Malaya J. Mat. 4(3)(2016) 380–387

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Reciprocal Graphs

G. Indulal[a,*] and A.Vijayakumar[b]

[a]Department of Mathematics, St.Aloysius College, Edathua, Alappuzha - 689573, India.

[b]Department of Mathematics, Cochin University of Science and Technology, Cochin-682 022, India.

Abstract

Eigenvalue of a graph is the eigenvalue of its adjacency matrix. A graph G is reciprocal if the reciprocal of each of its eigenvalue is also an eigenvalue of G. The Wiener index $W(G)$ of a graph G is defined by $W(G) = \frac{1}{2}\sum\limits_{d\in D} d$ where D is the distance matrix of G. In this paper some new classes of reciprocal graphs and an upperbound for their energy are discussed. Pairs of equienergetic reciprocal graphs on every $n \equiv 0 \bmod (12)$ and $n \equiv 0 \bmod (16)$ are constructed. The Wiener indices of some classes of reciprocal graphs are also obtained.

Keywords: Eigenvalue, Energy, Reciprocal graphs, splitting graph, Wiener index.

2010 MSC: 39B55, 39B52, 39B82.

1 Introduction

Let G be a graph of order n and size m with the vertex set $V(G)$ labelled as $\{v_1, v_2, \ldots, v_n\}$. The set of eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of an adjacency matrix A of G is called its spectrum and is denoted by $spec(G)$. Non-isomorphic graphs with the same spectrum are called cospectral. Studies on graphs with a specific pattern in their spectrum have been of interest. Gutman and Cvetkovic studied the spectral structure of graphs having a maximal eigenvalue not greater than 2 in [5] and Balinska et.al have studied graphs with integral spectra in [2]. In [12] some new constructions of integral graphs are provided. Dias in [6] has identified graphs with complementary pairs of eigenvalues(eigenvalues λ_1 and λ_2 with $\lambda_1 + \lambda_2 = -1$). A graph G is reciprocal [20] if the reciprocal of each of its eigenvalue is also an eigenvalue of G. The first reference of a reciprocal graph appeared in the work of J.R. Dias in [6, 7] and the chemical molecules of Dendralene and Radialene have been discussed there in. In [20] some classes of reciprocal graphs have been identified. In [3] reciprocal graphs are also referred to as graphs with property R.

The energy of a graph G [1], denoted by $E(G)$ is the sum of the absolute values of its eigenvalues. Non-cospectral graphs with the same energy are called equienergetic. In [8, 9, 15] some bounds on energy are described. In [1] and [22, 23] a pair of equienergetic graphs are constructed for every $n \equiv 0\,(\text{mod } 4)$ and $n \equiv 0\,(\text{mod } 5)$ and in [10] we have extended it for $n = 6,\ 14,\ 18$ and $n \geq 20$. In [17] a pair of equienergetic graphs within the family of iterated line graphs of regular graphs and in [11] a pair of equienergetic graphs obtained from the cross product of graphs are described. In [13] a pair of equienergetic self-complementary graphs on n vertices is constructed for every $n = 4k$ and $n = 24t + 1, k \geq 2,\ t \geq 3$. A plethora of papers have been appeared dealing with this parameter in recent years.

The distance matrix of a connected graph G, denoted by $D(G)$ is defined as $D(G) = [d(v_i, v_j)]$ where $d(v_i, v_j)$ is the distance between v_i and v_j. The Wiener index $W(G)$ is defined by

*Corresponding author.
E-mail address: indulalgopal@gmail.com (G. Indulal) and vambat@gmail.com (A.Vijayakumar).

$W(G) = \frac{1}{2} \sum_{d \in D} d$. The chemical applications of this index are well established in [16, 18].

In this paper, we construct some new classes of reciprocal graphs and an upperbound for their energy is obtained. Pairs of equienergetic reciprocal graphs on $n \equiv 0 \bmod (12)$ and $n \equiv 0 \bmod (16)$ are constructed. The Wiener indices of some classes of reciprocal graphs are also obtained. These results are not found so far in literature.

2 Some new classes of reciprocal graphs

If A and B are two matrices then $A \otimes B$ denote the tensor product of A and B. We use the following properties of block matrices[4].

Lemma 2.1. *Let M, N, P and Q be matrices with M invertible. Let $S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}$. Then $|S| = |M| \left|Q - PM^{-1}N\right|$. Moreover if M and P commutes then $|S| = |MQ - PN|$ where the symbol $|.|$ denotes the determinant.*

We consider the following operations on G.

Operation 1. *Attach a pendant vertex to each vertex of G. The resultant graph is called the pendant join graph of G.[Also referred to as G corona K_1 in [3].]*

Operation 2. *[19] Introduce n isolated vertices u_i, $i = 1$ to n and join u_i to the neighbors of v_i. The resultant graph is called the splitting graph of G.*

Operation 3. *In addition to G introduce two sets of n isolated vertices $U = \{u_i\}$ and $W = \{w_i\}$ corresponding to $V = \{v_i\}$, $i = 1$ to n. Join u_i and w_i to the neighbors of v_i and then w_i to the vertices in U corresponding to the neighbors of v_i in G for each $i = 1$ to n. The resultant graph is called the double splitting graph of G.*

Operation 4. *In addition to G introduce two more copies of G on $U = \{u_i\}$ and $W = \{w_i\}$ corresponding to $V = \{v_i\}$, $i = 1$ to n. Join u_i to the neighbors of v_i and then w_i to u_i for each $i = 1$ to n. The resultant graph is called the composition graph of G.*

Operation 5. *In addition to G introduce two more copies of G on $U = \{u_i\}$ and $W = \{w_i\}$ corresponding to $V = \{v_i\}$, $i = 1$ to n. Join w_i to the neighbors of v_i and vertices in U corresponding to the neighbors of v_i in G for each $i = 1$ to n.*

Lemma 2.2. *Let G be a graph on n vertices with $spec(G) = \{\lambda_1, \ldots, \lambda_n\}$ and H_i be the graph obtained from Operation i, $i = 1$ to 5. Then*

$$spec(H_1) = \left\{ \frac{\lambda_i \pm \sqrt{\lambda_i^2 + 4}}{2} \right\}_{i=1}^{n}$$

$$spec(H_2) = \left\{ \left(\frac{1 \pm \sqrt{5}}{2} \right) \lambda_i \right\}_{i=1}^{n}$$

$$spec(H_3) = \left\{ -\lambda_i, \left(1 \pm \sqrt{2}\right) \lambda_i \right\}_{i=1}^{n}$$

$$spec(H_4) = \left\{ \lambda_i, \lambda_i \pm \sqrt{\lambda_i^2 + 1} \right\}_{i=1}^{n}$$

$$spec(H_5) = \left\{ \lambda_i, \left(1 \pm \sqrt{2}\right) \lambda_i \right\}_{i=1}^{n}$$

Proof. The proof follows from Table 1 which gives the adjacency matrix of $H_i s$ for $i = 1$ to 5 and its spectrum, obtained using Lemma 2.1 and the spectrum of tensor product of matrices.

Table 1

Graph	Adjacency matrix	Spectrum
H_1	$\begin{bmatrix} A & I \\ I & 0 \end{bmatrix}$	$\left\{\frac{\lambda_i \pm \sqrt{\lambda_i^2+4}}{2}\right\}_{i=1}^{n}$
H_2	$\begin{bmatrix} A & A \\ A & 0 \end{bmatrix} = A \otimes \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$	$\left\{\left(\frac{1\pm\sqrt{5}}{2}\right)\lambda_i\right\}_{i=1}^{n}$
H_3	$\begin{bmatrix} A & A & A \\ A & 0 & A \\ A & A & 0 \end{bmatrix} = A \otimes \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	$\left\{-\lambda_i, \left(1 \pm \sqrt{2}\right)\lambda_i\right\}_{i=1}^{n}$
H_4	$\begin{bmatrix} A & A & 0 \\ A & A & I \\ 0 & I & A \end{bmatrix}$	$\left\{\lambda_i,\ \lambda_i \pm \sqrt{\lambda_i^2+1}\right\}_{i=1}^{n}$
H_5	$\begin{bmatrix} A & 0 & A \\ 0 & A & A \\ A & A & A \end{bmatrix} = A \otimes \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	$\left\{\lambda_i, \left(1 \pm \sqrt{2}\right)\lambda_i\right\}_{i=1}^{n}$

□

Note: $H_3 = H_5$ when G is bipartite.

Theorem 2.1. *The pendant join graph of a graph G is reciprocal if and only if G is bipartite.*

Proof. Let G be a bipartite graph and H, its pendant join graph. Then, corresponding to a non-zero eigenvalue λ of G, $-\lambda$ is also an eigenvalue of G [4].
By Lemma 2.2, $spec(H) = \{\frac{\lambda \pm \sqrt{\lambda^2+4}}{2},\ \lambda \in\ spec(G)\}$. Let $\alpha = \frac{\lambda+\sqrt{\lambda^2+4}}{2}$ be an eigenvalue of H. Then

$$\begin{aligned}\frac{1}{\alpha} &= \frac{2}{\lambda + \sqrt{\lambda^2+4}} \\ &= \frac{2\left(\lambda - \sqrt{\lambda^2+4}\right)}{\left(\lambda + \sqrt{\lambda^2+4}\right)\left(\lambda - \sqrt{\lambda^2+4}\right)} \\ &= \frac{2\left(\lambda - \sqrt{\lambda^2+4}\right)}{-4} \\ &= \frac{(-\lambda) + \sqrt{(-\lambda)^2+4}}{2}\end{aligned}$$

is an eigenvalue of H as $-\lambda$ is an eigenvalue of G. Similarly for $\alpha = \frac{\lambda-\sqrt{\lambda^2+4}}{2}$ also. The eigenvalues of H corresponding to the zero eigenvalues of G if any, are 1 and -1 which are self reciprocal. Therefore H is a reciprocal graph.
The converse can be proved by retracing the argument. □

Note 1. *This theorem enlarges the classes of reciprocal graphs mentioned in [20]. The claim in [20] that the pendant join graph of C_n is reciprocal for every n is not correct as C_n is not bipartite for odd n.*

Definition 2.1. *A graph G is partially reciprocal if $\frac{-1}{\lambda} \in spec(G)$ for every $\lambda \in spec(G)$.*

Examples:-

- Pendant join graph of any graph.
- Splitting graph of any reciprocal graph.

Theorem 2.2. *The splitting graph of G is reciprocal if and only if G is partially reciprocal.*

Proof. Let G be partially reciprocal and H be its splitting graph. Let $\alpha \in spec(H)$. Then by Lemma 3, $\alpha = \left(\frac{1\pm\sqrt{5}}{2}\right)\lambda,\ \lambda \in\ spec(G)$. Without loss of generality, take $\alpha = \left(\frac{1+\sqrt{5}}{2}\right)\lambda$. Then $\frac{1}{\alpha} = \left(\frac{1-\sqrt{5}}{2}\right)\frac{-1}{\lambda}$. Thus $\frac{1}{\alpha} \in spec(H)$ as G is partially reciprocal and hence H is reciprocal.
Conversely assume that H is reciprocal. Then by the structure of $spec(H)$ as given by Lemma 2.2, G is partially reciprocal. □

Theorem 2.3. *Let G be a reciprocal graph. Then the double splitting graph and the composition graph of G are reciprocal if and only if G is bipartite.*

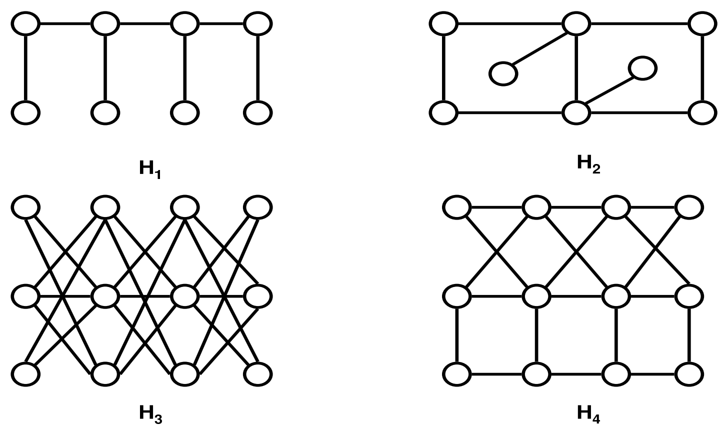
Proof. Let G be a bipartite reciprocal graph. Then $\lambda \in spec(G) \Rightarrow -\lambda, \frac{1}{\lambda}, \frac{-1}{\lambda} \in spec(G)$. Let H and H' respectively denote the double splitting graph and composition graph of G. Then using Lemma 2.2 and Table 2 it follows that H and H' are reciprocal.

Table 2

$Spec(H)$	$\frac{1}{spec(H)}$	$Spec(H')$	$\frac{1}{spec(H')}$
$\left\{-\lambda, \left(1 \pm \sqrt{2}\right) \lambda\right\}$	$\left\{-\frac{1}{\lambda}, \left(1 \pm \sqrt{2}\right) \frac{-1}{\lambda}\right\}$	$\left\{\lambda, \lambda \pm \sqrt{\lambda^2+1}\right\}$	$\left\{\frac{1}{\lambda}, -\lambda \pm \sqrt{(-\lambda)^2+1}\right\}$

Converse also follows. □

Illustration: The following graphs are reciprocal when $G = P_4$.



3 An upperbound for the energy of reciprocal graphs

The following bounds on the energy of a graph are known.

1. [15]$\sqrt{2m + n(n-1)\left|\det A\right|^{\frac{2}{n}}} E(G) \sqrt{2mn}$

2. [8] $E(G) \frac{2m}{n} + \sqrt{(n-1)\left(2m - 4\frac{m^2}{n^2}\right)}$

3. [9] $E(G) \frac{4m}{n} + \sqrt{(n-2)\left(2m - 8\frac{m^2}{n^2}\right)}$, if G is bipartite.

In this section we derive a better upperbound for the energy of a reciprocal graph and prove that the bound is best possible. A graph of order n and size m is referred to as an (n, m) graph.

Theorem 3.4. *Let G be an (n, m) reciprocal graph. Then $E(G) \leq \sqrt{\frac{n(2m+n)}{2}}$ and the bound is best possible for $G = tK_2$ and tP_4.*

Proof. Let G be an (n, m) reciprocal graph with $spec(G) = \{\lambda_1, \ldots, \lambda_n\}$.
Therefore $\sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{n} \frac{1}{|\lambda_i|} = E$ and $\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} \frac{1}{\lambda_i^2} = 2m$.

Now we have [21]the following inequality for real sequences a_i, b_i and c_i, $1 \le i \le n$

$$\sum_{i=1}^{n} a_i c_i \sum_{i=1}^{n} b_i c_i \le \frac{1}{2}\left\{\sum_{i=1}^{n} a_i b_i + \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} b_i^2\right)^{1/2}\right\}\sum_{i=1}^{n} c_i^2$$

Taking $a_i = |\lambda_i|, b_i = \frac{1}{|\lambda_i|}$ and $c_i = 1\ \forall i = 1,2,\ldots,n$,
we have $[E(G)]^2 \le \frac{1}{2}[n+2m]\,n$ and hence $E(G) \le \sqrt{\frac{n(2m+n)}{2}}$.
When $G = tK_2$, $n = 2t$, $m = t$, $E(G) = 2t$ and when $G = tP_4$, $n = 4t$, $m = 3t$, $E(G) = 2t\sqrt{5}$. □

4 Equienergetic reciprocal graphs

In this section we prove the existence of a pair of equienergetic reciprocal graphs on every $n = 12p$ and $n = 16p$, $p \ge 3$.

Theorem 4.5. *Let G be K_p and F_1 be the graph obtained by applying Operations 3, 1 and 2 on G and F_2, the graph obtained by applying Operations 5, 1 and 2 on G successively. Then F_1 and F_2 are reciprocal and equienergetic on $12p$ vertices.*

Proof. Let $G = K_p$. We have $spec(K_p) = \begin{pmatrix} p-1 & -1 \\ 1 & p-1 \end{pmatrix}$.
Let G_3 be the graph obtained by applying Operation 3 on G. Then by Lemma 2.2,
$spec(G_3) = \begin{pmatrix} -(p-1) & 1 & \left(1\pm\sqrt{2}\right)(p-1) & -\left(1\pm\sqrt{2}\right) \\ 1 & p-1 & \text{each once} & \text{each } p-1 \text{ times} \end{pmatrix}$.
Now, let G_{31} be the graph obtained by applying Operation 1 on G_3. Then by Lemma 2.2 $spec(G_{31})$

$$= \begin{pmatrix} \frac{p-1\pm\sqrt{(p-1)^2+4}}{2} & \frac{-1\pm\sqrt{5}}{2} & \frac{(1+\sqrt{2})(p-1)\pm\sqrt{\{(1+\sqrt{2})(p-1)\}^2+4}}{2} \\ \text{each once} & \text{each } p-1 \text{ times} & \text{each once} \\ & & \\ \frac{(1-\sqrt{2})(p-1)\pm\sqrt{\{(1-\sqrt{2})(p-1)\}^2+4}}{2} & \frac{(1+\sqrt{2})\pm\sqrt{\{(1+\sqrt{2})\}^2+4}}{2} & \frac{(1-\sqrt{2})\pm\sqrt{\{(1-\sqrt{2})\}^2+4}}{2} \\ \text{each once} & \text{each } p-1 \text{ times} & \text{each } p-1 \text{ times} \end{pmatrix}$$

Then

$$\begin{aligned} E(G_{31}) =& \sqrt{(p-1)^2+4} + \sqrt{5}\,(p-1) + \sqrt{\left\{\left(1+\sqrt{2}\right)(p-1)\right\}^2+4} \\ &+ \sqrt{\left\{\left(1-\sqrt{2}\right)(p-1)\right\}^2+4} + (p-1)\left[\sqrt{\left(1+\sqrt{2}\right)^2+4} + \sqrt{\left(1-\sqrt{2}\right)^2+4}\right] \\ =& \sqrt{(p-1)^2+4} + \sqrt{5}\,(p-1) + (p-1)\sqrt{14+2\sqrt{41}} \\ &+ \sqrt{6\,(p-1)^2+8+2\sqrt{(p-1)^4+24\,(p-1)^2+16}} \end{aligned}$$

Now, let F_1 be the graph obtained by applying Operation 2 on G_{31}. Then by Lemma 2.2,
$E(F_1) = \sqrt{5}E(G_{31})$. Let G_{51} be the graph obtained by applying Operations 5 and 1 on G successively and F_2 be that obtained by applying Operation 2 on G_{51}. Then we have
$E(F_2) = \sqrt{5}E(G_{51}) = \sqrt{5}E(G_{31}) = E(F_1)$. Also by Theorem 2, F_1 and F_2 are reciprocal. Thus the theorem follows. □

Lemma 4.3. *Let G be a non-bipartite graph on p vertices with $spec(G) = \{\lambda_1,\ldots,\lambda_p\}$ and an adjacency matrix A. Then the spectra of graphs whose adjacency matrices are*

$$F' = \begin{bmatrix} A & A & A & A \\ A & A & 0 & A \\ A & 0 & A & A \\ A & A & A & 0 \end{bmatrix} \text{ and } H' = \begin{bmatrix} 0 & A & A & A \\ A & 0 & A & A \\ A & A & A & A \\ A & A & A & 0 \end{bmatrix} \text{ are}$$

$\left\{\lambda_i, -\lambda_i, \left(\frac{3\pm\sqrt{13}}{2}\right)\lambda_i\right\}_{i=1}^{p}$ *and* $\left\{-\lambda_i, -\lambda_i, \left(\frac{3\pm\sqrt{13}}{2}\right)\lambda_i\right\}_{i=1}^{p}$ *respectively .*

Theorem 4.6. *Let G be K_p. Let T_1 and T_2 be the graphs obtained by applying Operations 1 and 2 successively on graphs associated with F' and H' respectively. Then T_1 and T_2 are reciprocal and equienergetic on $16p$ vertices.*

Proof. Let the graph associated with F' be also denoted by F' and F_1', the graph obtained by applying Operation 1 on F'. Then by a similar computation as in Theorem 5,

$$E(F_1') = 2\sqrt{(p-1)^2+4} + 2\sqrt{5}\,(p-1) + \sqrt{\left(\frac{11+3\sqrt{13}}{2}\right)(p-1)^2+4}$$
$$+\sqrt{\left(\frac{11-3\sqrt{13}}{2}\right)(p-1)^2+4} + (p-1)\left[\sqrt{\left(\frac{11+3\sqrt{13}}{2}\right)+4} + \sqrt{\left(\frac{11-3\sqrt{13}}{2}\right)+4}\right]$$

and $E(T_1) = \sqrt{5}E(F_1') = \sqrt{5}E(H_1') = E(T_2)$, by Lemma 2.2. Also by Theorem 2, T_1 and T_2 are reciprocal. Hence the theorem. □

5 Wiener index of some reciprocal graphs

In this section we derive the Wiener indices of some classes of reciprocal graphs described in the earlier section. We shall denote by $D(G) = D$, the distance matrix of G and d_i, the sum of entries in the i^{th} row of D. The following theorem generalizes the results in [14].

Theorem 5.7. *Let G be a graph with Wiener index $W(G)$. Let H be the pendant join graph of G. Then $W(H) = 4W(G) + n(2n-1)$.*

Proof. We have, $W(G) = \frac{1}{2}\sum_{i=1}^{n} d_i$.
Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $U = \{u_1, u_2, \ldots, u_n\}$ be the corresponding vertices used in the pendant join of G. Then the distance matrix of H is as follows.

$$\left[\begin{array}{cccc|cccc} 0 & d(v_1,v_2) & \ldots & d(v_1,v_n) & 1 & 1+d(v_1,v_2) & \ldots & 1+d(v_1,v_n) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ d(v_n,v_1) & \ldots & \ldots & 0 & 1+d(v_n,v_1) & \ldots & \ldots & 1 \\ \hline 1 & 1+d(v_1,v_2) & \ldots & 1+d(v_1,v_n) & 0 & 2+d(v_1,v_2) & \ldots & 2+d(v_1,v_n) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1+d(v_n,v_1) & \ldots & \ldots & \ldots & 2+d(v_n,v_1) & \ldots & \ldots & 0 \end{array}\right]$$

$$\begin{aligned} \text{since } d(v_i,u_j) &= 1;\ \text{if } i = j \\ &= 1 + d(v_i,v_j);\ i \neq j \text{and} \\ d(u_i,u_j) &= d(u_i,v_i) + d(v_i,v_j) + d(v_j,u_j) \\ &= 2 + d(v_i,v_j) \end{aligned}$$

The row sum matrix of H is $\begin{bmatrix} 2d_1 + n \\ \vdots \\ 2d_n + n \\ 2d_1 + 3n - 2 \\ \vdots \\ 2d_n + 3n - 2 \end{bmatrix}$.

$$\text{Then } W(H) = \frac{1}{2}\left[\sum_{i=1}^{n}(2d_i + n) + \sum_{i=1}^{n}(2d_i + 3n - 2)\right]$$
$$= 4W(G) + n(2n - 1). \text{ Hence the theorem.}$$

□

The proof techniques of the following theorems are on similar lines.

Theorem 5.8. *Let G be a triangle free* (n, m) *graph and H, its splitting graph. Then* $W(H) = 4W(G) + 2(m + n)$.

Corollory 5.1. *Let G be a triangle free* (n, m) *graph and F, the splitting graph of the pendant join graph of G. Then* $W(F) = 2[8W(G) + 4n^2 + (m + n)]$.

Theorem 5.9. *Let G be a triangle free* (n, m) *graph and H, its double splitting graph. Then* $W(H) = 9W(G) + 4m + 6n$.

Theorem 5.10. *Let G be a triangle free* (n, m) *graph and H, its composition graph. Then* $W(H) = 9W(G) + 2n^2 + 4n$.

References

[1] R. Balakrishnan, *The energy of a graph*, Lin. Algebra Appl., **387** (2004), 287–295.

[2] K. Balińska, D.M. Cvetković, Z. Radosavljević, S. Simić, D. Stevanović, *A Survey on integral graphs*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat., **13** (2002), 42–65.

[3] S. Barik, S. Pati, B.K. Sarma, *The spectrum of the corona of two graphs*, SIAM J. Discrete Math., **21**(2007), 47-56.

[4] D.M. Cvetkovi?, M. Doob, H. Sachs, Spectra of Graphs-Theory and Applications, Academic Press, (1980).

[5] D.M. Cvetković, I. Gutman, *On spectral structure of graphs having the maximal eigenvalue not greater than two*, Publ. Inst. Math., **18**(1975), 39–45.

[6] J.R. Dias, *Properties and relationships of right-hand mirror-plane fragments and their eigenvectors : the concept of complementarity of molecular graphs*, Mol. Phys., **88** (1996), 407–417.

[7] J.R. Dias, *Properties and Relationships of Conjugated Polyenes Having a Reciprocal Eigenvalue Spectrum Dendralene and Radialene Hydrocarbons*, Cro. Chem.Acta, **77** (2004), 325–330.

[8] J. Koolen, V. Moulton, *Maximal energy graphs*, Adv.Appl.Math., **26**(2001), 47–52.

[9] J. Koolen, V. Moulton, *Maximal energy bipartite graphs*, Graphs and Combin., **19**(2003), 131–135.

[10] G. Indulal, A. Vijayakumar, *On a pair of equienergetic graphs*, MATCH Commun. Math. Comput. Chem., **55**(2006), 83 - 90.

[11] G. Indulal, A. Vijayakumar, *Energies of some non-regular graphs*, J. Math. Chem. **42** (2007), 377–386.

[12] G. Indulal, A. Vijayakumar, *Some new integral graphs*, Applicable Analysis and Discrete Mathematics,**1** (2007), 420–426.

[13] G. Indulal, A. Vijayakumar, *Equienergetic self-complementary graphs*, Czechoslovak Math J. **58** (2008), 911–919.

[14] B. Mandal, M. Banerjee, A.K. Mukherjee, *Wiener and Hosoya indices of reciprocal graphs*, Mol. Phys., **103**(2005), 2665– 2674.

[15] B.J. McClelland, *Properties of the latent roots of a matrix:the estimation of $\pi-$ electron energy*, J. Chem. Phys., **54(2)**(1971), 640–643.

[16] S. Nikolić, N. Trinajstić, M. Randić, *Wiener index revisited*, Chem. Phys. Lett., **33**(2001), 319–321.

[17] H.S. Ramane, H.B. Walikar, S.B. Rao, B.D. Acharya, I. Gutman, P.R. Hampiholi, S.R. Jog, *Equienergetic graphs*, Krajugevac. J. Math., **26**(2004), 5–13.

[18] M. Randić, X. Guo, T. Oxley, H.K. Krishnapriyan, *Wiener Matrix:Source of novel graph invarients*, J. Chem. Inf. Comp. Sci., **33(5)**(1993), 709–716.

[19] E. Sampathkumar, H.B. Walikar, *On the splitting graph of a graph*, Karnatak Univ. J. Sci., **35/36** (1980–1981), 13–16.

[20] J. Sarkar, A.K. Mukherjee, *Graphs with reciprocal pairs of eigenvalues*, Mol. Phys., **90**(1997), 903–907 .

[21] J.M. Steele, The Cauchy-Schwarz Master Class, Cambridge University Press (2004).

[22] D. Stevanović, *When is NEPS of graphs connected?*, Linear Algebra Appl., **301**(1999), 137–144.

[23] D. Stevanović,*Energy and NEPS of graphs*, Linear Multilinear Algebra, **53**(2005), 67–74.

Received: November 12, 2015; *Accepted*: March 25, 2016

UNIVERSITY PRESS

Website: http://www.malayajournal.org/