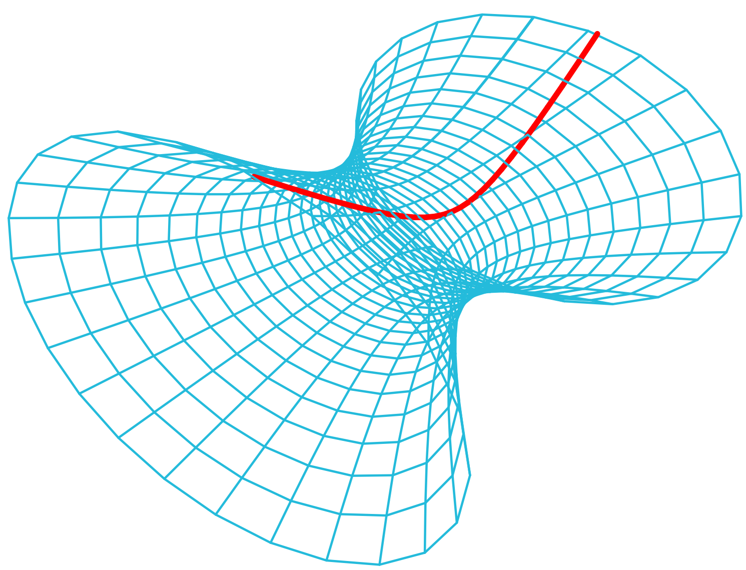
Corollary 2.5. *If the natural lift $\overline{\alpha}_C$ of α_C is an integral curve of the geodesic spray on the tangent bundle $T\left(H_0^2\right)$ then α is a part of a spacelike circular helix,*

$$\alpha\left(s\right)=\frac{1}{K}\left(\sqrt{\tau^2K}s,\kappa\cos\left(\sqrt{K}s\right),\kappa\sin\left(\sqrt{K}s\right)\right)$$

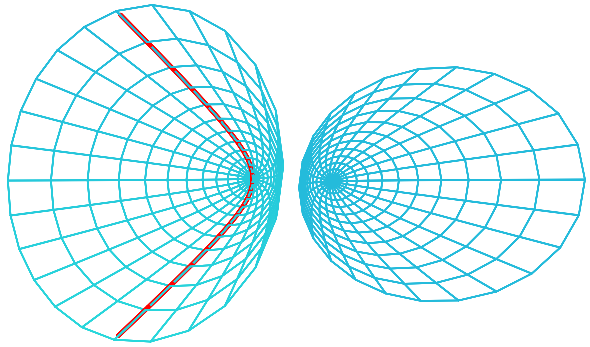
with $K=\kappa^2-\tau^2$.

Example 2.1. *Let* $\alpha(s) = \left(\cosh\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}, \sinh\left(\frac{s}{\sqrt{2}}\right)\right)$ *be a unit speed spacelike hyperbolic helix with*

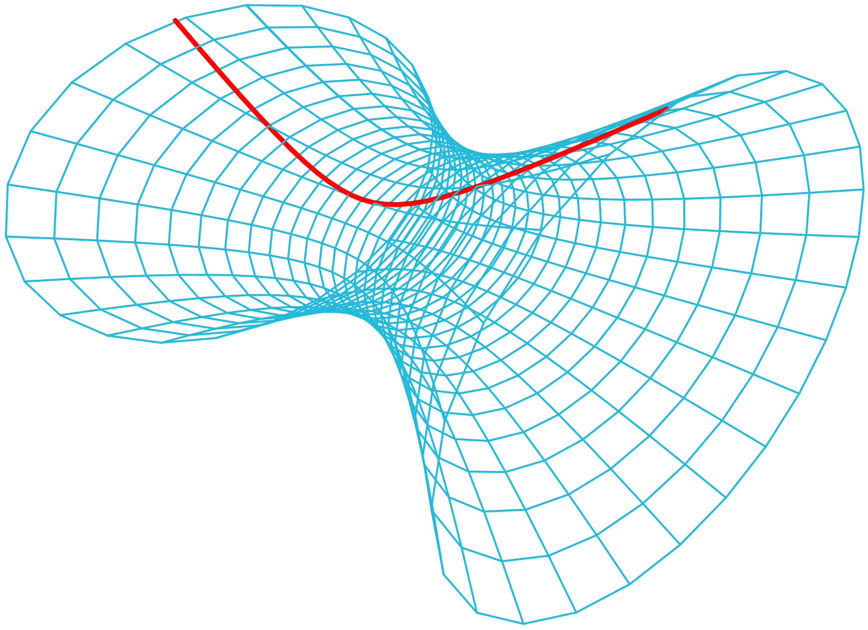
$$
\begin{aligned}
T(s) &= \left(\frac{1}{\sqrt{2}}\sinh\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cosh\left(\frac{s}{\sqrt{2}}\right)\right), \\
N(s) &= \left(\cosh\left(\frac{s}{\sqrt{2}}\right), 0, \sinh\left(\frac{s}{\sqrt{2}}\right)\right), \\
B(s) &= \left(-\frac{1}{\sqrt{2}}\sinh\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\cosh\left(\frac{s}{\sqrt{2}}\right)\right), \\
C(s) &= \left(\sinh\left(\frac{s}{\sqrt{2}}\right), 0, \cosh\left(\frac{s}{\sqrt{2}}\right)\right), \\
\alpha_T(s) &= \left(\frac{1}{\sqrt{2}}\sinh\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cosh\left(\frac{s}{\sqrt{2}}\right)\right), \\
\alpha_B(s) &= \left(-\frac{1}{\sqrt{2}}\sinh\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\cosh\left(\frac{s}{\sqrt{2}}\right)\right), \\
\alpha_C(s) &= \left(\sinh\left(\frac{s}{\sqrt{2}}\right), 0, \cosh\left(\frac{s}{\sqrt{2}}\right)\right),
\end{aligned}
$$



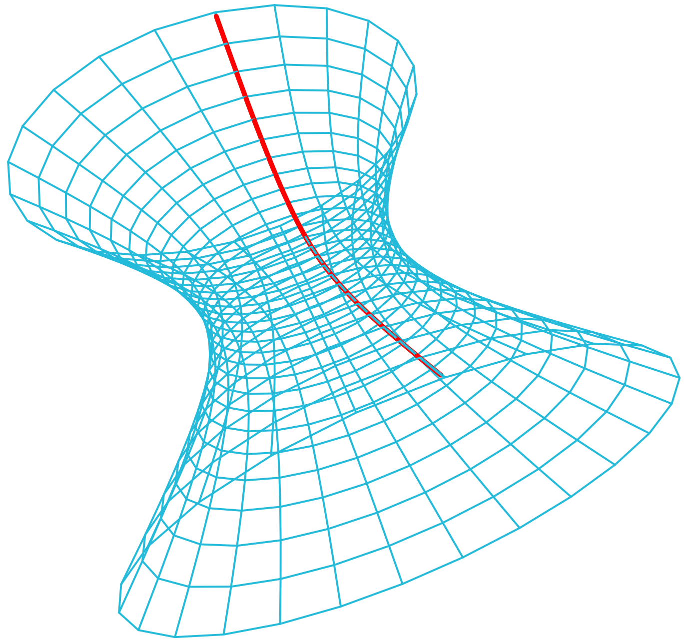
Tangent indicatrix of α.



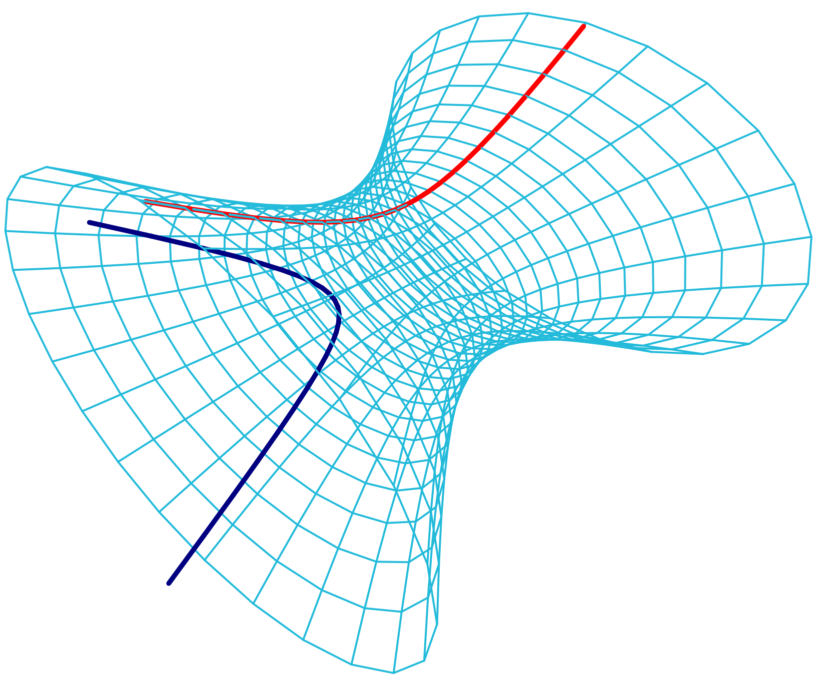
Principal normal indicatrix of α.



Binormal indicatrix of α.



Fixed cenroid of α.



Fixed cenroid of α and its natural lift curve.

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