Existence results for nonlinear fractional differential equation with nonlocal integral boundary conditions

Renu Chaudhary[a] and Dwijendra N Pandey[a,*]

[a] *Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee-247667, Uttarakhand, India.*

Abstract

In this paper, we shall study a nonlinear fractional differential equation with nonlocal integral boundary conditions. We have used fixed point theorems and Laray-Schauder nonlinear alternative to study the existence and uniqueness of solutions to the given equation. In the last, we have given examples to illustrate the applications of the abstract results.

Keywords: Fractional differential equations, Fixed point theorems, Laray-Schauder nonlinear alternative, Nonlocal boundary conditions.

2010 MSC: 34A08, 34B10, 34G20.

1 Introduction

Fractional differential equations are the generalization of ordinary differential equations to arbitrary non integer orders. The fact, that the fractional derivative(integral) is an operator which includes integer order derivatives(integrals) as special cases, is the reason why in present fractional differential equations becomes very popular and many applications are available. The fractional differential equations are of great importance because these are more precise in the modeling of many phenomenon, for instance, the nonlinear oscillations of earthquake can be described by the fractional differential equations. These differential equations are also very important to describe the memory and hereditary properties of various materials and phenomenon, this characteristic of fractional differential equations makes the fractional-order models more realistic and practical than the classical integer-order models. Recent work on fractional differential equations shows an overwhelming interest in this direction, for instance see [1–12] and the references cited therein. There have been many good books and monographs available on this field see [13–17].

On the other hand, the differential equations with a deviating argument are generalization of differential equations in which we permit the unknown function and its derivative to appear under different values of the argument. It is very important and significant branch of nonlinear analysis with numerous applications to physics, mechanics, control theory, biology, ecology, economics, theory of nuclear reactors, engineering, natural sciences and many other areas of science and technology. For a good introduction see [8, 18–21] and references cited therein.

The boundary value problem of fractional differential equations have been one of the hottest problems. Many problems related to blood flow, chemical engineering, thermo-elasticity, underground water flow, population dynamics, and so on can be reduced to nonlocal integral boundary problems. As a matter of fact, there are many papers dealing with the investigations on boundary value problems for some kinds of fractional differential equation with specific configurations covering theoretical as well as application aspects

---

*Corresponding author.

E-mail address: rrenu94@gmail.com (Renu Chaudhary), dwij.iitk@gmail.com (Dwijendra N Pandey)

of the subject. In this consequence, Bai and Lu [12] studied the existence of positive solutions for the fractional boundary value problem using Krasnoselskii's fixed point theorem and the Leggett-William's fixed point theorem. They established the criteria on the existence of at least one or three positive solutions for the boundary value problem. Later on, Kaufmann and Mboumi[4] discussed the existence of positive solutions for the fractional boundary value problem and provide sufficient conditions for the existence of at least one and at least three positive solutions to the nonlinear fractional boundary value problem. In [23] Ahmad et. al investigated a boundary value problem of Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions using Krasnoselskii's fixed point theorem. In [7] Yan et. al studied the boundary value problems for fractional differential equations subject to nonlocal boundary condition using Banach's fixed point theorem and Schaefer's fixed point theorem. In [11] Zhong et. al investigated nonlocal and multiple-point boundary value problem for fractional differential equations and establish the conditions for the uniqueness of solutions as well as the existence of at least one solution. In [9] Murad et. al investigated the existence and uniqueness of solutions to the nonlinear fractional differential equation of an arbitrary order with integral boundary condition using Schauder fixed point theorem and the Banach contraction principle. In [1] Ahmad et. al discussed a new class of fractional boundary value problems and establish the results using Banach and Krasnoselskii's fixed point theorem. Authors in [1] also studied Riemann-Liouville fractional nonlocal integral boundary value problems in [2] by means of classical fixed point theorems. In [10] Ntouyas et. al. studied the boundary value problems for nonlinear fractional differential equations and inclusions with nonlocal and fractional integral boundary conditions and obtained some new existence and uniqueness results by using fixed point theorems. In [6] Nyamoradi et. al investigate the existence of solutions for the multipoint boundary value problem of a fractional order differential inclusion on an infinite interval using suitable fixed point theorems. In [3] Ahmad et. al investigate the existence of solutions for higher order fractional differential inclusions with fractional integral boundary conditions involving nonintersecting finite many strips of arbitrary length using some standard fixed point theorems for multivalued maps. Akiladevi et.al [5] discuss the existence and uniqueness of solutions to the nonlinear neutral fractional boundary value problem using fixed point theorems. Recently, Zhao [25] studied triple positive solutions for two classes of delayed nonlinear fractional differential equation with nonlinear integral boundary value conditions using Leggett-Williams fixed point theorem and a generalization of Leggett-Williams fixed point theorem.

Motivated by the aforementioned techniques and papers, we have come to the conclusion that, although the fractional boundary value problems have been studied by many authors, but there is few gap in the literature on the boundary value problems with integral boundary conditions. In order to enhance the theoretical knowledge of the above, in this paper we intend to investigate the existence and uniqueness of solutions to the following Caputo-type fractional differential equation with deviated argument and nonlocal integral boundary conditions:

$$\begin{cases} {}^c\mathbf{D}^\gamma[z(t) - \mathcal{G}(t,z(t))] = \mathcal{F}(t,z(t),z[k(t,z(t))]), & 1 < \gamma \leq 2, \quad t \in (0,1) \\ z(0) = 0, \quad z(\tau) = \alpha \int_\eta^1 z(v)dv, & 0 < \tau < \eta < 1, \end{cases} \tag{1.1}$$

where ${}^c\mathbf{D}^\gamma$ is the Caputo fractional derivative of order γ. $\mathcal{F}, \mathcal{G}$ and k are suitably defined functions satisfying certain conditions to be stated later and α is a positive real constant. The nonlocal integral boundary condition $z(\tau) = \alpha \int_\eta^1 z(v)dv$ shows that the integration over a sub-strip $(\eta, 1)$ of an unknown function is proportional to the value of the unknown function at a nonlocal point $\tau \in (0,1)$ with $\tau < \eta < 1$.

In this work, our main aim is to establish some existence and uniqueness results for the system (1.1) by using fixed point techniques which will provide an effective way to deal with such problems. Most of the existing articles are only devoted to study of fractional differential equation with nonlocal integral boundary conditions up until now Caputo-type fractional differential equation with deviated argument and nonlocal integral boundary conditions, has not been considered in the literature. In this paper, the first sufficient condition proving existence and uniqueness of the mild solution of (1.1) is derived by utilizing Banach fixed point theorem under Lipschitz continuity of nonlinear terms. The second sufficient condition proving existence of the mild solution of (1.1) is obtained via Krasnoselskii's fixed point theorem. The third sufficient condition is obtained by using Laray-Schauder nonlinear alternative under non-Lipschitz continuity of nonlinear terms.

2   Preliminaries

In this segment we discuss some basic definitions of fractional integration and differentiation and some lemmas which plays an important role in the further sections.

Definition 2.1. *[17] For a function $f \in L^1(\mathbb{R}^+)$, the fractional integral of order γ is described by*

$$I_{0+}^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-v)^{\gamma-1} f(v) dv, \quad t > 0, \quad \gamma > 0.$$

Definition 2.2. *[13] For a function $f \in C^{m-1}(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, the Caputo fractional derivative of order γ is described by*

$$^{c}D_{0+}^{\gamma} f(t) = \frac{1}{\Gamma(m-\gamma)} \int_0^t (t-v)^{m-\gamma-1} f^m(v) dv,$$

where $m - 1 < \gamma < m$, $m = [\gamma] + 1$ and $[\gamma]$ denotes the integral part of the real number γ.

Lemma 2.1. *[14] Let $q > 0$, then*

$$D^{-\gamma} D^{\gamma} f(t) = f(t) + C_1 t^{\gamma-1} + C_2 t^{\gamma-2} + \ldots + C_n t^{\gamma-1},$$

for arbitrary $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $n = [\gamma] + 1$.

Lemma 2.2. *For any functions $\mathcal{F} \in C([0,1], \mathbb{R})$ and $\mathcal{G} \in C^1([0,1], \mathbb{R})$, the solution of following linear fractional boundary value problem*

$$^{c}D^{\gamma}[z(t) - \mathcal{G}(t)] = \mathcal{F}(t), \quad 1 < \gamma \leq 2, \quad t \in (0,1) \tag{2.2}$$

$$z(0) = 0, \quad z(\tau) = \alpha \int_{\eta}^{1} z(v) dv, \quad 0 < \eta < 1, \tag{2.3}$$

is defined by

$$z(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-v)^{\gamma-1} \mathcal{F}(v) dv - \mathcal{G}(0) + \mathcal{G}(t)$$

$$+ \frac{t}{\Lambda} \left\{ \mathcal{G}(0)(1 - \alpha(1-\eta)) - \mathcal{G}(\tau) - \frac{1}{\Gamma(\gamma)} \int_0^{\tau} (\tau - v)^{\gamma-1} \mathcal{F}(v) dv \right.$$

$$\left. + \alpha \int_{\eta}^{1} \mathcal{G}(v) dv + \frac{\alpha}{\Gamma(\gamma)} \int_{\eta}^{1} \left(\int_0^v (v-u)^{\gamma-1} \mathcal{F}(u) du \right) dv \right\}, \tag{2.4}$$

where

$$\Lambda = \tau - \frac{\alpha}{2}(1 - \eta^2) \neq 0. \tag{2.5}$$

Proof. Using Lemma(2.1), the solution z of (2.2) given by

$$z(t) = I^{\gamma} \mathcal{F}(t) - \mathcal{G}(0) + \mathcal{G}(t) + C_2 t + C_1, \tag{2.6}$$

for some constants $C_1, C_2 \in \mathbb{R}$.
On applying the boundary conditions (2.3), we get $C_1 = 0$ and

$$C_2 = \frac{1}{(\tau - \frac{\alpha}{2}(1-\eta^2))} \left\{ \mathcal{G}(0)(1 - \alpha(1-\eta)) - \mathcal{G}(\tau) - \frac{1}{\Gamma(\gamma)} \int_0^{\tau} (\tau - v)^{\gamma-1} \mathcal{F}(v) dv \right.$$

$$\left. + \alpha \int_{\eta}^{1} \mathcal{G}(v) dv + \frac{\alpha}{\Gamma(\gamma)} \int_{\eta}^{1} \left(\int_0^v (v-u)^{\gamma-1} \mathcal{F}(u) du \right) dv \right\}.$$

Substituting the values of C_1 and C_2 in (2.6), we get (2.4). $\square$

3 Existence and Uniqueness Results

Let $\mathcal{C} = C([0,1], \mathbb{R})$ be the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ equipped with the norm

$$\|z\| = \sup_{t \in [0,1]} |z(t)|, \quad z \in \mathcal{C}.$$

Set,

$$\mathfrak{B} = \{z \in \mathcal{C} : |z(t) - z(v)| \le L|t - v| \, \forall \, t, v \in [0,1]\},$$

where L is a positive constant.

With the help of Lemma (2.2), we introduce an operator $\Phi : \mathfrak{B} \to \mathfrak{B}$ as

$$(\Phi z)(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t - l)^{\gamma - 1} \mathcal{F}(l, z(l), z[k(l, z(l))]) dl + [\frac{t}{\Lambda}(1 - \alpha(1 - \eta)) - 1] \mathcal{G}(0, z(0))$$

$$+ \mathcal{G}(t, z(t)) + \frac{t}{\Lambda} \left\{ -\mathcal{G}(\tau, z(\tau)) - \frac{1}{\Gamma(\gamma)} \int_0^\tau (\tau - l)^{\gamma - 1} \mathcal{F}(l, z(l), z[k(l, z(l))]) dl \right.$$

$$\left. + \alpha \int_\eta^1 \mathcal{G}(l, z(l)) dl + \frac{\alpha}{\Gamma(\gamma)} \int_\eta^1 \left(\int_0^l (l - y)^{\gamma - 1} \mathcal{F}(y, z(y), z[k(y, z(y))]) dy \right) dl \right\}, \quad (3.7)$$

where Λ is given by (2.5). Here note that the boundary value problem (1.1) has solutions if and only if the operator Φ has fixed points.

Now, we introduce some assumptions which are required for the existence and uniqueness of the solution to boundary value problem (1.1).

(H1) The continuous function k is defined from $[0,1] \times \mathbb{R}$ to $\mathbb{R}$ with a constant $L_k > 0$ such that

$$|k(t, z) - k(t, x)| \le L_k |z - x|.$$

(H2) The continuous function $\mathcal{F}$ is defined from $[0,1] \times \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ with a constant $L_f > 0$ such that

$$|\mathcal{F}(t, z, z[k(t, z(t))]) - \mathcal{F}(t, x, x[k(t, x(t))])| \le L_f(2 + LL_k)|z - x|.$$

(H3) The continuously differentiable function $\mathcal{G}$ is defined from $[0,1] \times \mathbb{R}$ to $\mathbb{R}$ with a constant $L_g > 0$ such that

$$|\mathcal{G}(t, z) - \mathcal{G}(t, x)| \le L_g|z - x|.$$

(H4) There exists $M_1(t)$ and $M_2(t) \in \mathcal{C}$ such that

$$|\mathcal{F}(t, z, z[k(t, z(t))])| \le M_1(t),$$

and

$$|\mathcal{G}(t, z)| \le M_2(t).$$

Theorem 3.1. *Suppose* $(H1) - (H3)$ *hold with* $\delta_1 = L_f(2 + LL_k)\mu_1 + L_g\mu_2 < 1$, *where*

$$\mu_1 = \frac{1}{|\Lambda|} \left(\frac{(|\Lambda| + \tau^\gamma)}{\Gamma(\gamma + 1)} + \frac{\alpha(1 - \eta^{\gamma+1})}{\Gamma(\gamma + 2)} \right) \text{ and } \mu_2 = \left(1 + \frac{1}{|\Lambda|}(1 + \alpha(1 - \eta)) \right).$$

Then the boundary value problem (1.1) has a unique solution.

Proof. Let $\sup_{t \in [0,1]} |\mathcal{F}(t, 0, 0)| = N_1$, $\sup_{t \in [0,1]} |\mathcal{G}(t, 0)| = N_2$ and $B_r = \{z \in \mathfrak{B} : \|z\| \le r\}$, where $r \ge \frac{\delta_2}{1 - \delta_1}$ with

$$\delta_2 = N_1\mu_1 + N_2\mu_2 + \frac{1}{|\Lambda|}((1 - \alpha(1 - \eta)) - 1)|\mathcal{G}(0, z(0))|).$$

Now we will show that $\Phi B_r \subset B_r$. For $z \in B_r, 0 \leq t \leq 1$, we have

$$
\begin{aligned}
\|(\Phi z)(t)\| &\leq \sup_{t\in[0,1]} \left\{ \frac{1}{\Gamma(\gamma)} \int_0^t (t-l)^{\gamma-1} |\mathcal{F}(l,z(l),z[k(l,z(l))]) - \mathcal{F}(l,0,0) + \mathcal{F}(l,0,0)|dl \right. \\
&\quad + [\frac{t}{|\Lambda|}(1-\alpha(1-\eta))-1]|\mathcal{G}(0,z(0))| + |\mathcal{G}(t,z(t)) - \mathcal{G}(t,0) + \mathcal{G}(t,0)| \\
&\quad + \frac{t}{|\Lambda|}\left\{ |\mathcal{G}(\tau,z(\tau)) - \mathcal{G}(\tau,0) + \mathcal{G}(\tau,0)| + \frac{1}{\Gamma(\gamma)} \int_0^\tau (\tau-l)^{\gamma-1}|\mathcal{F}(l,z(l),z[k(l,z(l))])\right. \\
&\quad \left. - \mathcal{F}(l,0,0) + \mathcal{F}(l,0,0)|dl + \alpha \int_\eta^1 |\mathcal{G}(l,z(l)) - \mathcal{G}(l,0) + \mathcal{G}(l,0)|dl \right. \\
&\quad \left.\left. + \frac{\alpha}{\Gamma(\gamma)} \int_\eta^1 \left(\int_0^l (l-y)^{\gamma-1}|\mathcal{F}(y,z(y),z[k(y,z(y))]) - \mathcal{F}(y,0,0) + \mathcal{F}(y,0,0)|dy \right)dl \right\} \right\} \\
&\leq (L_f(2+LL_k)r + N_1)\mu_1 + (L_g r + N_2)\mu_2 + \frac{1}{|\Lambda|}((1-\alpha(1-\eta))-1)|\mathcal{G}(0,z(0))| \\
&\leq (L_f(2+LL_k)\mu_1 + L_g\mu_2)r + \left[N_1\mu_1 + N_2\mu_2 + \frac{1}{|\Lambda|}((1-\alpha(1-\eta))-1)|\mathcal{G}(0,z(0))| \right] \\
&\leq \delta_1 r + \delta_2 \leq r.
\end{aligned}
$$

Thus $\Phi B_r \subset B_r$. Now for $z, x \in B_r$ and $t \in [0,1]$, we have

$$
\begin{aligned}
\|\Phi z - \Phi x\| &\leq \sup_{t\in[0,1]} \left\{ \frac{1}{\Gamma(\gamma)} \int_0^t (t-l)^{\gamma-1}|\mathcal{F}(l,z(l),z[k(l,z(l))]) - \mathcal{F}(l,x(l),x[k(l,x(l))])|dl \right. \\
&\quad + |\mathcal{G}(t,z(t)) - \mathcal{G}(t,x(t))| + \frac{t}{|\Lambda|}\left\{ |\mathcal{G}(\tau,z(\tau)) - \mathcal{G}(\tau,x(\tau))| \right. \\
&\quad + \frac{1}{\Gamma(\gamma)} \int_0^\tau (\tau-l)^{\gamma-1}|\mathcal{F}(l,z(l),z[k(l,z(l))]) - \mathcal{F}(l,x(l),x[k(l,x(l))])|dl \\
&\quad + \alpha \int_\eta^1 |\mathcal{G}(l,z(l)) - \mathcal{G}(l,x(l))|dl + \frac{\alpha}{\Gamma(\gamma)} \int_\eta^1 \left(\int_0^l (l-y)^{\gamma-1}|\mathcal{F}(y,z(y),z[k(y,z(y))]) \right. \\
&\quad \left.\left. - \mathcal{F}(y,x(y),x[k(y,x(y))])|dy \right)dl \right\} \\
&\leq [L_f(2+LL_k)\mu_1 + L_g\mu_2]|z-x| \\
&\leq \delta_1|z-x|.
\end{aligned}
$$

Since $\delta_1 < 1$, $\|\Phi z - \Phi x\| < |z-x|$ i.e. Φ is a contraction mapping. Therefore by Banach contraction principle, the boundary value problem (1.1) has a unique solution. $\square$

Krasnoselskii combined two main result(Schauder's theorem and the contraction mapping principle) of fixed-point theory and gave a new theorem called Krasnoselskii's fixed point theorem. Now we show existence of solution with the help of Krasnoselskii's fixed point theorem [24].

Theorem 3.2. (Krasnoselskii fixed point theorem [24]) *Let X be a Banach space and B be a nonempty, closed and convex subset of X. Let Q_1 and Q_2 be two operators which maps B into X such that*

1. $Q_1 x + Q_2 y \in B$, *whenever* $x, y \in B$,

2. Q_1 *is completely continuous,*

3. Q_2 *is a contraction mapping.*

Then there exists $z \in B$ such that $z = Q_1 z + Q_2 z$.

Theorem 3.3. *Let $(H1) - (H4)$ hold with*

$$
\delta = \left(\frac{(L_f(2+LL_k))}{|\Lambda|} \left[\frac{\tau^\gamma}{\Gamma(\gamma+1)} + \frac{\alpha(1-\eta^{\gamma+1})}{\Gamma(\gamma+2)} \right] + L_g\left[1 + \frac{1}{|\Lambda|}(1+\alpha(1-\eta)) \right] \right) < 1.
$$

Then there exists at least one solution on $[0,1]$ of the given boundary value problem (1.1).

Proof. Let $\sup_{t\in[0,1]} |M_i(t)| = \|M_i\|$ for $i = 1, 2$, $M = \max\{M_1, M_2, \mathcal{G}(0, z(0))\}$ and $B_r = \{z \in \mathfrak{B} : \|z\| \leq r\}$, choose r such that

$$r \geq \|M\|\left[\mu_1 + \mu_2 + \frac{1}{|\Lambda|}(1 - \alpha(1 - \eta)) - 1\right].$$

Now, introduce the decomposition of the map Φ into Φ_1 and Φ_2 on B_r for $t \in [0,1]$ such that

$$(\Phi_1 z)(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t - l)^{\gamma-1} \mathcal{F}(l, z(l), z[k(l, z(l))]) dl,$$

$$(\Phi_2 z)(t) = [\frac{t}{\Lambda}(1 - \alpha(1 - \eta)) - 1]\mathcal{G}(0, z(0)) + \mathcal{G}(t, z(t))$$

$$+ \frac{t}{\Lambda}\left\{ -\mathcal{G}(\tau, z(\tau)) - \frac{1}{\Gamma(\gamma)} \int_0^\tau (\tau - l)^{\gamma-1} \mathcal{F}(l, z(l), z[k(l, z(l))]) dl \right.$$

$$\left. + \alpha \int_\eta^1 \mathcal{G}(l, z(l)) dl + \frac{\alpha}{\Gamma(\gamma)} \int_\eta^1 \left(\int_0^l (l - y)^{\gamma-1} \mathcal{F}(y, z(y), z[k(y, z(y))]) dy \right) dl \right\}.$$

For $y, x \in B_r$, we have

$$\|\Phi_1 z + \Phi_2 x\| \leq \sup_{t\in[0,1]} \left\{ \frac{1}{\Gamma(\gamma)} \int_0^t (t - l)^{\gamma-1} |\mathcal{F}(l, z(l), z[k(l, z(l))])| dl + [\frac{t}{|\Lambda|}(1 - \alpha(1 - \eta)) - 1]|\mathcal{G}(0, x(0))| \right.$$

$$+ |\mathcal{G}(t, x(t))| + \frac{t}{|\Lambda|}\left[|\mathcal{G}(\tau, x(\tau))| + \frac{1}{\Gamma(\gamma)} \int_0^\tau (\tau - l)^{\gamma-1} |\mathcal{F}(l, x(l), x[k(l, x(l))])| dl \right.$$

$$\left. \left. + \alpha \int_\eta^1 |\mathcal{G}(l, x(l))| dl + \frac{\alpha}{\Gamma(\gamma)} \int_\eta^1 \left(\int_0^l (l - y)^{\gamma-1} |\mathcal{F}(y, x(y), x[k(y, x(y))])| dy \right) dl \right] \right\}$$

$$\leq \|M_1\|\mu_1 + \|M_2\|\mu_2 + [\frac{1}{|\Lambda|}(1 - \alpha(1 - \eta)) - 1]|\mathcal{G}(0, z(0))|$$

$$\leq \|M\|\left[\mu_1 + \mu_2 + \frac{1}{|\Lambda|}(1 - \alpha(1 - \eta)) - 1\right]$$

$$\leq r.$$

Thus $\Phi_1 z + \Phi_2 x \in B_r$. Now to show Φ_1 is continuous and compact. The continuity of $\mathcal{F}$ implies the continuity of Φ_1. Also

$$\|(\Phi_1 z)(t)\| \leq \sup_{t\in[0,1]} \left\{ \frac{1}{\Gamma(\gamma)} \int_0^t (t - l)^{\gamma-1} |\mathcal{F}(l, z(l), z[k(l, z(l))])| dl \right\}$$

$$\leq \frac{\|M_1\|}{\Gamma(\gamma + 1)},$$

i.e. map Φ_1 is uniformly bounded on B_r.

Now, we show that $\{\Phi_1 z(t) : z \in B_r\}$ is equicontinuous. Clearly $\{\Phi_1 z(t) : z \in B_r\}$ are equicontinuous at $t = 0$. For $t < t + h \leq 1, h > 0$, we have

$$\|\Phi_1 z(t + h) - \Phi_1 z(t)\| \leq \frac{1}{\Gamma(\gamma)} \| \int_0^{t+h} (t + h - l)^{\gamma-1} \mathcal{F}(l, z(l), z[k(l, z(l))]) dl$$

$$- \int_0^t (t - l)^{\gamma-1} \mathcal{F}(l, z(l), z[k(l, z(l))]) dl \|$$

$$\leq \frac{1}{\Gamma(\gamma)} \int_0^t \left[(t + h - l)^{\gamma-1} - (t - l)^{\gamma-1} \right] \|\mathcal{F}(l, z(l), z[k(l, z(l))])\| dl$$

$$+ \frac{1}{\Gamma(\gamma)} \int_t^{t+h} (t + h - l)^{\gamma-1} \|\mathcal{F}(l, z(l), z[k(l, z(l))])\| dl,$$

which tends to zero as $h \to 0$, thus the set $\{\Phi_1 z(t) : z \in B_r\}$ is equicontinuous. Therefore by Arzelà-Ascoli's theorem Φ_1 is completely continuous.

Next we prove that Φ_2 is a contraction. For this

$$
\begin{aligned}
\|\Phi_2 z - \Phi_2 x\| \;\leq\; &\sup_{t\in[0,1]} \Bigg\{ |\mathcal{G}(t,z(t)) - \mathcal{G}(t,x(t))| + \frac{t}{|\Lambda|}\Bigg\{ |\mathcal{G}(\tau,z(\tau)) - \mathcal{G}(\tau,x(\tau))| + \frac{1}{\Gamma(\gamma)}\int_0^\tau (\tau - l)^{\gamma-1} \\
&|\mathcal{F}(l,z(l),z[k(l,z(l))]) - \mathcal{F}(l,x(l),x[k(l,x(l))])|dl + \alpha\int_\eta^1 |\mathcal{G}(l,z(l)) - \mathcal{G}(l,x(l))|dl \\
&+\frac{\alpha}{\Gamma(\gamma)}\int_\eta^1 \bigg(\int_0^l (l-y)^{\gamma-1}|\mathcal{F}(y,z(y),z[k(y,z(y))]) - \mathcal{F}(y,x(y),x[k(y,x(y))])|dy \bigg) dl \Bigg\} \\
\leq\; &\bigg(\frac{(L_f(2 + LL_k))}{|\Lambda|}\bigg[\frac{\tau^\gamma}{\Gamma(\gamma+1)} + \frac{\alpha(1-\eta^{\gamma+1})}{\Gamma(\gamma+2)} \bigg] + L_g\bigg[1 + \frac{1}{|\Lambda|}(1 + \alpha(1-\eta))\bigg] \bigg) |z - x| \\
\leq\; &\delta|z - x|.
\end{aligned}
$$

Since $\delta < 1$, $\|\Phi_2 z - \Phi_2 x\| < |z - x|$ i.e. Φ_2 is a contraction. Therefore by Krasnoselskii fixed point theorem, there exists at least one solution on $[0,1]$ of boundary value problem (1.1). □

In our next result we show the existence of solution with the help of Laray-Schauder nonlinear alternative [22].

Theorem 3.4. *(Laray-Schauder nonlinear alternative [22]) Let U and $\overline{U}$ denote respectively the open and closed subset of a nonempty, closed and convex set B of a Banach space X such that $0 \in U$. Let $T : \overline{U} \to B$ be a continuous and compact operator. Then either*

(i) *T has a fixed point in $\overline{U}$, or*

(ii) *there exists a point $u \in \partial U$ such that $u = \varepsilon Tu$ for some $\varepsilon \in (0,1)$, where ∂U is the boundary of U.*

Theorem 3.5. *Let the following assumptions hold.*

(H5) *There exists continuous nondecreasing functions $\psi_1, \psi_2 : [0,\infty) \to (0,\infty)$ and $\theta_1, \theta_2 \in L^1([0,1], \mathbb{R}^+)$ such that*

   (i) $|\mathcal{F}(t,z,x)| \leq \theta_1(t)\psi_1(\|z\| + \|x\|)$,
   (ii) $|\mathcal{G}(t,z)| \leq \theta_2(t)\psi_2(\|z\|)$.

(H6) *There exists a constant $P > 0$ such that $\frac{P}{\Theta} \geq 1$, where*

$$
\begin{aligned}
\Theta = \psi(\|P\|)\bigg[&\theta_2(1) + I^\gamma\bigg(\theta_2(1) + \frac{1}{|\Lambda|}\Big(\theta_1(\tau) + \alpha\int_\eta^1 \theta_1(l)dl\Big) \bigg) \\
&+\frac{1}{|\Lambda|}\bigg(((1 - \alpha(1-\eta)) - 1) + \theta_2(\tau) \bigg) + \alpha\int_\eta^1 \theta_2(l)dl \bigg].
\end{aligned}
$$

Then there exists at least one solution on $[0,1]$ of the given boundary value problem (1.1).

Proof. Clearly the operator $\Phi : \mathfrak{B} \to \mathfrak{B}$ defined by (3.7) is continuous. Firstly we show that the bounded sets in $\mathfrak{B}$ are mapped into the bounded sets in $\mathfrak{B}$ by the mapping Φ. For $r > 0$, let $B_r = \{z \in \mathfrak{B} : \|z\| \leq r\}$ be a bounded set in $\mathfrak{B}$. Thus for $z \in B_r$, we get

$$
\begin{aligned}
\|(\Phi z)(t)\| \;\leq\; &\sup_{t\in[0,1]} \Bigg\{ \frac{1}{\Gamma(\gamma)}\int_0^t (t-l)^{\gamma-1}|\mathcal{F}(l,z(l),z[k(l,z(l))])|dl + [\frac{t}{|\Lambda|}(1 - \alpha(1-\eta)) - 1]|\mathcal{G}(0,z(0))| \\
&+|\mathcal{G}(t,z(t))| + \frac{t}{|\Lambda|}\Bigg\{ |\mathcal{G}(\tau,z(\tau))| + \frac{1}{\Gamma(\gamma)}\int_0^\tau (\tau-l)^{\gamma-1}|\mathcal{F}(l,z(l),z[k(l,z(l))])|dl \\
&+\alpha\int_\eta^1 |\mathcal{G}(l,z(l))|dl + \frac{\alpha}{\Gamma(\gamma)}\int_\eta^1 \bigg(\int_0^l (l-y)^{\gamma-1}|\mathcal{F}(y,z(y),z[k(y,z(y))])|dy \bigg) dl \Bigg\} \\
\leq\; &\psi_1(2\|r\|)\int_0^1 \frac{(1-l)^{\gamma-1}}{\Gamma(\gamma)}\theta_1(l)dl + \frac{1}{|\Lambda|}((1 - a(1-\eta)) - 1)|\mathcal{G}(0,z(0))| + \psi_2(\|r\|)\theta_2(1) \\
&+\frac{1}{|\Lambda|}\Bigg\{ \psi_2(\|r\|)\theta_2(\tau) + \psi_1(2\|r\|)\int_0^\tau \frac{(\tau-l)^{\gamma-1}}{\Gamma(\gamma)}\theta_1(l)dl \\
&+\alpha\psi_2(\|r\|)\int_\eta^1 \theta_2(l)dl + \alpha\psi_1(2\|r\|)\int_\eta^1 \bigg(\int_0^l \frac{(l-y)^{\gamma-1}}{\Gamma(\gamma)}\theta_1(y)dy \bigg) dl \Bigg\},
\end{aligned}
$$

choose $\psi(r) \leq \max\{\psi_1(2\|r\|), \mathcal{G}(0, z(0)), \psi_2(\|r\|)\}$, we obtain

$$\|(\Phi z)(t)\| \leq \psi(r) \left[\theta_2(1) + I^\gamma \left(\theta_2(1) + \frac{1}{|\Lambda|} (\theta_1(\tau) + \alpha \int_\eta^1 \theta_1(l)dl) \right) \right.$$
$$\left. + \frac{1}{|\Lambda|} \left(((1 - \alpha(1 - \eta)) - 1) + \theta_2(\tau) \right) + \alpha \int_\eta^1 \theta_2(l)dl \right]. \tag{3.8}$$

Next, we will show that Φ maps bounded sets into equicontinuous sets in B_r. For this, let $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$ and $z \in B_r$, then

$$\|(\Phi z)(t_2) - (\Phi z)(t_1)\| \leq \int_0^{t_2} \frac{(t_2 - l)^{\gamma-1}}{\Gamma(\gamma)} |\mathcal{F}(l, z(l), z[k(l, z(l))])| dl + |\mathcal{G}(t_2, z(t_2))|$$
$$- \int_0^{t_1} \frac{(t_1 - l)^{\gamma-1}}{\Gamma(\gamma)} |\mathcal{F}(l, z(l), z[k(l, z(l))])| dl - |\mathcal{G}(t_1, z(t_1))|$$
$$+ \frac{(t_2 - t_1)}{|\Lambda|} \left[(1 - \alpha(1 - \eta))\mathcal{G}(0, z(0)) + \mathcal{G}(\tau, z(\tau)) + \alpha \int_\eta^1 |\mathcal{G}(l, z(l))| dl \right.$$
$$+ \int_0^\tau (\tau - l)^{\gamma-1} |\mathcal{F}(l, z(l), z[k(l, z(l))])| dl$$
$$\left. + \alpha \int_\eta^1 \left(\int_0^l (l - y)^{\gamma-1} |\mathcal{F}(y, z(y), z[k(y, z(y))])| dy \right) dl \right]$$
$$\leq \psi_1(2\|r\|) \left[\int_0^{t_1} \frac{(t_2 - l)^{\gamma-1} - (t_1 - l)^{\gamma-1}}{\Gamma(\gamma)} \theta_1(l) dl + \int_{t_1}^{t_2} \frac{(t_2 - l)^{\gamma-1}}{\Gamma(\gamma)} \theta_1(l) dl \right.$$
$$\left. + \frac{|t_2 - t_1|}{|\Lambda|} \left(\int_0^\tau \frac{(\tau - l)^{\gamma-1}}{\Gamma(\gamma)} \theta_1(l) dl + \alpha \int_\eta^1 \left(\int_0^l \frac{(l - y)^{\gamma-1}}{\Gamma(\gamma)} \theta_1(y) dy \right) dl \right) \right]$$
$$+ (\theta_2(t_2) - \theta_1(t_1))\psi_2(\|r\|) + \frac{|t_2 - t_1|}{|\Lambda|} \left(1 - \alpha(1 - \eta) \right) |\mathcal{G}(0, z(0))|$$
$$+ |\mathcal{G}(\tau, z(\tau))| + \alpha\psi_2(\|r\|) \int_\eta^1 \theta_2(l) dl \right).$$

Clearly, the right hand side does not depend on $z \in B_r$ and tends to zero as $t_2 \to t_1$. Thus by Arzelà-Ascoli theorem, Φ is compact and continuous.

Now, suppose z be the solution of the given problem. Then for $\varepsilon \in (0, 1)$ and using (3.8), we get

$$\|z(t)\| = \|\varepsilon(\Phi z)(t)\| \leq \psi(\|z\|) \left[\theta_2(1) + I^\gamma \left(\theta_2(1) + \frac{1}{|\Lambda|} (\theta_1(\tau) + \alpha \int_\eta^1 \theta_1(l)dl) \right) \right.$$
$$\left. + \frac{1}{|\Lambda|} \left(((1 - \alpha(1 - \eta)) - 1) + \theta_2(\tau) \right) + \alpha \int_\eta^1 \theta_2(l)dl \right],$$

which implies

$$\|z\| \leq \psi(\|z\|) \left[\theta_2(1) + I^\gamma \left(\theta_2(1) + \frac{1}{|\Lambda|} (\theta_1(\tau) + \alpha \int_\eta^1 \theta_1(l)dl) \right) \right.$$
$$\left. + \frac{1}{|\Lambda|} \left(((1 - \alpha(1 - \eta)) - 1) + \theta_2(\tau) \right) + \alpha \int_\eta^1 \theta_2(l)dl \right].$$

Using assumption $(H6)$, we get P such that $\|z\| \neq P$. Set $V = \{z \in \mathcal{C} : \|z\| < P\}$.

Here the operator $\Phi : \overline{V} \to \mathcal{C}$ is continuous and completely continuous. For any V, there is no $z \in \partial V$ such that $z = \varepsilon \Phi z$ for some $\varepsilon \in (0, 1)$. Using Laray-Schauder nonlinear alternative, we conclude that there exists a fixed point $z \in \overline{V}$ of operator Φ and this z is a solution of boundary value problem (1.1). $\qquad \square$

4 Examples

In this section, we present some examples, which indicate how our abstract result can be applied to the problem.

Chaudhary and Pandey /Existence results for nonlinear ...

Example(1): Consider the following fractional boundary value problem

$$\begin{cases} {}^c\mathbf{D}^{3/2}\left[z(t) - \frac{e^{-t}}{1+16e^{-t}}\frac{|z(t)|+1}{2+|z(t)|}\right] = \frac{1}{(t+7)^2}\left(|z(t)| + |t(|z(t)|+1)| + 2\right), \\ z(0) = 0, \quad z(1/4) = \int_{1/2}^1 z(l)dl. \end{cases}$$

(4.9)

Here $\gamma = 3/2$, $\tau = 1/4$, $\alpha = 2$, $\eta = 1/2$, $\mathcal{G}(t,z(t)) = \frac{e^{-t}}{1+16e^{-t}}\frac{(|z(t)|+1)}{(2+|z(t)|)}$, $k(t,z(t)) = \frac{t}{(t+7)^2}(|z(t)|+1)$ and $\mathcal{F}(t,z(t),z[k(t,z(t))]) = \frac{1}{(t+7)^2}\left(|z(t)| + |t(|z(t)|+1)| + 2\right)$. Here $\Lambda = \tau - \frac{\alpha}{2}(1-\eta^2) = -1/2 \neq 0$.

Observe that

$$|k(t,z(t)) - k(t,x(t))| \leq \frac{1}{49}|z-x|,$$

$$|\mathcal{F}(t,z,z[k(t,z(t))]) - \mathcal{F}(t,x,x[k(t,x(t))])| \leq \frac{1}{(t+7)^2}\left[|z|-|x| + |t|(|z|-|x|)\right]$$

$$\leq \frac{2}{49}|z-x|,$$

$$|\mathcal{G}(t,z(t)) - \mathcal{G}(t,x(t))| \leq \left|\frac{e^{-t}}{1+16e^{-t}}\right|\left|\frac{|z(t)|+1}{2+|z(t)|} - \frac{|x(t)|+1}{2+|x(t)|}\right|$$

$$\leq \frac{1}{17}|z-x|.$$

Thus assumptions $(H1)-(H3)$ holds with $L_f(2+LL_k) = 2/49$ and $L_g = 1/17$ and we get $\delta_1 = .2210 < 1$. Using Theorem (3.1) we get (4.9) has a unique solution.

Example(2): Consider the fractional boundary value problem given by

$$\begin{cases} {}^c\mathbf{D}^{3/2}\left[z(t) - \frac{1}{(t+7)^2}\sin z\right] = \frac{1}{\pi^2\sqrt{(1+t)}}\left(\sin z + \sin(t\sin z)\right), \\ z(0) = 0, \quad z(1/4) = \int_{1/2}^1 z(l)dl. \end{cases}$$

(4.10)

Here $\gamma = 3/2$, $\tau = 1/4$, $\alpha = 1$, $\eta = 1/2$, $\mathcal{G}(t,z(t)) = \frac{1}{(t+7)^2}\sin z$, $k(t,z(t)) = \frac{1}{\pi^2\sqrt{(1+t)}}t\sin z$ and $\mathcal{F}(t,z(t),z[k(t,z(t))]) = \frac{1}{\pi^2\sqrt{(1+t)}}\left(\sin z + \sin(t\sin z)\right)$. Here $\Lambda = \tau - \frac{\alpha}{2}(1-\eta^2) = -1/8 \neq 0$.

Observe that

$$|k(t,z(t)) - k(t,x(t))| \leq \frac{1}{\pi^2}|z-x|,$$

$$|\mathcal{F}(t,z,z[k(t,z(t))]) - \mathcal{F}(t,x,x[k(t,x(t))])| \leq \frac{2}{\pi^2}|z-x|,$$

$$|\mathcal{G}(t,z(t)) - \mathcal{G}(t,x(t))| \leq \frac{1}{49}|z-x|,$$

$$|\mathcal{F}(t,z,z[k(t,z(t))])| \leq \frac{2}{\pi^2\sqrt{(1+t)}} = M_1(t),$$

$$|\mathcal{G}(t,z(t))| \leq \frac{1}{(t+7)^2} = M_2(t).$$

Thus conditions $(H1)-(H4)$ holds with $L_f(2+LL_k) = 2/\pi^2$ and $L_g = 1/49$ and we get $\delta = .8186 < 1$. Clearly the assumptions $(H1)-(H4)$ of Theorem (3.3) are satisfied. Therefore (4.10) has at least one solution on $[0,1]$.

Example(3): Consider the following fractional boundary value problem

$$\begin{cases} {}^c\mathbf{D}^{3/2}\left[z(t) - \frac{1}{(t+11)^2}(|z|+1)\right] = \frac{1}{(t+7)^2}\left[|z| + |\sin(|z|+1)| + 2\right], \\ z(0) = 0, \quad z(1/2) = \int_{3/4}^1 z(l)dl. \end{cases}$$

(4.11)

Here $\gamma = 3/2$, $\tau = 1/4$, $\alpha = 1$, $\eta = 3/4$, $\mathcal{G}(t,z(t)) = \frac{1}{(t+11)^2}(|z|+1)$, $k(t,z(t)) = \frac{1}{(t+7)^2}\sin(|z|+1)$ and $\mathcal{F}(t,z(t),z[k(t,z(t))]) = \frac{1}{(t+7)^2}\left[|z| + |\sin(|z|+1)| + 2\right]$. Here $\Lambda = \tau - \frac{\alpha}{2}(1-\eta^2) = 9/32 \neq 0$.

Observe that

$$|\mathcal{F}(t,z,z[k(t,z(t))])| \leq \frac{1}{49}(2|z|+3),$$

$$|\mathcal{G}(t,z(t))| \leq \frac{1}{121}(|z|+1).$$

From $(H5)$ we get $\theta_1(t) = 1$, $\psi_1(\|z\| + \|x\|) = \frac{1}{49}(2|z|+3)$, $\theta_2(t) = 1$ and $\psi_2(\|z\|) = \frac{1}{121}(|z|+1)$. Also

$$
\begin{aligned}
\Theta &= \psi(\|M\|)\Bigg[\theta_2(1) + I^\gamma\left(\theta_2(1) + \frac{1}{|\Lambda|}\left(\theta_1(\tau) + \alpha \int_\eta^1 \theta_1(l)dl\right)\right) \\
&\quad + \frac{1}{|\Lambda|}\left(((1-\alpha(1-\eta))-1) + \theta_2(\tau)\right) + \alpha \int_\eta^1 \theta_2(l)dl\Bigg] \\
&= \psi(\|M\|)(8.0012).
\end{aligned}
$$

Using condition $\frac{P}{\Theta} \geq 1$, we found that there exists a constant P such that $P \geq .7274 > 0$, therefore assumptions $(H5)$ and $(H6)$ of Theorem (3.5) are fulfilled. Therefore (4.11) has at least one solution on $[0,1]$.

5 Conclusion

This paper has investigated the existence and uniqueness of solution to the Caputo-type fractional differential equation with deviated argument and nonlocal integral boundary conditions. The first sufficient condition proving existence and uniqueness of the mild solution of (1.1) is derived by utilizing Banach fixed point theorem under Lipschitz continuity of nonlinear terms. The second sufficient condition proving existence of the mild solution of (1.1) is obtained via Krasnoselskii's fixed point theorem. The third sufficient condition is obtained by using Laray-Schauder nonlinear alternative under non-Lipschitz continuity of nonlinear terms. At last, examples are provided to illustrate the applications of the abstract results.

Acknowledgment

References

[1] Ahmad, Bashir; Alsaedi, A.; Assolami, A.; Agarwal, Ravi P., A new class of fractional boundary value problems, Adv. Diff. Equ. **273**(2013), 1-8.

[2] Ahmad, Bashir; Alsaedi, A.; Assolami, A.; Agarwal, Ravi P., A study of Riemann-Liouville fractional nonlocal integral boundary value problems, Adv. Diff. Equ. **274**(2013), 1-9.

[3] Ahmad, Bashir; Ntouyas, Sotiris K., Existence results for higher order fractional differential inclusions with multi-strip fractional integral boundary conditions. Electron. J. Qual. Theory Differ. Equ. 2013, No. 20, 19 pp.

[4] Kaufmann, Eric R.; Mboumi, Ebene., Positive solutions of a boundary value problem for a nonlinear fractional differential equation. Electron. J. Qual. Theory Differ. Equ. 2008, No. 3, 11 pp.

[5] Akiladevi, K. S.; Balachandran, K.; Kim, J. K., Existence results for neutral fractional integrod-ifferential equations with fractional integral boundary conditions, Nonlinear Func. Anal. and App., **19**(2014), no. 2, 251-270.

[6] Nyamoradi, Nemat; Baleanu, Dumitru; Agarwal, Ravi P., On a multipoint boundary value problem for a fractional order differential inclusion on an infinite interval. Adv. Math. Phys. 2013, Art. ID 823961, 9 pp.

[7] Yan, R.; Sun, S.; Sun, Y.; Han, Z., Boundary value problems for fractional differential equations with nonlocal boundary conditions, Adv. Diff. Equ. **176**(2013), 1-12.

[8] Kumar, Pradeep; Pandey, Dwijendra N.; Bahuguna, D. Approximations of solutions to a fractional differential equation with a deviating argument. Differ. Equ. Dyn. Syst. **22** (2014), no. 4, 333-352.

[9] Murad, S. A.; Hadid,S.B., Existence and uniqueness theorem for fractional differential equation with integral boundary condition, J. Frac. Calc. Appl., **3** (2012), 1-9.

[10] Ntouyas, Sotiris K, Boundary value problems for nonlinear fractional differential equations and inclusions with nonlocal and fractional integral boundary conditions. Opuscula Math. **33** (2013), no. 1, 117-138.

[11] Zhong, Wenyong; Lin, Wei, Nonlocal and multiple-point boundary value problem for fractional differential equations. Comput. Math. Appl. **59** (2010), no. 3, 1345-1351.

[12] Bai, Zhanbing; Lu, Haishen, Positive solutions for boundary value problem of nonlinear fractional differential equation. J. Math. Anal. Appl. **311** (2005), no. 2, 495-505.

[13] Kilbas, Anatoly A.; Srivastava, Hari M.; Trujillo, Juan J., Theory and applications of fractional differential equations. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[14] Podlubny, Igor, Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Mathematics in Science and Engineering, 198. Academic Press, Inc., San Diego, CA, 1999.

[15] Oldham, Keith B.; Spanier, Jerome. The fractional calculus. Theory and applications of differentiation and integration to arbitrary order. Mathematics in Science and Engineering, Vol. 111. Academic Press, New York-London, 1974.

[16] Miller, Kenneth S.; Ross, Bertram, An introduction to the fractional calculus and fractional differential equations. A Wiley-Interscience Publication. John Wiley and Sons, Inc., New York, 1993.

[17] Samko, Stefan G.; Kilbas, Anatoly A.; Marichev, Oleg I. Fractional integrals and derivatives. Theory and applications. Gordon and Breach Science Publishers, Yverdon, 1993.

[18] Gal, Ciprian G., Nonlinear abstract differential equations with deviated argument. J. Math. Anal. Appl. **333** (2007), no. 2, 971-983.

[19] Gal, Ciprian G., Semilinear abstract differential equations with deviated argument. Int. J. Evol. Equ. **2** (2008), no. 4, 381-386.

[20] Elsgolc, L. E., Introduction to the theory of differential equations with deviating arguments, Holden-Day, San Francisco, CA, 1966.

[21] Oberg, Robert J., On the local existence of solutions of certain functional-differential equations. Proc. Amer. Math. Soc. **20** (1969), 295-302.

[22] Granas, Andrzej; Dugundji, James., Fixed point theory. Springer Monographs in Mathematics. Springer-Verlag, New York, 2003.

[23] Ahmad, Bashir; Nieto, Juan J., Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions. Bound. Value Probl. 2011, 2011:36, 9 pp.

[24] Smart, D. R., Fixed point theorems. Cambridge Tracts in Mathematics, No. 66. Cambridge University Press, London-New York, 1974.

[25] Zhao, Kaihong., Triple positive solutions for two classes of delayed nonlinear fractional FDEs with nonlinear integral boundary value conditions. Bound. Value Probl. 2015, 2015:181, 20 pp.

UNIVERSITY PRESS

Website: http://www.malayajournal.org/