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Existence of solutions of q-functional integral equations with deviated argument

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Abstract

In this paper, we study the existence of solutions for q-functional integral equations in Banach space $C[0,T]$. The existence and uniqueness of solutions for the problems are proved by means of the Banach contraction principle.

Keywords: q-functional integral equations; Banach contraction principle; Deviated argument; existence.

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1 Introduction

The quantum calculus or q-difference calculus is an old subject that was first developed by Jackson ([12],[13]), while basic definitions and properties can be found in [15]. Studies on q-difference equations appeared already at the beginning of the last century in intensive works especially by F H Jackson [14], R D Carmichael [6], T E Mason [19], C R Adams [1], W J Trjitzinsky [21] and other authors [5].
Recently, q-calculus has served as abridge between mathematics and physics. It has a lot of applications in mathematics and physics([7]-[9],[17],[22]).

In this paper, we are concerned with the q-functional integral equations

$$x(t) = g(t) + \int_0^t f_1\,(t,\ s,\ x(\phi(s)))\,d_qs, \quad t \in [0,T] \tag{1.1}$$

and

$$x(t) = g(t) + f_2\,(t,\ \int_0^t g(s,\ x(\phi(s)))\,d_qs), \quad t \in [0,T] \tag{1.2}$$

where ϕ is deviated function. The existence of continuous solutions of the q-functional integral equation (1.1) in the Banach space $C[0,T]$ will be proved. The monotonicity of the solution of the equation (1.1) will be studied. The existence of continuous solutions of the q-functional integral equation (1.2) in Banach space $C[0,T]$ will be proved.

2 preliminaries

Here, we give the definition of q-derivative and q-integral and some of their properties which is referred to ([2],[15]).

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Let $q \in (0,1)$ and define

$$[n]_q = \frac{q^n - 1}{q-1} = 1 + q + q^2 + \cdots + q^{n-1}, \quad n \in \mathbf{R}$$

which is called The q- analogue of n.

Definition 2.1. *The q-derivative of a real valued function f is defined by*

$$D_q f(t) = \frac{d_q f(t)}{d_q t} = \frac{f(qt) - f(t)}{qt - t}, \qquad D_q f(0) = \lim_{t \to 0} D_q f(t)$$

Note that $\lim_{q \to 1} D_q f(t) = f'(t)$ if $f(t)$ is differentiable.
The higher order q-derivative are defined as

$$D_q^0 f(t) = f(t), \qquad D_q^n f(t) = D_q D_q^{n-1} f(t), \qquad n \in \mathbf{N}.$$

Definition 2.2. *Suppose $0 < a < b$. The definite q-integral is defined as*

$$I_q f(x) = \int_0^b f(x)\, d_q x = (1-q) b \sum_{j=0}^{\infty} q^j f(q^j b).$$

and

$$\int_a^b f(x)\, d_q x = \int_0^b f(x)\, d_q x - \int_0^a f(x)\, d_q x.$$

Similarly, we have

$$I_q^0 f(t) = f(t), \qquad I_q^n f(t) = I_q I_q^{n-1} f(t), \qquad n \in \mathbf{N}.$$

Theorem 2.1 (see [15]). **(Fundamental Theorem of q-Calculus)**
If $F(x)$ is an antiderivative of $f(x)$, and $F(x)$ is continuous at $x = 0$, then

$$\int_a^b f(x) d_q x = F(b) - F(a), \qquad 0 \le a < b \le \infty.$$

Theorem 2.2. *(see [4],[15]) For any function f one has*

$$D_q I_q f(x) = f(x). \tag{2.3}$$

Theorem 2.3. *(see [2]) Let f be a function defined on $[a,b]$, $0 \le a \le b$, and c is a fixed point in $[a,b]$. Assume that there exists, $0 \le \gamma < 1$ such that $x^\gamma f(x)$ is continuous on $[a,b]$. Let*

$$F(x) = \int_c^x f(t)\, d_q t, \qquad x \in [a,b].$$

Then $F(x)$ is a continuous function on $[a,b]$.

Lemma 2.1. *If*

$$F(t) = \int_0^t f(s)\, d_q s, \qquad \text{for} \quad t \in [a,b],$$

is continuous, then for every $\epsilon > 0 \; \exists \; \delta > 0$, such that $t_2, t_2 \in [0,T]$, $| t_2 - t_1 | < \delta$, then

$$|F(t_2) - F(t_1)| < \epsilon$$

i.e.,

$$| \int_0^{t_2} f(s)\, d_q s - \int_0^{t_1} f(s)\, d_q s| < \epsilon.$$

Lemma 2.2. *(see [18])*
(1) *If f and g are q-integrable on $[a,b]$, $\alpha \in R$, $c \in [a,b]$, then*

(i) $\int_a^b [f(x) + g(x)]\, d_q x = \int_a^b f(x)\, d_q x + \int_a^b g(x)\, d_q x,$

(ii) $\int_a^b \alpha f(x)\, d_q x = \alpha \int_a^b f(x)\, d_q x,$

(iii) $\int_a^b f(x)\, d_qx = \int_a^c f(x)\, d_qx + \int_c^b f(x)\, d_qx.$

(2) *If* $|f|$ *is* q*-integrable on the interval* $[0,x]$, *then*

$$\left| \int_0^x f(x)\, d_qx \right| \le \int_0^x |f(x)|\, d_qx.$$

.

(3) *If* f *and* g *are* q*-integrable on* $[0,x]$, $f(x) \le g(x)$, *for all* $x \in [0,x]$, *then*

$$\int_0^x f(x)\, d_qx \le \int_0^x g(x)\, d_qx.$$

3 Main results

Let X be the class of all continuous functions, $x \in C[0,T]$ with the norm

$$\|x\| = \sup_{t\in[0,T]} |\, x(t)\, | \,.$$

First, we study the existence and uniqueness of the solution of the q-functional integral equation (1.1) and then we proved the monotonicity for the solution.

Consider the q-functional integral equation (1.1) under the following assumptions

(i) $g : [0,T] \to R$ is continuous.

(ii) $f_1 : [0,T] \times [0,T] \times R \to R$ is continuous.

(iii) f_1 satisfies the Lipschitz condition

$$|f_1(t,s,x) - f_1(t,s,y)| \le k(t,s)\, |x-y|.$$

(iv)

$$\sup_t \int_0^t k(t,s)\, d_qs \le K$$

Now for the existence of a unique continuous solution of the q-functional integral equation (1.1) we have the following theorem.

Theorem 3.4. *Let the assumptions (i)-(iv) be satisfied. If* $K < 1$, *then the* q*-functional integral equation (1.1) has a unique solution* $x \in C[0,T]$.

Proof. Define the operator F associated with the q-functional integral equation (1.1) by

$$Fx(t) = g(t) + \int_0^t f_1(t,\, s,\, x(\phi(s)))\, d_qs.$$

To show that $F : C[0,T] \to C[0,T]$, let $x \in C[0,T]$, $t_1, t_2 \in [0,T]$, then

$$\begin{aligned}
|Fx(t_2) - Fx(t_1)| &= |g(t_2) - g(t_1) + \int_0^{t_2} f_1(t_2,\, s,\, x(\phi(s)))d_qs - \int_0^{t_1} f_1(t_1,\, s,\, x(\phi(s)))d_qs| \\
&\le |g(t_2) - g(t_1)| + |\int_0^{t_2} f_1(t_2,\, s,\, x(\phi(s)))\, d_qs - \int_0^{t_1} f_1(t_1,\, s,\, x(\phi(s)))\, d_qs| \\
&\le |g(t_2) - g(t_1)| + |\int_0^{t_2} f_1(t_2,\, s,\, x(\phi(s)))\, d_qs - \int_0^{t_2} f_1(t_1,\, s,\, x(\phi(s)))\, d_qs| \\
&\quad + |\int_0^{t_2} f_1(t_1,\, s,\, x(\phi(s)))d_qs - \int_0^{t_1} f_1(t_1,\, s,\, x(\phi(s)))d_qs| \\
&\le |g(t_2) - g(t_1)| + \int_0^{t_2} |\, f_1(t_2,\, s,\, x(\phi(s))) - f_1(t_1,\, s,\, x(\phi(s)))\, |\, d_qs \\
&\quad + |\int_0^{t_2} f_1(t_1,\, s,\, x(\phi(s)))d_qs - \int_0^{t_1} f_1(t_1,\, s,\, x(\phi(s)))d_qs\, |
\end{aligned}$$

applying Theorem (2.3) and Lemma (2.1), then we deduce that

$$F : C[0,T] \rightarrow C[0,T].$$

Let $x, y \in C[0,T]$, we have

$$\begin{aligned}
|Fx(t) - Fy(t)| &= |g(t) + \int_0^t f_1(t, s, x(\phi(s)))\, d_q s - g(t) - \int_0^t f_1(t, s, y(\phi(s)))\, d_q s| \\
&= |\int_0^t f_1(t, s, x(\phi(s)))\, d_q s - \int_0^t f_1(t, s, y(\phi(s)))\, d_q s| \\
&\leq \int_0^t |f_1(t, s, x(\phi(s))) - f_1(t, s, y(\phi(s)))|\, d_q s \\
&\leq \int_0^t k(t,s)\, |x(\phi(s)) - y(\phi(s))|\, d_q s \\
&\leq \|x - y\| \int_0^t k(t,s)\, d_q s \\
&\leq K\, \|x - y\|.
\end{aligned}$$

This means that F is contraction.
Applying Banach contraction principle ([10],[16]), then we deduce that there exists a unique solution $x \in C[0,T]$ of the q-functional integral equation (1.1). □

The following theorem prove the monotonicity for the solution of the q-functional integral equation (1.1).

Theorem 3.5. *Let the assumptions (i)-(iv) of Theorem (3.1) be satisfied. If $f_1(t,s,x(\phi(s)))$ and $g(t)$ are monotonic nonincreasing(nondecreasing) in t for each $t \in [0,T]$, then the q-integral equation (1.1) has a unique monotonic nonincreasing(nondecreasing) solution $x \in C[0,T]$.*

Proof. Let f, g be monotonic nonincreasing functions in $t \in [0,T]$, then for $t_2 > t_1$

$$\begin{aligned}
x(t_2) &= g(t_2) + \int_0^{t_2} f_1(t_2, s, x(\phi(s)))\, d_q s \\
&\leq g(t_1) + \int_0^{t_1} f_1(t_1, s, x(\phi(s)))\, d_q s \\
&= x(t_1).
\end{aligned}$$

Hence,

$$x(t_2) \leq x(t_1).$$

Also, If f_1, g are monotonic nondecreasing functions in $t \in [0,T]$, then for $t_2 > t_1$

$$\begin{aligned}
x(t_2) &= g(t_2) + \int_0^{t_2} f_1(t_2, s, x(\phi(s)))\, d_q s \\
&\geq g(t_1) + \int_0^{t_1} f_1(t_1, s, x(\phi(s)))\, d_q s \\
&= x(t_1).
\end{aligned}$$

Hence

$$x(t_2) \ \geq \ x(t_1).$$

□

Now, we study the existence and uniqueness of the solution of the q-functional integral equation

$$x(t) \ = \ g(t) \ + \ f_2\ (t, \ \int_0^t g(s, \ x(\phi(s)))\ d_q s), \quad t \in [0,T]$$

Consider the q-functional integral equation (1.2) under the following assumptions

(i) $g \ : \ [0,T] \ \to R$ is continuous.

(ii) $f_2 : [0,T] \times R \to R$ is continuous.

(iii) f_2 satisfies the Lipschitz condition

$$|f_2(t,x(t)) - f_2(t,y(t))| \ \leq \ k\ |x(t) - y(t)|.$$

(iv) g satisfies the Lipschitz condition

$$|g(s,x(t)) - g(s,y(t))| \ \leq \ l\ |x(t) - y(t)|.$$

For the existence of a unique continuous solution of the q-functional integral equation (1.2), we have the following theorem.

Theorem 3.6. *Let the assumptions (i)-(iv) be satisfied. If $klT \ < \ 1$, then the q-functional integral equation (1.2) has a unique solution $x \ \in \ C[0,T]$.*

Proof. Define the operator F associated with the q-functional integral equation (1.2) by

$$Fx(t) \ = \ g(t) \ + \ f_2(t, \ \int_0^t g(s, \ x(\phi(s)))\ d_q s).$$

To show that $F : C[0,T] \ \to C[0,T]$, let $x \ \in \ C[0,T]$, $t_1, t_2 \ \in \ [0,T]$, then

$$\begin{aligned}
|Fx(t_2) - Fx(t_1)| \quad &= \quad |(g(t_2) - g(t_1)) \ + \ (f_2(t_2, \int_0^{t_2} g(s,x(\phi(s)))d_q s) - f_2(t_1, \int_0^{t_1} g(s,x(\phi(s)))d_q s))| \\
&\leq \quad |\ g(t_2) - g(t_1)\ | \ + \ |f_2(t_2, \int_0^{t_2} g(s,x(\phi(s)))d_q s) - f_2(t_1, \int_0^{t_1} g(s,x(\phi(s)))d_q s)| \\
&\leq \quad |\ g(t_2) - g(t_1)\ | \ + \ |f_2(t_2, \int_0^{t_2} g(s,x(\phi(s)))d_q s) - f_2(t_1, \int_0^{t_2} g(s,x(\phi(s)))d_q s)| \\
&\qquad + \ |f_2(t_1, \int_0^{t_2} g(s,x(\phi(s)))d_q s) - f_2(t_1, \int_0^{t_1} g(s,x(\phi(s)))d_q s)| \\
&\leq \quad |\ g(t_2) - g(t_1)\ | \ + \ |f_2(t_2, \int_0^{t_2} g(s,x(\phi(s)))d_q s) - f_2(t_1, \int_0^{t_2} g(s,x(\phi(s)))d_q s)| \\
&\qquad + \ |\int_0^{t_2} g(s,x(\phi(s)))d_q s \ - \ \int_0^{t_1} g(s,x(\phi(s)))d_q s|
\end{aligned}$$

applying Theorem (2.3) and Lemma (2.1), then we deduce that

$$F : C[0,T] \ \to C[0,T].$$

Let $x, y \in C[0, T]$, we have

$$
\begin{aligned}
|Fx(t) - Fy(t)| &= | \, g(t) + f_2(t, \int_0^t g(s, x(\phi(s))) \, d_q s) - g(t) - f_2(t, \int_0^t g(s, y(\phi(s))) \, d_q s) \, | \\
&= | \, f_2(t, \int_0^t g(s, x(\phi(s))) \, d_q s) - f_2(t, \int_0^t g(s, y(\phi(s))) \, d_q s) \, | \\
&\leq k \, | \int_0^t g(s, x(\phi(s))) \, d_q s - \int_0^t g(s, y(\phi(s))) \, d_q s \, | \\
&\leq k \int_0^t | \, g(s, x(\phi(s))) - g(s, y(\phi(s))) \, | \, d_q s \\
&\leq kl \int_0^t | \, x(\phi(s)) - y(\phi(s)) \, | \, d_q s \\
&\leq klT \, \|x - y\|.
\end{aligned}
$$

This means that F ([10]) is contraction .
Then F has a fixed point $x \in C[0, T]$ which proves that there exists a unique solution of the q-functional integral equation (1.2). □

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