Malaya J. Mat. 4(3)(2016) 388–391

Malaya Journal of Matematik

www.malayajournal.org

MJM

an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

A Generalization of Natural Density

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Abstract

The concept of natural density is generalized. It is proved that the new theory is consistent with the existing theory in the literature. Many new results were obtained. A theorem analogous to the Riemann's theorem on rearrangement of non-absolutely convergent series is proved in the sense of generalized natural density. Some more possible generalizations are suggested.

Keywords: Natural Density.

2010 MSC: 11B99.

1 Introduction

We know that the set of even natural numbers and the set of natural numbers have same cardinality. In other words both the sets have equal number of elements and they have the same size. But intuitively we feel that the set of natural numbers is one half of the set of integers. This intuition is made into a mathematical concept called natural density [1]. In this paper we generalize this concept and derive some interesting results. We also suggest some more possible generalizations. Now we give some preliminary concepts which are available in the literature. As usual we use $\mathbb{N}$ to denote the set of natural numbers and $|S|$ to denote the cardinality of the set S.

Definition 1.1. *Let $A \subseteq \mathbb{N}$. Let $A(n) = \{1, 2, \ldots, n\} \cap A$ for all $n \in \mathbb{N}$. The upper density and the lower density of A are defined as $\limsup\limits_{n\to\infty} \frac{|A(n)|}{n}$ and $\liminf\limits_{n\to\infty} \frac{|A(n)|}{n}$ respectively; they are denoted by $\bar{d}(A)$ and $\underline{d}(A)$ respectively. The natural density $d(A)$ of A is defined as $\lim\limits_{n\to\infty} \frac{|A(n)|}{n}$ if the limit exists.*

A has natural density if and only if $\bar{d}(A) = \underline{d}(A)$. We have some classical results:

- For any finite set A, $d(A) = 0$.
- for any $k \in \mathbb{N}$, $d(k\mathbb{N}) = \frac{1}{k}$ where $n\mathbb{N}$ is the set of all positive multiples of k.
- the infinite set $\{n^2 : n \in \mathbb{N}\}$ has density 0.

Further for any subsets A and B of $\mathbb{N}$, if $d(A)$ and $d(B)$ exist, then

- $d(A^c) = 1 - d(A)$.
- for any finite set F, $d(A - F) = d(A)$.
- $d(A \cup B) = d(A) + d(B) - d(A \cap B)$.

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- $d(kA) = \frac{1}{k}d(A)$, for $k \in \mathbb{N}$.
- $d(A+c) = d(A)$ for all constant $c \in \mathbb{N}$ where
$$A + c = \{a + c/a \in A\}.$$
- If
$$A = \bigcup_{n=0}^{\infty}\{2^{2n}, 2^{2n}+1, \ldots, 2^{2n+1}-1\},$$
then $\overline{d}(A) = \frac{2}{3}$ and $\underline{d}(A) = \frac{1}{3}$; this shows the existence of a set for which natural density does not exist.

In Section 2, we give a generalization of the concept of natural density and in Section 3, we prove a theorem very similar to the Riemann's theorem on rearrangement of nonabsolutely convergent series. This very interesting theorem suggests us the generalization is a natural one and also that many classical theorems may have similar interpretations.

2 Generalization of Natural Density

We observe that the expression $\frac{|A(n)|}{n}$ is equal to $\frac{|A \cap X_n|}{|X_n|}$ where X_n is the set $\{1, 2, \ldots, n\}$ and that the sets X_n form an increasing sequence of subsets of the natural numbers whose union is the whole set of natural numbers. This motivates us the following definitions.

Definition 2.2. *Let $\mathscr{C} = \{X_n\}$ be any sequence of subsets of $\mathbb{N}$ such that $X_1 \subseteq X_2 \subseteq X_3 \subseteq \ldots$ and $\cup X_n = \mathbb{N}$. Then $\mathscr{C}$ is called a cover for $\mathbb{N}$.*

We simply write 'cover' instead of writing 'cover for $\mathbb{N}$'. We define the natural density in a generalized form in the following definition.

Definition 2.3. *The Upper density $\overline{d}_{\mathscr{C}}(A)$ and the lower density $\underline{d}_{\mathscr{C}}(A)$ of a subset A of $\mathbb{N}$ with respect to a cover $\mathscr{C}$ are defined as*
$$\overline{d}_{\mathscr{C}}(A) = \limsup_{n\to\infty} \frac{|A \cap X_n|}{|X_n|} \text{ and } \underline{d}_{\mathscr{C}}(A) = \liminf_{n\to\infty} \frac{|A \cap X_n|}{|X_n|}.$$
The density $d_{\mathscr{C}}(A)$ of A with respect to $\mathscr{C}$ is defined as
$$d_{\mathscr{C}}(A) = \lim_{n\to\infty} \frac{|A \cap X_n|}{|X_n|}$$
provided the limit exists.

If $X_n = \{1, 2, \ldots, n\}$, then we get the theory of natural density which is available in the literature. So the concept of natural density becomes a particular case of the new concept and the new theory is consistent with that available in the literature.

If $\mathscr{C}$ is any cover for $\mathbb{N}$ and if A and B are subsets of $\mathbb{N}$ such that $d_{\mathscr{C}}(A)$ and $d_{\mathscr{C}}(B)$ exist, then the following results follow from the definition.

- $d_{\mathscr{C}}(\mathbb{N}) = 1$.
- $d_{\mathscr{C}}(A^c) = 1 - d_{\mathscr{C}}(A)$ where A^c denote the complement of A in $\mathbb{N}$.
- for any finite set F, $d_{\mathscr{C}}(F) = 0$.
- for any finite set F, $d_{\mathscr{C}}(A - F) = d_{\mathscr{C}}(A)$.
- $d_{\mathscr{C}}(A \cup B) = d_{\mathscr{C}}(A) + d_{\mathscr{C}}(B) - d_{\mathscr{C}}(A \cap B)$.

Example 2.1. *Let $A = 2\mathbb{N}$. Let $X_n = \{1, 2, \ldots, n\}$ and $\mathscr{C}$ be the cover $\{X_n\}$. Then $d_{\mathscr{C}}(A)$ is the natural density, which is equal to $\frac{1}{2}$. Let $\mathscr{D}$ be the cover $\{X_n\}$ where*
$$X_n = \{1, 2, 3, \ldots, 2n+1, 2n+3, \ldots, 4n-1\}.$$
Then the sequence $\left(\frac{|A \cap X_n|}{|X_n|}\right)$ is, $\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \ldots$ which converge to $\frac{1}{3}$. That is, $d_{\mathscr{D}}(A) = \frac{1}{3}$.

This example shows that the density of a set may vary as the cover varies. In Theorem 3.2, we prove that for any real number α, $0 \leq \alpha \leq 1$, if A is an infinite set whose complement is also infinite, there is a cover $\mathscr{C}$ so that $d_{\mathscr{C}}(A) = \alpha$.

Let us consider another example.

Example 2.2. *Let $A = \{1, 3, 5, \dots\}$ and let $\mathscr{C}$ be the cover $\{X_n\}$ where*

$$X_n = \{1, 2, 3, \dots, 2n, 2(n+1), 2(n+2), \dots, 4n\}.$$

Then $d_{\mathscr{C}}(A)$ is $\frac{1}{3}$ and $d_{\mathscr{C}}(2A) = \frac{1}{3}$.

This example shows that, $d_{\mathscr{C}}(kA)$ need not be equal to $\frac{1}{k}d_{\mathscr{C}}(A)$ in contrast with the classical result $d(kA) = \frac{1}{k}d(A)$. Also it is easy to verify that $d_{\mathscr{C}}(A+1) = \frac{2}{3}$ which shows that, $d_{\mathscr{C}}(A)$ need not be equal to $d_{\mathscr{C}}(A+1)$ in contrast with the classical result $d(A+c) = d(A)$ for all constant $c \in \mathbb{N}$.

Theorem 2.1. *Let A be a subset of $\mathbb{N}$. Let $m_1 < m_2 < m_3 < \dots$ be an increasing sequence of natural numbers. Let $X_n = \{1, 2, \dots, m_n\}$ and $\mathscr{C} = \{X_n\}$. Then $\mathscr{C}$ is a cover of $\mathbb{N}$ and $d_{\mathscr{C}}(A) = d(A)$ provided $d(A)$ exists.*

Proof. Let $d(A)$ exist. Let $a_n = \frac{|A \cap \{1,2,\dots,n\}|}{n}$ and $b_n = \frac{|A \cap \{1,2,\dots,m_n\}|}{m_n}$. Then $d(A) = \lim_{n\to\infty} a_n$ which exists by our assumption. As (b_n) is a subsequence of (a_n), $d_{\mathscr{C}}(A) = \lim_{n\to\infty} b_n$ exists and is equal to $d(A)$. □

3 The Major Theorem

In this section, we prove a theorem which resembles the Riemann's theorem on rearrangements of series. First we recall Riemann's theorem on rearrangement of Series: If Σa_n is a nonabsolutely convergent series (Σa_n is convergent and $\Sigma |a_n|$ is not convergent) of real numbers and $-\infty \leq \alpha \leq \beta \leq \infty$, then there exists a rearrangement Σb_n of Σa_n with partial sum sequence (t_n) such that $\liminf_{n\to\infty} t_n = \alpha$ and $\limsup_{n\to\infty} t_n = \beta$.

We now state our main theorem.

Theorem 3.2. *If A is an infinite subset of $\mathbb{N}$ whose complement is also an infinite set and $\alpha, \beta \in [0,1]$ with $\alpha \leq \beta$, then there exists a cover $\mathscr{C}$ such that $\underline{d}_{\mathscr{C}}(A) = \alpha$ and $\overline{d}_{\mathscr{C}}(A) = \beta$.*

Proof. There exists a sequence of rational numbers in $[0,1]$ whose limit infimum is α limit supremum is β. Indeed if, $p_1, p_2, \dots$ and $q_1, q_2, \dots$ are sequences of rational numbers in $[0,1]$ converging to α and β respectively, then the sequence $p_1, q_1, p_2, q_2, \dots$ has the required property.

Let a and b be two rational numbers in $[0,1]$. Let a representation $\frac{m}{n}$ for a be given. Then we claim that there exists a representation $\frac{m'}{n'}$ for b such that $m \leq m'$ and $n < n'$. If $b = \frac{p}{q}$ is any representation of b, and if $m' = pmn$ and $n' = qmn$, then $b = \frac{m'}{n'}$ is a required representation of b, if at least one of m and n is different from 1. If $m = n = 1$, then $\frac{2p}{2q}$ will be a representation of b with the required property.

We claim that there exists a sequence

$$\frac{m_1}{n_1}, \frac{m_2}{n_2}, \frac{m_3}{n_3}, \dots$$

of rational numbers such that

$$m_1 \leq m_2 \leq m_3 \leq \dots, \quad n_1 < n_2 < n_3 < \dots,$$

and $m_i \leq n_i$ for all i, so that

$$\liminf_{k\to\infty} \frac{m_k}{n_k} = \alpha \text{ and } \limsup_{k\to\infty} \frac{m_k}{n_k} = \beta.$$

To prove this claim let $\alpha_1, \alpha_2, \alpha_3, \dots$ be a sequence of rational numbers in $[0,1]$ such that $\liminf_{n\to\infty} \alpha_n = \alpha$ and $\limsup_{n\to\infty} \alpha_n = \beta$. Taking α_1 and α_2 as a and b with the representation $\alpha_1 = \frac{m_1}{n_1}$ in our first claim, we get a representation $\alpha_2 = \frac{m_2}{n_2}$ such that $m_1 \leq m_2$ and $n_1 < n_2$. Taking α_2 and α_3 as a and b with the representation $\alpha_2 = \frac{m_2}{n_2}$ in the same claim we get a representation $\alpha_3 = \frac{m_3}{n_3}$ such that $m_2 \leq m_3$ and $n_2 < n_3$. Continuing in this way, we get a sequence with the required properties.

Let $B = \mathbb{N} - A$. Since A and B are infinite subset of $\mathbb{N}$, we can write the elements of the sets as infinite sequences:

$$A: \ a_1 < a_2 < a_3 < \ldots, \text{ and } B: \ b_1 < b_2 < b_3 < \ldots.$$

Let

$$X_k = \{a_1, a_2, a_3, \ldots, a_{m_k}, b_1, b_2, \ldots, b_{n_k - m_k}\}$$

for $k = 1, 2, 3, \ldots$. Then $\mathscr{C} = \{X_k\}$ is a cover with $\underline{d}_{\mathscr{C}}(A) = \alpha$ and $\overline{d}_{\mathscr{C}}(A) = \beta$. □

Corollary 3.1. *If A an infinite subset of $\mathbb{N}$ whose complement is also an infinite set and if $\alpha \in [0,1]$, then there exists a cover $\mathscr{C}$ such that $d_{\mathscr{C}}(A) = \alpha$.*

Conclusion

The theory developed here can be viewed as way to find the density of a set after assigning some weights to the natural numbers. If for some k and ℓ in $\mathbb{N}$, there is an n such that $k \in X_n$ and $\ell \notin X_n$, we may consider the weight of k is larger (or equal) than the weight of ℓ.

Moreover, in the existing literature our intuition that the set of positive even integers is half of the set of positive integers, is given a mathematical meaning. In the new theory the intuition by which the theory started fails. This is not an odd one in mathematics.

We started topology generalizing the concept of metric spaces. In the metric space $\mathbb{R}$, with usual topology the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \ldots$$

converges to 0 and only to 0. But the same sequence on $\mathbb{R}$ with the topology $\tau = \{\mathbb{R}, \emptyset, \{0\}\}$ converges to all real numbers other than 0 and it does not converge to 0, breaking our intuition that the sequence tends to 0. Likewise our theory also breaks some intuitions. Through this happens, the theory developed in this work has many similarities with the theory available in the literature of other branches of mathematics like Riemann's theorem on rearrangement of non-absolutely converging series. Some other types of densities and many open problems were discussed in [2, 3] and some of them can be studied in this new context.

We have discussed a generalization of the concept of natural density by replacing $\frac{|A \cap \{1,2,3,\ldots,n\}|}{n}$ by $\frac{|A \cap X_n|}{|X_n|}$ where $\{X_n\}$ is a sequence of subsets of $\mathbb{N}$ satisfying certain properties. Replacing $\frac{|A \cap \{1,2,3,\ldots,n\}|}{n}$ by $\frac{\mu(A \cap X_n)}{\mu(X_n)}$ where A and X_n are subsets of a measure space (X, μ), we can further generalize the concept of natural density to a very large setup. For example one may take $X = \mathbb{R}$, the Lebesgue measure on $\mathbb{R}$ as μ, and $\{X_n\}$ as an increasing sequence of sets with finite measure whose union is $\mathbb{R}$, and obtain new results like the set of positive real numbers is one half of the set of all real numbers and so on.

References

[1] E. Artin and P. Seherk, *On the Sum of Two Sets of Integers*, Ann. of Math. 44 (1943), 138-142.

[2] P. Erdos and J. Suranyi, *Topics in the Theory of Numbers*, Springer (2003).

[3] G. Grekos, *Open problems on densities, in Number Theory and Applications, Proceedings of the International Conference on Number Theory and Cryptography* (Allahabad, India, February 23-27, 2007), Hindustan Book Agency, New Delhi, 2009, 55-63.

Received: November 24, 2015; *Accepted*: April 15, 2016