On the Probabilistic Stability of the 2-variable k-AC-mixed Type Functional Equation

K. Ravi[a,*], R. Jamuna[b], Matina J. Rassias[c], Yanhui Zhang[d] and R. Kishore Kumar[e]

[a]*Department of Mathematics, Sacred Heart College, Tirupattur - 635601, Tamilnadu, India.*

[b]*Research Scholar, Research and Development Centre, Bharathiar University, Coimbatore - 641046, India.*

[c]*Department of Statistical Science, University College London, 1-19 Torrington Place, #140, London, WC1E 7HB, UK.*

[d]*Department of Mathematics, Beijing Technology and Business University, China.*

[e]*Research Scholar, Indian Institute of Technology, Kharagpur, India.*

Abstract

In this paper, we obtain the general solution and the generalized Ulam-Hyers stability of the 2-variable k-AC mixed type functional equation

$$f(x + ky, z + kw) + f(x - ky, z - kw)$$
$$= k^2[f(x + y, z + w) + f(x - y, z - w)] + 2(1 - k^2)f(x, z).$$

for any $k \in Z - \{0, \pm 1\}$ in α-Šerstnev Menger Probabilistic normed spaces.

Keywords: Generalized Hyers-Ulam-Rassias stability, k-AC mixed type functional equation, α-Šerstnev Menger Probabilistic normed spaces.

2010 MSC: 39B55, 39B52, 39B82.

1 Introduction

Menger introduced probabilistic metric space in 1942 [16]. A probabilistic normed space (PN space) is a natural generalization of an ordinary normed linear space. Such spaces were first introduced by Šerstnev in 1963, (see, [28]). Alsina et al. generalized the definition of PN space [1]. This definition became the standard one and has been adopted by all researchers, who after them have investigated the properties of PN spaces. In this article, we adopt the new definition of α-Šerstnev PN spaces (or generalized Šerstnev PN spaces) given in the paper [14] by Lafuerza-Guillén and Rodríguez.

The problem of Ulam-Hyers stability for functional equations concerns deriving conditions under which, given an approximate solution of a functional equation, one may find an exact solution that is near it in some sense. The problem was first stated by Ulam [30] in 1940 for the case of group homomorphisms, and solved by Hyers [9] in the setting of Banach spaces. Hyers result has since then seen many significant generalizations, both in terms of the control condition used to define the concept of approximate solution ([2, 7, 22]) and in terms of the methods used for the proof ([4, 6, 8, 10, 29]). Many interesting results concerning this problem can be found, for example, in [11–13, 15, 17–20, 23, 24].

The stability of generalized mixed type functional equation of the form

$$f(x + ky) + f(x - ky) = k^2[f(x + y) + f(x - y)] + 2(1 - k^2)f(x) \tag{1.1}$$

*Corresponding author.

E-mail address: shckravi@yahoo.co.in (K. Ravi), rjamche31@gmail.com (R. Jamuna), matina@stats.ucl.ac.uk (Matina J. Rassias), zhangyanhui@th.btbu.edu.cn (Yanhui Zhang), rskishorekumar@yahoo.co.in (R. Kishore Kumar).

for fixed integers k and $k \neq 0, \pm 1$ in quasi-Banach spaces was introduced by M. Eshaghi Gordji and H. Khodaie [5]. The mixed type functional equation (1.1) is having the property additive, quadratic and cubic.

J.H. Bae and W.G. Park proved the general solution and investigated the generalized Hyers-Ulam stability of the 2-variable quadratic functional equation

$$f(x + y, z + w) + f(x - y, z - w) = 2f(x, z) + 2(y, w). \tag{1.2}$$

The functional equation (1.2) has solution

$$f(x, y) = ax^2 + bxy + cy^2 \tag{1.3}$$

The general solution and generalized Hyers-Ulam stability of a 3-variable quadratic functional equation

$$f(x + y, z + w, u + v) + f(x - y, z - w, u - v) = 2f(x, z, u) + 2(y, w, v) \tag{1.4}$$

was discussed by K. Ravi and M. Arun Kumar [25]. The solution of (1.4) is of the form

$$f(x, y, z) = ax^2 + by^2 + cz^2 + dxy + eyz + fzx \tag{1.5}$$

Very recently, M. Aruk Kumar et al., introduced and investigated the solution and generalized Ulam-Hyers stability of a 2-varibale AC-mixed type functional equation

$$f(2x + y, 2z + w) - f(2x - y, 2z - w) = 4[f(x + y, z + w) - f(x - y, z - w)] - 6f(y, w) \tag{1.6}$$

having solutions

$$f(x, y) = ax + by \tag{1.7}$$

and

$$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 \tag{1.8}$$

in Banach spaces [3] and Quasi-Beta normed space [21].

Following the same approach, in this paper, we investigate the general solution and establish that generalized Ulam-Hyers stability of the 2-variable k-AC mixed type functional equation

$$f(x + ky, z + kw) + f(x - ky, z - kw)$$
$$= k^2[f(x + y, z + w) + f(x - y, z - w)] + 2(1 - k^2)f(x, z) \tag{1.9}$$

having solutions

$$f(x, y) = ax + by \tag{1.10}$$

and

$$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 \tag{1.11}$$

for fixed integers k with $k \neq 0, \pm 1$ in α-Šestnev (or generalized Šerstnev) Menger Probabilistic normed spaces.

Δ^+ is the space of distribution functions that is, the space of all mappings $F : R \cup \{-\infty, \infty\} \to [0, 1]$ that is non-decreasing, left-continuous on R and such that $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions F for which $\lim_{x \to +\infty} F(x) = 1$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions. The maximal element for Δ^+ in this order is the distribution function ϵ_0 given by

$$\epsilon_0(t) = \begin{cases} 0, \text{if } t \leq 0 \\ 1, \text{if } t > 0 \end{cases}$$

Definition 1.1. *[26, 27] A triangle function is a mapping $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ such that, for all F, G, H, K in Δ^+,*

(1) $\tau(F, \epsilon_0) = F$,

(2) $\tau(F, G) = \tau(G, F)$,

(3) $\tau(F, G) \leq \tau(H, K)$ whenever $F \leq H, G \leq K$,

(4) $\tau(\tau(F, G), H) = \tau(F, \tau(G, H))$.

Moreover, a triangle function is continuous if it is continuous in the metric space (Δ^+, d_s).
Typical continuous triangle functions are

$$\tau_T(F,G)(x) := \sup_{s+t=x} T(F(s), G(t)) \tag{1.12}$$

and

$$\tau_{T^*}(F,G)(x) := \inf_{s+t=x} T^*(F(s), G(t)) \tag{1.13}$$

for all $F, G \in \Delta^+$ *and all* $x \in R$. *Here, T is a continuous t-norm and* T^* *is the corresponding continuous t-conorm, i.e., both are continuous binary operations on [0, 1] that are commutative, associative, and non decreasing in each variable; T has 1 as identity and* T^* *has 0 as identity. Also* $T^*(x, y) = 1 - T(1 - x, 1 - y)$.

Definition 1.2 (PN spaces redefined [1]). *A PN space is a quadruple* (V, ν, τ, τ^*), *where V is a real vector space, τ and τ^* are continuous triangle functions such that* $\tau \leq \tau^*$, *and the mapping* $\nu : V \to \Delta^+$ *satisfies, for all p and q in V, the conditions:*

(N1) $\nu_p = \epsilon_0$ *if, and only if,* $p = \theta$ *(θ is the null vector in V);*

(N2) $\forall\ p \in V, \nu_{-p} = \nu_p;$

(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q);$

(N4) $\forall\ \alpha \in [0, 1], \nu_p \leq \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p}).$

A PN space is called a Šerstnev-space if it satisfies (N1), (N3) and the following condition:

$$(\check{S})\ \ \nu_{\alpha p}(x) = \nu_p\left(\frac{x}{|\alpha|}\right) \tag{1.14}$$

holds for every $\alpha \neq 0 \in \mathbb{R}$ *and* $x > 0$.
If $\tau = \tau_T$ *and* $\tau^* = \tau_{T^*}$ *for some continuous t-norm T and its t-conorm* T^*, *then the PN space* $(V, \nu, \tau_T, \tau_{T^*})$ *is called Menger PN space (briefly, MPN space), and is denoted by* (V, ν, T).
Let $\phi : [0, +\infty] \to [0, +\infty]$ *be a non-decreasing, left-continuous function with* $\phi(0) = 0$, $\phi(+\infty) = +\infty$ *and* $\phi(x) > 0$ *for* $x > 0$. *Let* $\hat{\phi}$ *be the (unique) quasi-inverse of* ϕ *which is left-continuous.* $\hat{\phi}$ *is defined by* $\hat{\phi}(0) = 0$, $\hat{\phi}(+\infty) = +\infty$ *and* $\hat{\phi}(t) = sup\{u : \phi(u) < t\}$ *for all* $0 < t < +\infty$. *It follows that* $\hat{\phi}(\phi(x)) \leq x$ *and* $\phi(\hat{\phi}(y)) \leq y$ *for all x and y.*

Definition 1.3. *[14] A quadruple* (V, ν, τ, τ^*) *satisfy the*

$$(\phi - \check{S})\ \ \nu_{\lambda p}(x) = \nu_p\left(\hat{\phi}\left(\frac{\phi(x)}{|\lambda|}\right)\right) \tag{1.15}$$

for all $x \in \mathbb{R}^+$, $p \in V$ *and* $\lambda \in \mathbb{R}\backslash\{0\}$ *is called a ϕ-Šerstnev PN space (generalized Šerstnev space).*
If $\phi(x) = x^{1/\alpha}$ *for a fixed positive real number α, the condition* $(\phi - \check{S})$ *takes the form*

$$(\alpha - \check{S})\ \ \nu_{\lambda p}(x) = \nu_p\left(\frac{x}{|\lambda|^\alpha}\right) \tag{1.16}$$

for every $p \in V$, *for every* $x > 0$ *and* $\lambda \in \mathbb{R}\backslash\{0\}$.
PN spaces satisfying the condition $(\alpha - \check{S})$ *are called α-Šerstnev PN spaces.*

Definition 1.4. *Let* (V, ν, τ) *be a PN space and* $\{x_n\}$ *be a sequence in V. Then* $\{x_n\}$ *is said to be convergent if there exists* $x \in V$ *such that*

$$\lim_{n \to \infty} \nu_{x_n - x}(t) = 1 \tag{1.17}$$

for all $t > 0$. *In this case x is called the limit of* $\{x_n\}$.

Definition 1.5. *The sequence* $\{x_n\}$ *in* (V, ν, τ) *is called a Cauchy sequence if, for every* $\epsilon > 0$ *and* $\delta > 0$, *there exists a positive integer* n_0 *such that* $\nu(x_n - x_m)(\delta) > 1 - \epsilon$ *for all* $m, n \geq n_0$. *Clearly, every convergent sequence in a PN-space is Cauchy. If every Cauchy sequence is convergent in a PN-space* (V, ν, τ), *then* (V, ν, τ) *is called a probabilistic Banach space (PB-space).*

2 General Solution

Through out this section let U and V be real vector spaces and we present the solution of (1.9) using Lemma 2.1, 2.2, 2.3.

Lemma 2.1. *If $f : U^2 \to V$ is a mapping satisfying (1.9) and let $g : U^2 \to V$ be a mapping given by*

$$g(x, x) = f(2x, 2x) - 8f(x, x) \tag{2.1}$$

for all $x \in U$ then

$$g(2x, 2x) = 2g(x, x) \tag{2.2}$$

for all $x \in U$ such that g is additive.

Proof. Letting (x, y, z, w) by $(0, 0, 0, 0)$ in (1.9), we get

$$f(0, 0) = 0 \tag{2.3}$$

Setting (x, y, z, w) by (y, x, w, z) in (1.9), we obtain

$$f(y + kx, w + kz) + f(y - kx, w - kz) = k^2[f(x + y, w + z) + f(y - x, w - z)] + 2(1 - k^2)f(z, x) \tag{2.4}$$

for all $x, y, z, w \in U$.
Replacing (x, y, z, w) by $(x, -y, z, -w)$ in (2.4), we get

$$f(-y + kx, -w + kz) + f(-y - kz, -w - kz) = k^2[f(x - y), (w - z)) + f(-y - x, -w - z)] + 2(1 - k^2)f(z, x) \tag{2.5}$$

for all $x, y, z, w \in U$.
From (2.4) and (2.5) we arrive at

$$f(y + kx, w + kz) + f(y - kx, w - kz) + f(-y + kx, -w + kz) \\ + f(-y - kx, -w - kz) = k^2[f(x + y, w + z) + f(y - x, w - z) \\ + f(x - y, z - w) + f(-y - x, -w - z)] + 4(1 - k^2)f(z, x) \tag{2.6}$$

Now, letting (x, y, z, w) by $(0, y, 0, y)$ in (2.6), we obtain

$$2[k^2 - 1][f(y, y) + f(-y, -y)] = 0$$

which implies

$$f(y, y) = -f(-y, -y) \tag{2.7}$$

for all $y \in U$.
Replacing (x, y, z, w) by (x, x, x, x) in (1.9), we get

$$f((1 + k)x, (1 + k)x) + f((1 - k)x, (1 - k)x) = k^2 f(2x, 2x) + 2(1 - k^2)f(x, x) \tag{2.8}$$

for all $x \in U$. Now, replacing x by $2x$ in (2.8), we have

$$f(2(1 + k)x, 2(1 + k)x) + f(2(1 - k)x, 2(1 - k)x) = k^2 f(4x, 4x) + 2(1 - k^2)f(2x, 2x) \tag{2.9}$$

for all $x \in U$. Again replacing (x, y, z, w) by $(2x, x, 2x, x)$ in (1.9), we obtain

$$f((2 + k)x, (2 + k)x) + f((2 - k)x, (2 - k)x) = k^2 f(3x, 3x) + k^2 f(x, x) + 2(1 - k^2)f(2x, 2x) \tag{2.10}$$

for all $x \in U$.

Replacing (x, y, z, w) by $(x, 2x, x, 2x)$ in (1.9), we get

$$f((1 + 2k)x, (1 + 2k)x) + f((1 - 2k)x, (1 - 2k)x)$$
$$= k^2 f(3x, 3x) - k^2 f(x, x) + 2(1 - k^2) f(x, x) \tag{2.11}$$

for all $x \in U$. Replacing (x, y, z, w) by $(x, 3x, x, 3x)$ in (1.9), we obtain

$$f((1 + 3k)x, (1 + 3k)x) + f((1 - 3k)x, (1 - 3k)x)$$
$$= k^2 f(4x, 4x) - k^2 f(2x, 2x) + 2(1 - k^2) f(x, x) \tag{2.12}$$

for all $x \in U$. We substitute (x, y, z, w) by $((1 + k)x, x, (1 + k)x, x)$ in (1.9) and then (x, y, z, w) by $((1 - k)x, x, (1 - k)x, x)$ in (1.9) to obtain

$$f((1 + 2k)x, (1 + 2k)x) + f(x, x) = k^2 f((2 + k)x, (2 + k)x)$$
$$+ k^2 f(kx, kx) + 2(1 - k^2) f((1 + k)x, (1 + k)x) \tag{2.13}$$

and

$$f((1 - 2k)x, (1 - 2k)x) + f(x, x) = k^2 f((2 - k)x, (2 - k)x)$$
$$- k^2 f(kx, kx) + 2(1 - k^2) f((1 - k)x, (1 - k)x) \tag{2.14}$$

for all $x \in U$. Then, by adding (2.13) to (2.14), we have

$$f((1 + 2k)x, (1 + 2k)x) + f((1 - 2k)x, (1 - 2k)x) + 2f(x, x)$$
$$= k^2 f((2 + k)x, (2 + k)x) + k^2 f((2 - k)x, (2 - k)x)$$
$$+ 2(1 - k^2)[f((1 + k)x, (1 + k)x) + f((1 - k)x, (1 - k)x)] \tag{2.15}$$

for all $x \in U$. Now, substitute (x, y, z, w) by $((1 + 2k)x, x, (1 + 2k)x, x)$ in (1.9) and (x, y, z, w) by $((1 - 2k)x, x, (1 - 2k)x, x)$ in (1.9) to obtain

$$f((1 + 3k)x, (1 + 3k)x) + f((1 + k)x, (1 + k)x)$$
$$= k^2 f(2(1 + k)x, 2(1 + k)x) + k^2 f(2kx, 2kx)$$
$$+ 2(1 - k^2) f((1 + 2k)x, (1 + 2k)x) \tag{2.16}$$

and

$$f((1 - 3k)x, (1 - 3k)x) + f((1 - k)x, (1 - k)x)$$
$$= k^2 f(2(1 - k)x, 2(1 - k)x) - k^2 f(2kx, 2kx)$$
$$+ 2(1 - k^2) f((1 - 2k)x, (1 - 2k)x) \tag{2.17}$$

for all $x \in U$. Now, adding (2.16) to (2.17), we have,

$$f((1 + 3k)x, (1 + 3k)x) + f((1 - 3k)x, (1 - 3k)x) + f((1 + k)x, (1 + k)x)$$
$$+ f((1 - k)x, (1 - k)x) = k^2 f(2(1 + k)x, 2(1 + k)x)$$
$$+ k^2 f(2(1 - k)x, 2(1 - k)x)$$
$$+ 2(1 - k^2)[f((1 + 2k)x, (1 + 2k)x) + f((1 - 2k)x, (1 - 2k)x)] \tag{2.18}$$

for all $x \in U$. From (2.8), (2.10), (2.11) and (2.15), we arrive at

$$f(3x, 3x) = 4f(2x, 2x) - 5f(x, x) \tag{2.19}$$

for all $x \in U$. From (2.9), (2.11), (2.8), (2.12) and (2.18), we have

$$f(4x, 4x) = 2f(2x, 2x) + 2f(3x, 3x) - 6f(x, x) \tag{2.20}$$

for all $x \in U$. Using (2.19) in (2.20), we obtain

$$f(4x, 4x) = 10f(2x, 2x) - 16f(x, x) \tag{2.21}$$

for all $x \in U$. From (2.21), we establish

$$f(4x, 4x) - 8f(2x, 2x) = 2f(2x, 2x) - 16f(x, x) \tag{2.22}$$

for all $x \in U$. Using (2.1) in (2.22), we get our desired result. $\qquad \square$

Lemma 2.2. *If $f : U^2 \to V$ be a mapping satisfying (1.9) and let $h : U^2 \to V$ be a mapping given by*

$$h(x, x) = f(2x, 2x) - 2f(x, x) \tag{2.23}$$

for all $x \in U$ then

$$h(2x, 2x) = 8h(x, x) \tag{2.24}$$

for all $x \in U$ such that h is cubic.

Proof. Proceeding as in Lemma 2.1, it follows from (2.21)

$$f(4x, 4x) - 2f(2x, 2x) = 8f(2x, 2x) - 16f(x, x) \tag{2.25}$$

for all $x \in U$. Using (2.23) in (2.25), we arrive at our desired result. $\qquad \square$

Remark 2.1. *If $f : U^2 \to V$ be a mapping satisfying (1.9) let $g, h : U^2 \to V$ be mappings defined by (2.1) and (2.23) then*

$$f(x, x) = \frac{1}{6}(h(x, x) - g(x, x)) \tag{2.26}$$

for all $x \in U$.

Lemma 2.3. *If $f : U^2 \to V$ is a mapping satisfying (1.9) and let $t : U \to V$ be a mapping given by*

$$t(x) = f(x, x) \tag{2.27}$$

for all $x \in U$, then t satisfies

$$t(x + ky) + t(x - ky) = k^2[t(x + y) + t(x - y)] + 2(1 - k^2)t(x) \tag{2.28}$$

for all $x, y \in U$.

Proof. From (1.9) and (2.27), we get

$$\begin{aligned} t(x + ky) + t(x - ky) &= f(x + ky, x + ky) - f(x - ky, x - ky) \\ &= k^2[f(x + y, x + y) + f(x - y, x - y)] + 2(1 - k^2)f(x, x) \\ &= k^2[t(x + y) + t(x - y)] + 2(1 - k^2)t(x) \end{aligned}$$

for all $x, y \in U$. $\qquad \square$

3 Stability Results : Direct Method

In this section, we investigate the generalized Ulam-Hyers stability problem of (1.9) using direct method. Let U be a real linear space and (Y, ν, τ_T) be a α-Šerstnev MPB space. Now, we define a difference operator $\Delta f : U^4 \to Y$ by

$$\begin{aligned} \Delta f(x, y, z, w) = {} & f(x + ky, z + kw) + f(x - ky, z - kw) - k^2 f(x + y, z + w) \\ & - k^2 f(x - y, z - w) - 2(1 - k^2)f(x, z) \end{aligned} \tag{3.1}$$

$\forall \; x, y, z, w \in U$, where $f : U^2 \to Y$ is a mapping.

Theorem 3.1. *Let $f : U^2 \to Y$ be a mapping for which there exist a function $\xi : U^4 \to D^+$ with the condition*

$$\lim_{m\to\infty} \tau_T \left[\xi_{(2^m x, 2^m y, 2^m z, 2^m w)}(2^{m\alpha} t), \xi_{(2^m x, 2^m y, 2^m z, 2^m w)}(2^{(m-3)\alpha-1} t) \right] = 1 \tag{3.2}$$

such that the functional inequality

$$\nu_{\Delta f(x,y,z,w)}(t) \geq \xi_{x,y,z,w}(t) \tag{3.3}$$

for all $x, y, z, w \in U$, $t > 0$ and $\alpha > 0$. Then there exists a unique 2-variable additive mapping $A(x,x) : U^2 \to Y$ satisfying (1.9) and

$$\nu_{f(2x,2x)-8f(x,x)-A(x,x)}(t) \geq \tilde{\Phi} \tag{3.4}$$

where

$$A(x,x) = \lim_{n\to\infty} \frac{f(2^{(n+1)} x, 2^{(n+1)} x) - 8 f(2^n x, 2^n x)}{2^n} \tag{3.5}$$

$$\begin{cases} \tilde{\Phi} = \lim_{n\to\infty} \Phi_n = 1 \\ \Phi_n = \tau_T \left[\tilde{\tau}_{T(2^{n-1} x)}(t), \Phi_{n-1} \right], & \text{for } n > 1 \end{cases} \tag{3.6}$$

$$\Phi_1 = \tilde{\tau}_{T(x)}(t) \tag{3.7}$$

and

$$\begin{aligned}
\tilde{\tau}_{T(x)}(t) = \tau_T \left(\tau_T \left(\tau_T \left(\tau_T \left(\xi_{(x,2x,x,2x)} \left(\frac{k^{2\alpha} t}{2^4 2^{\alpha}} \right), \right.\right.\right.\right. \\
\xi_{((1-2k)x,x,(1-2k)x,x)} \left(\frac{k^{2\alpha} |k^2-1|^{\alpha} t}{2^4} \right) \right), \tau_T \left(\xi_{((1+2k)x,x,(1+2k)x,x)} \left(\frac{k^{2\alpha} |k^2-1|^{\alpha} t}{2^4} \right), \\
\xi_{(x,x,x,x)} \left(\frac{k^{2\alpha} |k^2-1|^{\alpha} t}{2^4} \right) \right) \right), \tau_T \left(\xi_{(2x,2x,2x,2x)} \left(\frac{|k^2-1|^{\alpha} t}{2^3} \right), \\
\xi_{(x,3x,x,3x)} \left(\frac{k^{2\alpha} |k^2-1|^{\alpha} t}{2^3} \right), \tau_T \left(\tau_T \left(\tau_T \left(\xi_{(x,x,x,x)} \left(\frac{k^{2\alpha} t}{2^4 2^{2\alpha}} \right), \right.\right.\right. \\
\xi_{((1-k)x,x,(1-k)x,x)} \left(\frac{k^{2\alpha} |k^2-1|^{\alpha} t}{2^4 2^{\alpha}} \right) \right), \tau_T \left(\xi_{((1+k)x,x,(1+k)x,x)} \left(\frac{k^{2\alpha} |k^2-1|^{\alpha} t}{2^4 2^{\alpha}} \right) \\
\xi_{(x,2x,x,2x)} \left(\frac{k^{2\alpha} |k^2-1|^{\alpha} t}{2^4 2^{\alpha}} \right) \right), \xi_{(2x,x,2x,x)} \left(\frac{|k^2-1|^{\alpha} t}{2^4 2^{\alpha}} \right) \right) \right),
\end{aligned} \tag{3.8}$$

for all $x \in U$, $t > 0$ and $\alpha > 0$.

Proof. Letting (x, y, z, w) by (x, x, x, x) in (3.3), we obtain

$$\begin{aligned}
\nu_{f((1+k)x,(1+k)x)+f((1-k)x,(1-k)x)-k^2 f(2x,2x)-2(1-k^2)f(x,x)}(t) \\
\geq \xi_{(x,x,x,x)}(t), \; \forall\, x \in U, t > 0.
\end{aligned} \tag{3.9}$$

It follows from (3.9) that

$$\begin{aligned}
\nu_{f(2(1+k)x,2(1+k)x)+f(2(1-k)x,2(1-k)x)-k^2 f(4x,4x)-2(1-k^2)f(2x,2x)}(t) \\
\geq \xi_{(2x,2x,2x,2x)}(t), \; \forall\, x \in U, t > 0.
\end{aligned} \tag{3.10}$$

Replacing (x, y, z, w) by $(2x, x, 2x, x)$ in (3.3), respectively, we have

$$\begin{aligned}
\nu_{f((2+k)x,(2+k)x)+f((2-k)x,(2-k)x)-k^2 f(3x,3x)-k^2 f(x,x)-2(1-k^2)f(2x,2x)}(t) \\
\geq \xi_{(2x,x,2x,x)}(t), \; \forall\, x \in U, t > 0.
\end{aligned} \tag{3.11}$$

Setting (x, y, z, w) by $(x, 2x, x, 2x)$ in (3.3) gives

$$\nu_{f((1+2k)x,(1+2k)x)+f((1-2k)x,(1-2k)x)-k^2 f(3x,3x)-k^2 f(x,x)-2(1-k^2)f(x,x)}(t)$$
$$\geq \xi_{(x,2x,x,2x)}(t), \ \forall \ x \in U, t > 0. \tag{3.12}$$

Replacing (x, y, z, w) by $(x, 3x, x, 3x)$ in (3.3), we obtain

$$\nu_{f((1+3k)x,(1+3k)x)+f((1-3k)x,(1-3k)x)-k^2 f(4x,4x)+k^2 f(2x,2x)-2(1-k^2)f(x,x)}(t)$$
$$\geq \xi_{(x,3x,x,3x)}(t), \ \forall \ x \in U, t > 0. \tag{3.13}$$

Replacing (x, y, z, w) by $((1 + k)x, x, (1 + k)x, x)$ in (3.3), respectively, we get

$$\nu_{f((1+2k)x,(1+2k)x)+f(x,x)-k^2 f((2+k)x,(2+k)x)-k^2 f(kx,kx)-2(1-k^2)f((1+k)x,(1+k)x)}(t)$$
$$\geq \xi_{((1+k)x,x,(1+k)x,x)}(t), \ \forall \ x \in U, t > 0. \tag{3.14}$$

Replacing (x, y, z, w) by $((1 - k)x, x, (1 - k)x, x)$ in (3.3), respectively, one gets

$$\nu_{f((1-2k)x,(1-2k)x)+f(x,x)-k^2 f((2-k)x,(2-k)x)+k^2 f(kx,kx)-2(1-k^2)f((1-k)x,(1-k)x)}(t)$$
$$\geq \xi_{((1-k)x,x,(1-k)x,x)}(t), \ \forall \ x \in U, t > 0. \tag{3.15}$$

Replacing (x, y, z, w) by $((1 + 2k)x, x, (1 + 2k)x, x)$ in (3.3), respectively, we obtain

$$\nu_{f((1+3k)x,(1+3k)x)+f((1+k)x,(1+k)x)-k^2 f(2(1+k)x,2(1+k)x)-k^2 f(2kx,2kx)-2(1-k^2)f((1+2k)x,(1+2k)x)}(t)$$
$$\geq \xi_{((1+2k)x,x,(1+2k)x,x)}(t), \ \forall \ x \in U, t > 0. \tag{3.16}$$

Replacing (x, y, z, w) by $((1 - 2k)x, x, (1 - 2k)x, x)$ in (3.3), respectively, we have

$$\nu_{f((1-3k)x,(1-3k)x)+f((1-k)x,(1-k)x)-k^2 f(2(1-k)x,2(1-k)x)+k^2 f(2kx,2kx)-2(1-k^2)f((1-2k)x,(1-2k)x)}(t)$$
$$\geq \xi_{((1-2k)x,x,(1-2k)x,x)}(t), \ \forall \ x \in U, t > 0. \tag{3.17}$$

Thus it follows from (3.9), (3.11), (3.12), (3.14) and (3.15) that

$$\nu_{f(3x,3x)-4f(2x,2x)+5f(x,x)}(t)$$
$$\geq \tau_T \left(\tau_T \left(\tau_T \left(\xi_{(x,x,x,x)}\left(\frac{k^{2\alpha}t}{2^3 2^\alpha}\right), \ \xi_{((1-k)x,x,(1-k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3}\right)\right), \right.$$
$$\left. \tau_T \left(\xi_{((1+k)x,x,(1+k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3}\right), \xi_{(x,2x,x,2x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3}\right)\right)\right)$$
$$\xi_{(2x,x,2x,x)}\left(\frac{|k^2-1|^\alpha t}{2}\right)\right), \ \forall \ x \in U, t > 0 \text{ and } \alpha > 0. \tag{3.18}$$

Also, from (3.9), (3.10), (3.12), (3.13) (3.16) and (3.17), we have

$$\nu_{f(4x,4x)-2f(3x,3x)-2f(2x,2x)+6f(x,x)}(t)$$
$$\geq \tau_T \left(\tau_T \left(\tau_T \left(\xi_{(x,2x,x,2x)}\left(\frac{k^{2\alpha}t}{2^3 2^\alpha}\right), \ , \xi_{((1-2k)x,x,(1-2k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3}\right)\right), \right.$$
$$\left. \tau_T \left(\xi_{((1+2k)x,x,(1+2k)x,x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3}\right), \xi_{(x,x,x,x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3}\right)\right)\right)$$
$$\tau_T \left(\xi_{(2x,2x,2x,2x)}\left(\frac{|k^2-1|^\alpha t}{2^2}\right), \xi_{(x,3x,x,3x)}\left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^2}\right)\right), \tag{3.19}$$

for all $x \in U$, $t > 0$ and $\alpha > 0$.
Finally, by using (3.18) and (3.19), we obtain

$$\nu_{f(4x,4x)-10f(2x,2x)+16f(x,x)}(t) \geq \tilde{\tau}_{T(x)}(t) \tag{3.20}$$

where,

$$
\tilde{\tau}_{T(x)}(t)
$$

$$
= \tau_T \left(\tau_T \left(\tau_T \left(\tau_T \left(\xi_{(x,2x,x,2x)} \left(\frac{k^{2\alpha} t}{2^4 2^\alpha} \right), \quad \xi_{((1-2k)x,x,(1-2k)x,x)} \left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^4} \right) \right), \right. \right. \right.
$$

$$
\left. \tau_T \left(\xi_{((1+2k)x,x,(1+2k)x,x)} \left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^4} \right), \xi_{(x,x,x,x)} \left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^4} \right) \right) \right),
$$

$$
\tau_T \left(\xi_{(2x,2x,2x,2x)} \left(\frac{|k^2-1|^\alpha t}{2^3} \right), \xi_{(x,3x,x,3x)} \left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^3} \right) \right),
$$

$$
\tau_T \left(\tau_T \left(\tau_T \left(\xi_{(x,x,x,x)} \left(\frac{k^{2\alpha} t}{2^4 2^{2\alpha}} \right), \xi_{((1-k)x,x,(1-k)x,x)} \left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^4 2^{2\alpha}} \right) \right), \right. \right.
$$

$$
\tau_T \left(\xi_{((1+k)x,x,(1+k)x,x)} \left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^4 2^\alpha} \right), \xi_{(x,2x,x,2x)} \left(\frac{k^{2\alpha}|k^2-1|^\alpha t}{2^4 2^\alpha} \right) \right),
$$

$$
\left. \left. \xi_{(2x,x,2x,x)} \left(\frac{|k^2-1|^\alpha t}{2^2 2^\alpha} \right) \right) \right), \ \forall \, x \in U, t > 0 \text{ and } \alpha > 0. \tag{3.21}
$$

Let $g : U^2 \to Y$ be a function defined by

$$
g(x,x) = f(2x,2x) - 8f(2x,2x) \ \text{ for all } x \in U. \tag{3.22}
$$

From (3.20), we conclude that

$$
\nu_{\frac{g(2x,2x)}{2} - g(x,x)}(t) \geq \tilde{\tau}_{T(x)}(2^\alpha t) \geq \tilde{\tau}_{T(x)}(t), \ \forall \, x \in U, t > 0 \text{ and } \alpha > 0 \tag{3.23}
$$

which implies that

$$
\nu_{\frac{g(2^{\ell+1}x,2^{\ell+1}x)}{2^{\ell+1}} - \frac{g(2^\ell x,2^\ell x)}{2^\ell}}(t) \geq \tilde{\tau}_{T(2^\ell x)}(2^{(\ell+1)\alpha} t) \tag{3.24}
$$

for all $x \in U$, $t > 0$, $\alpha > 0$ and $\ell \in \mathbb{N}$. From the inequalities (3.23) and (3.24) we use iterative methods and induction on n and apply defined sequence in (3.6) and (3.7) to prove our next relation

$$
\nu_{\frac{g(2^n x,2^n x)}{2^n} - g(x,x)}(t) \geq \tau_T \left[\tilde{\tau}_{T(2^{n-1}x)}(t), \Phi_{n-1} \right] \ \forall \, x \in U, t > 0 \text{ and } \alpha > 0. \tag{3.25}
$$

So

$$
\nu_{\frac{g(2^{m+n}x,2^{m+n}x)}{2^{m+n}} - \frac{g(2^m x,2^m x)}{2^m}}(t) \geq \tau_T \left[\tilde{\tau}_{T(2^{(m+n)-1}x)}(2^{m\alpha} t), \Phi_{(m+n)-1} \right] \tag{3.26}
$$

for all non negative integers m and n and for all $x \in U$, $t > 0$. By assumptions (3.26) shows that the sequence $\left\{ \frac{g(2^n x,2^n x)}{2^n} \right\}$ is a Cauchy sequence in Y for all $x \in U$. Since Y is a α-Šerstnev MPB, it follows that the sequence $\left\{ \frac{g(2^n x,2^n x)}{2^n} \right\}$ converges for all $x \in U$. Therefore, one can define the function $A(x,x) : U^2 \to Y$ by

$$
A(x,x) = \lim_{n \to \infty} \frac{g(2^n x, 2^n x)}{2^n} \ \text{ for all } x \in U. \tag{3.27}
$$

Now, if we replace (x,y,z,w) by $(2^n x, 2^n y, 2^n z, 2^n w)$ in (3.3), respectively, then it follows that

$$
\nu_{\frac{\Delta g(2^n x,2^n y,2^n z,2^n w)}{2^n}}(t) = \nu_{\frac{\Delta f(2^{n+1}x,2^{n+1}y,2^{n+1}z,2^{n+1}w)}{2^n} - 8\frac{\Delta f(2^n x,2^n y,2^n z,2^n w)}{2^n}}(t)
$$

$$
\geq \tau_T \left[\nu_{\Delta f(2^{n+1}x,2^{n+1}y,2^{n+1}z,2^{n+1}w)}(2^{n\alpha-1} t), \nu_{\Delta f(2^n x,2^n y,2^n z,2^n w)}(2^{(n-3)\alpha-1} t) \right]
$$

$$
\geq \tau_T \left[\xi_{2^{n+1}x,2^{n+1}y,2^{n+1}z,2^{n+1}w}(2^{n\alpha-1} t), \xi_{2^n x,2^n y,2^n z,2^n w}(2^{(n-3)\alpha-1} t) \right] \tag{3.28}
$$

for all $x,y,z,w \in U$, $t > 0$ and $\alpha > 0$. By letting $n \to \infty$ in (3.28), we have $\nu_{\Delta A(x,y,z,w)}(t) = 1$ for all $t > 0$ and so $\Delta A(x,y,z,w) = 0$. Hence A satisfies (1.9) for all $x,y,z,w \in U$. To prove (3.4), if we take the limit as

$n \to \infty$ in (3.25), then we can get (3.4). Finally, to prove the uniqueness of the additive function A subject to (3.4), assume that there exists another 2-variable additive mapping A' which satisfies (3.4) and (1.9), then

$$
\begin{aligned}
\nu_{A(x,x)-A'(x,x)}(t) &= \nu_{\frac{A(2^n x,2^n x)}{2^n} - \frac{A'(2^n x,2^n x)}{2^n}}(t) \\
&= \nu_{A(2^n x,2^n x)-A'(2^n x,2^n x)}(2^{n\alpha} t) \\
&\geq \nu_{A(2^n x,2^n x)-g(2^n x,2^n x)+g(2^n x,2^n x)-A'(2^n x,2^n x)}(2^{n\alpha} t) \\
&\geq \lim_{n \to \infty} \tau_T \left[\tau_T \left[\tilde{\tau}_{T(2^{2n-1} x)}(2^{n\alpha-1} t), \Phi_{n-1} \right], \tau_T \left[\tilde{\tau}_{T(2^{2n-1} x)}(2^{n\alpha-1} t), \Phi_{n-1} \right] \right]
\end{aligned}
\tag{3.29}
$$

which tends to 1 as $n \to \infty$ for all $x \in U$. So we can conclude that $A = A'$. This completes the proof of the theorem. $\qquad\square$

Theorem 3.2. *Let $f : U^2 \to Y$ be a mapping for which there exist a function $\xi : U^4 \to D^+$ with the condition*

$$
\lim_{m \to \infty} \tau_T \left[\xi_{(2^m x,2^m y,2^m z,2^m w)}(2^{3m\alpha} t), \xi_{(2^m x,2^m y,2^m z,2^m w)}(2^{(3m-1)\alpha-1} t), \right]
\tag{3.30}
$$

such that the functional inequality (3.3) is satisfied for all $x, y, z, w \in U$, $t > 0$ and $\alpha > 0$. Then there exists a unique 2-variable cubic mapping $c(x, x) : U^2 \to Y$ satisfying (1.9) and

$$
\nu_{f(2x,2x)-2f(x,x)-c(x,x)}(t) \geq \tilde{\Psi}
\tag{3.31}
$$

where

$$
c(x, x) = \lim_{n \to \infty} \frac{f(2^{(n+1)} x, 2^{(n+1)} x) - 2f(2^n x, 2^n x)}{2^{3n}}
\tag{3.32}
$$

$$
\begin{cases}
\tilde{\Psi} = \lim_{n \to \infty} \Psi_n = 1 \\
\Psi_n = \tau_T \left[\tilde{\tau}_{T(2^{n-1} x)}(2^{2n\alpha} t), \Psi_{n-1} \right]
\end{cases}
\tag{3.33}
$$

$$
\Psi_1 = \tilde{\tau}_{T(x)}(2^{2\alpha} t), \ \forall \, x \in U, t > 0, \alpha > 0,
\tag{3.34}
$$

where $\tilde{\tau}_{T(x)}(t)$ is defined as in Theorem 3.1.

Proof. By the similar approach as in the proof of Theorem 3.1, we can obtain

$$
\nu_{f(4x,4x)-10f(2x,2x)+16f(x,x)}(t) \geq \tilde{\tau}_{T(x)}(t), \ \forall \, x \in U, t > 0.
$$

Let $h : U^2 \to Y$ be a function defined by

$$
h(x, x) = f(2x, 2x) - 2f(x, x), \text{ for all } x \in U
\tag{3.35}
$$

Thus from (3.20), we have

$$
\nu_{\frac{h(2x,2x)}{2^3}-h(x,x)}(t) \geq \tilde{\tau}_{T(x)}(2^{3\alpha} t) \geq \overline{\tau}_{T(x)}(2^{2\alpha} t), \ \forall \, x \in U, t > 0, \alpha > 0
\tag{3.36}
$$

which implies that

$$
\nu_{\frac{h(2^{\ell+1} x,2^{\ell+1} x)}{2^{3(\ell+1)}} - \frac{h(2^\ell x,2^\ell x)}{2^{3\ell}}}(t) \geq \overline{\tau}_{T(2^\ell x)}(2^{3(\ell+1)\alpha} t)
\tag{3.37}
$$

for all $x \in U$, $t > 0$, $\alpha > 0$ and $\ell \in \mathbb{N}$. Thus it follows from (3.37) and (N3)

$$
\nu_{\frac{h(2^n x,2^n x)}{2^{3n}} - h(x,x)}(t) \geq \tau_T \left[\tilde{\tau}_{T(2^{n-1} x)}(2^{2n\alpha} t), \Phi_{n-1} \right], \ \forall \, x \in U; t > 0, \alpha > 0.
\tag{3.38}
$$

In order to prove the convergence of the sequence $\left\{ \frac{h(2^n x,2^n x)}{2^{3n}} \right\}$ if we replace x with $2^m x$ in (3.38), then we get

$$
\nu_{\frac{h(2^{n+m} x,2^{n+m} x)}{2^{3(n+m)}} - \frac{h(2^m x,2^m x)}{2^{3m}}}(t) \geq \tau_T \left[\overline{\tau}_{T(2^{n+m-1} x)}(2^{(2n+3m)\alpha} t), \Phi_{n+m} \right]
\tag{3.39}
$$

for all non-negative integers m and n and $\forall\, x \in U, t > 0, \alpha > 0$.

Since the right hand side of the inequality tends to 1 as m and n tend to infinity, by assumptions, the sequence $\left\{ \frac{h(2^n x, 2^n x)}{2^{3n}} \right\}$ is a Cauchy sequence in Y for all $x \in U$. Since Y is a α-Šerstnev MPB, one can define the function $c(x, x) : U^2 \to Y$ by

$$c(x, x) = \lim_{n \to \infty} \frac{h(2^n x, 2^n x)}{2^{3n}} \text{ for all } x \in U. \tag{3.40}$$

Now, if we replace (x, y, z, w) by $(2^n x, 2^n y, 2^n z, 2^n w)$ in (3.3), respectively, then it follows that

$$
\begin{aligned}
\nu_{\frac{\Delta h(2^n x, 2^n y, 2^n z, 2^n w)}{2^{3n}}}(t) &= \nu_{\frac{\Delta f(2^{n+1} x, 2^{n+1} y, 2^{n+1} z, 2^{n+1} w)}{2^{3n}} - 2\frac{\Delta f(2^n x, 2^n y, 2^n z, 2^n w)}{2^{3n}}}(t) \\
&\geq \tau_T \left[\nu_{\Delta f(2^{n+1} x, 2^{n+1} y, 2^{n+1} z, 2^{n+1} w)}(2^{3n\alpha - 1} t), \nu_{\Delta f(2^n x, 2^n y, 2^n z, 2^n w)}(2^{(3n-2)\alpha - 1} t) \right] \\
&\geq \tau_T \left[\xi_{(2^{n+1} x, 2^{n+1} y, 2^{n+1} z, 2^{n+1} w)}(2^{3n\alpha - 1} t), \xi_{(2^n x, 2^n y, 2^n z, 2^n w)}(2^{(3n-1)\alpha - 1} t) \right]
\end{aligned}
\tag{3.41}
$$

for all $x, y, z, w \in U, t > 0$ and $\alpha > 0$. By letting $n \to \infty$ in (3.41), we find that $\nu_{\Delta c(x,y,z,w)}(t) = 1$ for all $t > 0$, which implies $\Delta c(x, y, z, w) = 0$ and so c satisfies (1.9) for all $x, y, z, w \in U$. To prove (3.31), if we take the limit as $n \to \infty$ in (3.38), then we get (3.31). The rest of the proof is similar to the proof of Theorem 3.1. This completes the proof. $\qquad\square$

Theorem 3.3. *Let $\xi : U^2 \to D^+$ be a function with the conditions given in (3.2) and (3.30) and $f : U^2 \to Y$ be a function which satisfies (3.3) for all $x, y, z, w \in U$ and $t > 0$. Then there exists a unique 2-variable additive mapping $A : U^2 \to Y$ and a unique 2-variable cubic mapping $C : U^2 \to Y$ satisfying (1.9) such that*

$$
\begin{aligned}
\nu_{f(x,x) - A(x,x) - C(x,x)}(t) &\geq \\
\lim_{n \to \infty} \tau_T &\left[\tau_T \left(\tilde{\tau}_{T(2^{n-1} x)}(3^\alpha 2^{\alpha - 1} t), \Phi_{n-1} \right), \tau_T \left(\tilde{\tau}_{T(2^{n-1} x)}(2^{(2n+1)\alpha - 1} 3^\alpha t), \Psi_{n-1} \right) \right]
\end{aligned}
\tag{3.42}
$$

for all $x \in U, t > 0$ and $\alpha > 0$, where Φ_n, $\tilde{\tau}_{T(x)}(t)$ is defined as in Theorem 3.1 and Ψ_n is defined as in Theorem 3.2.

Proof. By Theorems 3.1 and 3.2, there exist a unique 2-variable additive function $A_0 : U^2 \to Y$ and a unique 2-variable cubic function $C_0 : U^2 \to Y$ such that

$$\nu_{f(2x,2x) - 8f(x,x) - A_0(x,x)}(t) \geq \tilde{\Phi} \tag{3.43}$$

and

$$\nu_{f(2x,2x) - 2f(x,x) - C_0(x,x)}(t) \geq \tilde{\Psi}, \; \forall\, x \in U, t > 0. \tag{3.44}$$

Thus it follows from (3.43) and (3.44) that

$$
\begin{aligned}
\nu_{f(x,x) + \frac{1}{6} A_0(x,x) - \frac{1}{6} C_0(x,x)}(t) & \\
\geq \tau_T &\left[\nu_{f(2x,2x) - 8f(x,x) - A_0(x,x)}(3^\alpha 2^{\alpha - 1} t), \nu_{f(2x,2x) - 2f(x,x) - C_0(x,x)}(3^\alpha 2^{\alpha - 1} t) \right]
\end{aligned}
\tag{3.45}
$$

for all $x \in U, t > 0$ and $\alpha > 0$. Thus we obtain (3.42) by letting $A(x, x) = -\frac{1}{6} A_0(x, x)$ and $C(x, x) = \frac{1}{6} C_0(x, x)$ for all $x \in U$. This completes the proof of the stability of the functional equation (1.9) in α-Šerstnev MPN spaces. $\qquad\square$

4 Acknowledgement

References

[1] C. Alsina, B. Schweizer and A. Sklar, On the definition of a probabilistic normed space, *Aequationes Math.*, 46 (1993), 91–98.

[2] T. Aoki, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan*, 2 (1950), 64–66.1

[3] M. Arun Kumar, Matina J. Rassias, Yanhui Zhang, Ulam-Hyers stability of a 2-variable AC-mixed type functional equation: direct and fixed point methods, *Journal of Modern Mathematics Frontier*, 1(3) (2012), 10–26.

[4] P.W. Cholewa, Remarks on the stability of functional equations, *Aequationes Math.*, 27(1-2) (1984), 76–86.

[5] M. Eshagi Gordji and H. Khodaie, Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces, arxiv : 0812. 2939 VI Math FA (2008).

[6] M. Eshaghi Gordji and H. Khodaie, The fixed point method for fuzzy approximation of a functional equation associated with inner product spaces, *Discrete Dyn. Nat. Soc.*, 2010 (2010), Article ID 140767, 15 pages, doi: 10.1155 / 2010 / 14076.

[7] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.*, 184 (1994), 431–436.

[8] P. Găvruta and L. Găvruta, A new method for the generalized Hyers-Ulam-Rassias stability, *Int. J. Nonlinear Anal. Appl.*, 1(2) (2010), 11–18.

[9] D.H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci.*, USA, 27 (1941), 222–224.1.

[10] D.H. Hyers, G. Isac and T.M. Rassias, Stability of functional equations in several variables, Birkhauser, Basel, (1998).

[11] D.H. Hyers, G. Isac and Th.M. Rassias, On the asymptoticity aspect of Hyers-Ulam stability of mappings, *Proceedings of the American Mathematical Society*, 126(2) (1998), 425–430.

[12] K.W. Jun and H.M. Kim, On the Hyers-Ulam stability of a generalized quadratic and additive functional equation, *Bulletin of the Korean Mathematical Society*, 42(1) (2005), 133–148.

[13] K.W. Jun and H.M. Kim, Ulam stability problem for generalized *A*-quadratic mappings, *Journal of Mathematical Analysis and Applications*, 305(2) (2005), 466–476.

[14] B. Lafuerza-Guillén and J.L. Rodríguez, Boundedness in generalized Šerstnev spaces, http://front.math.UCdavis.edu/ math.PR/ 0408207.

[15] Y.H. Lee and K.W. Jun, A note on the Hyers-Ulam-Ravias stability of Penider equation, *Journal of the Korean Mathematical Society*, 37(1) (2000), 111-124.

[16] K. Menger, Statistical metrices, *Proc. Nat. Acad. Sci.*, USA, (28) (1942), 535–537.

[17] A. Najati, On the stability of a quartic functional equation, *Journal of Mathematical Analysis and Applications*, 340(1) (2008), 569–574.

[18] A. Najati and C. Park, Hyers-Ulam-Ravias stability of homomorphisms in quasi-Banach algebras associated to the Pexidesized Cauchy functional equation, *Journal of Mathematical Analysis and Applications*, 335(2) (2007), 763–778.

[19] C.G. Park, On the stability of the quadratic mappings in Banach modules, *Journal of Mathematical Analysis and Applications*, 276(1) (2002), 135–144.

[20] C.G. Park, On the stability Hyers-Ulam-Ravias stability of generalized quadratic mappings in Banach modules, *Journal of Mathematical Analysis and Applications*, 291(1) (2004), 214–223.

[21] J.M. Rassias, M. Arun Kumar, S. Ramamoorthi and s. Hemalatha, Ulam-Hyers stability of a 2-variable AC-mixed type functional equation in quasi-beta normed spaces: direct and fixed point methods, *Malaya Journal of Matematik*, 2(2) (2014), 108–128.

[22] Th. M. Rassias, On the stability of linear mappings in Banach spaces, *Proc. Amer. Math. Soc.*, 72(2) (1978), 297–300.1.

[23] Th. M. Rassias, On the stability of the quadratic functional equation and its applications, *Studia Mathematica*, Universitatis Babes-Bolyai, 43(3) (1998), 89–124.

[24] Th.M. Rassias and J. Tabe, Eds., Stability of mappings of Hyers-Ulam Type, Handronic Press Collection of Original articles, Handronic Press, Palm Harbour, Fla, USA, (1994).

[25] K. Ravi and M. Arun Kumar, Stability of a 3-variable quadratic functional equation, *Journal of Quality Measurement and Analysis*, (1) (2008), 97–107.

[26] S. Saminger-Platz and C. Sempi, A primer on triangle functions I, *Aequationes Math.*, 76 (2008), 201–240, doi: 10.1007 / S00010-008-2936-8.

[27] S. Saminger-Platz and C. Sempi, A primer on triangle functions II, *Aequationes Math.*, 80 (2008), 239–268, doi: 10.1007 / S00010-010-0038-X.

[28] A.N. Šerstnev, On the notion of a random normed space, *Dokl. Akad. Nauk*, SSSR, 149 (1963), 280–283.

[29] Stefan Czerwik, Functional Equations and Inequalities in Several Variables, *World Scientific*, London, 2002.

[30] S.M. Ulam, Problems in Modern Mathematics, Chapter VI, Science Editions, Wiley, New York, (1964).1.