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Interval criteria for oscillation of second-order impulsive delay differential equation with mixed nonlinearities

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Abstract

We obtain interval oscillation criteria for the second-order impulsive delay differential equation

$$\left(r(t)\Phi_\alpha(x'(t))\right)' + p(t)\Phi_\alpha(x(t-\tau)) + \sum_{i=1}^{n} q_i(t)\Phi_{\beta_i}(x(t-\tau)) = e(t),\ t \geq t_0,\ t \neq t_k,$$
$$x(t_k{}^+) = a_k x(t_k), \quad x'(t_k{}^+) = b_k x'(t_k),\ k = 1, 2, 3, \ldots .$$

The results obtained in this paper extend some of the existing results. We have given some examples to illustrate our results.

Keywords: Interval oscillation; Impulse; Delay; Mixed nonlinearities.

2010 MSC: 34C10, 34K11.

1 Introduction

Consider the second-order impulsive delay differential equation with mixed nonlinearities

$$\begin{aligned}\left(r(t)\Phi_\alpha(x'(t))\right)' + p(t)\Phi_\alpha(x(t-\tau)) + \sum_{i=1}^{n} q_i(t)\Phi_{\beta_i}(x(t-\tau)) = e(t),\ t \geq t_0,\ t \neq t_k,\\ x(t_k{}^+) = a_k x(t_k), \quad x'(t_k{}^+) = b_k x'(t_k),\ k = 1, 2, 3, \ldots\end{aligned} \tag{1.1}$$

where

$$x(t_k^-) := \lim_{t \to t_k^-} x(t), \quad x(t_k^+) := \lim_{t \to t_k^+} x(t),$$
$$x'(t_k^-) := \lim_{h \to 0^-} \frac{x(t_k + h) - x(t_k)}{h}, \quad x'(t_k^+) := \lim_{h \to 0^+} \frac{x(t_k + h) - x(t_k)}{h}.$$

$\Phi_*(s) := |s|^{*-1}s$, τ is a non negative constant, $\{t_k\}$ denotes the impulsive moment sequence with $0 \leq t_0 < t_1 < \cdots < t_k < \ldots$, $\lim_{k\to\infty} t_k = \infty$ and $t_{k+1} - t_k > \tau$ for $k = 1, 2, 3, \ldots$.
Let $J \subset \mathbb{R}$ be an interval, we define

$$PLC(J, \mathbb{R}) := \{h : J \to \mathbb{R} \mid h \text{ is continuous on each interval } (t_k, t_{k+1}),$$
$$h(t_k^\pm) \text{ exists and } h(t_k) = h(t_k^-) \text{ for all } k \in \mathbb{N}\}.$$

For given t_0 and $\phi \in PLC([t_0 - \tau, t_0], \mathbb{R})$, we say $x \in PLC([t_0 - \tau, \infty), \mathbb{R})$ is a solution of equation(1.1) with the initial value ϕ if $x(t)$ satisfies equation(1.1) for $t \geq t_0$ and $x(t) = \phi(t)$ for $t \in [t_0 - \tau, t_0]$.

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A nontrivial solution of equation(1.1) is called *oscillatory* if it has arbitrarily large zeros; otherwise, it is called *nonoscillatory*.

The theory of impulsive differential equations is an important branch of differential equations. The first paper in this theory is related to V. D. Milman and A. D. Mishkis in 1960 [14]. In recent years the oscillation theory of impulsive differential equations emerging as an important area of research, since such equations have applications in control theory, physics, biology, population dynamics, economics, etc. For further applications and questions concerning existence and uniqueness of solutions of impulsive differential equation, see for example Lakshmigantham et. al [10] and the references cited therein.

During the last decades, several oscillation results were established for different kinds of impulsive delay differential equations (see Agarwal and Karakoc [2]). Recently, interval oscillation of impulsive delay differential equations was attracting the interest of many researchers, see Guo et. al[5, 6] and Li and Cheung [11]. However, only very few interval oscillation results are available in the literature for " second order impulsive differential equations with delay ". For example, Huang and Feng [8] considered the second order delay differential equations with impulses

$$\begin{gathered} x''(t) + p(t)f(x(t-\tau)) = e(t),\ t \geq t_0,\ t \neq t_k, \\ x({t_k}^+) = a_k x(t_k), \quad x'({t_k}^+) = b_k x'(t_k),\ k = 1, 2, ... \end{gathered}$$

and established some interval oscillation criteria which developed some known results for the equations without delay or impulses [4, 12, 18].

In [5], Guo et. al considered the second order mixed nonlinear impulsive differential equations with delay

$$\begin{gathered} \big(r(t)\Phi_\alpha(x'(t))\big)' + p_0(t)\Phi_\alpha(x(t)) + \sum_{i=1}^{n} p_i(t)\Phi_{\beta_i}(x(t-\sigma)) = e(t),\ t \geq t_0,\ t \neq \tau_k, \\ x({\tau_k}^+) = a_k x(\tau_k), \quad x'({\tau_k}^+) = b_k x'(\tau_k),\ k = 1, 2, ... \end{gathered}$$

and obtained some interval oscillation criteria which generalized the results in [13, 15, 17].

In [11], Li and Cheung established some interval oscillation criteria for the second order impulsive delay differential equations of the form

$$\begin{gathered} \big(p(t)(x'(t))\big)' + q(t)(x(t-\tau)) + \sum_{i=1}^{n} q_i(t)\Phi_{\alpha_i}(x(t-\tau)) = e(t),\ t \geq t_0,\ t \neq t_k, \\ x({t_k}^+) = a_k x(t_k), \quad x'({t_k}^+) = b_k x'(t_k),\ k = 1, 2, ... \end{gathered}$$

Motivated mainly by [5, 6, 11], we establish some interval oscillation criteria for equation (1.1). We also provide two examples to illustrate the effectiveness of our results.

2 Main results

Throughout this paper, assume that the following conditions hold without further mention:

(A1) $r(t) \in C([t_0, \infty), (0, \infty))$ is non-decreasing, $p, q_i, e \in PLC([t_0, \infty), \mathbb{R})$, $i = 1, 2 \ldots, n$;

(A2) $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$ are constants;

(A3) α is a quotient of odd positive integers, $b_k \geq a_k > 0$, $k \in \mathbb{N}$ are constants.

let $k(s) := \max\{i : t_0 < t_i < s\}$ and for $c_j < d_j$, let $M_j := \max\{r(t) : t \in [c_j, d_j]\}$, $j = 1, 2$, $\Omega_j := \{\omega \in C^1[c_j, d_j] : \omega(t) \not\equiv 0,\ \omega(c_j) = \omega(d_j) = 0\}$, $j = 1, 2$. For two constants $c, d \notin \{t_k\}$ with $c < d$ and a function $\phi \in C([c, d], \mathbb{R})$, we define an operator $\Psi : C([c, d], \mathbb{R}) \to \mathbb{R}$ by

$$\Psi_c^d[\phi] = \begin{cases} 0, & \text{for } k(c) = k(d), \\ \phi(t_{k(c)+1})\theta(c) + \sum_{i=k(c)+2}^{k(d)} \phi(t_i)\varepsilon(t_i), & \text{for } k(c) < k(d), \end{cases}$$

where

$$\theta(c) = \frac{b_{k(c)+1}^\alpha - a_{k(c)+1}^\alpha}{(a_{k(c)+1}^\alpha (t_{k(c)+1} - c)^\alpha)}\ , \quad \varepsilon(t_i) = \frac{b_i^\alpha - a_i^\alpha}{(a_i^\alpha (t_i - t_{i-1})^\alpha)}.$$

where $\sum_s^t = 0$ if $s > t$.

In the discussion of the impulse moments of $x(t)$ and $x(t-\tau)$, we need to consider the following four cases for $k(c_j) < k(d_j)$,

$$(s_1)\ t_{k(c_j)} + \tau < c_j \text{ and } t_{k(d_j)} + \tau > d_j;\ (s_2)\ t_{k(c_j)} + \tau < c_j \text{ and } t_{k(d_j)} + \tau < d_j;$$
$$(s_3)\ t_{k(c_j)} + \tau > c_j \text{ and } t_{k(d_j)} + \tau > d_j;\ (s_4)\ t_{k(c_j)} + \tau > c_j \text{ and } t_{k(d_j)} + \tau < d_j,\ j = 1,2$$

and the three cases for $k(c_j) = k(d_j)$,

$$(\tilde{s_1})\ t_{k(c_j)} + \tau < c_j;\ (\tilde{s_2})\ t_{k(d_j)} + \tau < d_j;\ (\tilde{s_3})\ t_{k(d_j)} + \tau > d_j,\ j = 1,2.$$

Combining (s_*) with $(\tilde{s_*})$, we can get 12 cases. Throughout the paper, we study equation(1.1) under the case of combination of (s_1) with $(\tilde{s_1})$ only. The discussions for other cases are similar and so omitted.

Let us see some lemmas which will be useful to prove our main results.

Lemma 2.1. *[1] For any given n-tuple $\{\beta_1, \beta_2, \dots, \beta_n\}$ satisfying $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$, there corresponds an n-tuple $(\eta_1, \eta_2, \dots, \eta_n)$ such that*

$$\sum_{i=1}^{n} \beta_i \eta_i = \alpha, \qquad \sum_{i=1}^{n} \eta_i < 1, \quad 0 < \eta_i < 1. \tag{2.2}$$

Lemma 2.2. *[1] For any given n-tuple $\{\beta_1, \beta_2, \dots, \beta_n\}$ satisfying $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$, there corresponds an n-tuple $(\eta_1, \eta_2, \dots, \eta_n)$ such that*

$$\sum_{i=1}^{n} \beta_i \eta_i = \alpha, \qquad \sum_{i=1}^{n} \eta_i = 1, \quad 0 < \eta_i < 1. \tag{2.3}$$

Lemma 2.3. *[7] Suppose X and Y are non-negative, then*

$$\lambda X Y^{\lambda-1} - X^{\lambda} \le (\lambda - 1) Y^{\lambda}, \lambda > 1 \tag{2.4}$$

where equality holds if and only if $X = Y$.

Lemma 2.4. *Assume that for any $T \ge t_0$, there exists $c_j, d_j \notin \{t_k\}$, $j = 1,2$ such that $T < c_1 < d_1 \le c_2 < d_2$ and*

$$\begin{aligned} p(t), q_i(t) &\ge 0, \quad t \in [c_1 - \tau, d_1] \cup [c_2 - \tau, d_2] \backslash \{t_k\},\ i = 1,2,3,...,n \\ e(t) &\le 0, \quad t \in [c_1 - \tau, d_1] \backslash \{t_k\}, \\ e(t) &\ge 0, \quad t \in [c_2 - \tau, d_2] \backslash \{t_k\}. \end{aligned} \tag{2.5}$$

If $x(t)$ is a non-oscillatory solution of equation(1.1), then there exist the following estimations of $x(t-\tau)/x(t)$;

$$\begin{aligned} &(a)\, for \quad t \in (t_i + \tau, t_{i+1}], \quad \frac{x(t-\tau)}{x(t)} > \left(\frac{t - t_i - \tau}{t - t_i} \right), \\ &(b)\, for \quad t \in (t_i, t_i + \tau), \quad \frac{x(t-\tau)}{x(t)} > \left(\frac{t - t_i}{b_i(t + \tau - t_i)} \right), \\ &(c)\, for \quad t \in [c_j, t_{k(c_j)+1}], \quad \frac{x(t-\tau)}{x(t)} > \left(\frac{t - t_{k(c_j)} - \tau}{t - t_{k(c_j)}} \right), \\ &(d)\, for \quad t \in (t_{k(d_j)}, d_j], \quad \frac{x(t-\tau)}{x(t)} > \left(\frac{t - t_{k(d_j)}}{b_{k(d_j)}(t + \tau - t_{k(d_j)})} \right), \end{aligned} \tag{2.6}$$

where $i = k(c_j), ..., k(d_j) - 1$, $j = 1,2$.

Proof. Without loss of generality, we assume that $x(t) > 0$ and $x(t-\tau) > 0$ for $t \ge t_0$. In this case the selected interval of t is $[c_1, d_1]$. From equation(1.1) and (2.5), we obtain

$$\left[r(t) \Phi_\alpha(x'(t)) \right]' = e(t) - p(t) \Phi_\alpha(x(t-\tau)) - \sum_{i=1}^{n} q_i(t) \Phi_{\beta_i}(x(t-\tau)) \le 0 \tag{2.7}$$

Hence $r(t)\Phi_\alpha(x'(t))$ is non-increasing on the interval $[c_1, d_1]\backslash\{t_k\}$.
Case(a): $t_i + \tau < t \le t_{i+1}$.
Then $(t-\tau, t) \subset (t_i, t_{i+1}]$ and hence there is no impulsive moment in $(t-\tau, t)$. For any $s \in (t-\tau, t)$, we have

$$x(s) - x(t_i^+) = x'(\xi_1)(s - t_i),\ \xi_1 \in (t_i, s).$$

Because of the facts that $x(t_i^+) > 0, \phi_\alpha(*)$ is an increasing function and $r(s)\Phi_\alpha(x'(s))$ is non-increasing on (t_i, t_{i+1}), we have

$$\phi_\alpha(x(s)) > \phi_\alpha(x'(\xi_1)(s-t_i)) = \frac{r(\xi_1)}{r(\xi_1)}\phi_\alpha(x'(\xi_1))(s-t_i)^\alpha,$$

and hence

$$\Phi_\alpha(x(s)) \ge \frac{r(s)\Phi_\alpha(x'(s))}{r(\xi_1)}(s-t_i)^\alpha.$$

Since $r(s)$ is positive and non-decreasing, the above inequality becomes

$$\phi_\alpha(x(s)) \ge \phi_\alpha(x'(s)(s-t_i)),\ \xi_1 \in (t_i, s).$$

Thus, we have

$$\frac{x'(s)}{x(s)} < \frac{1}{(s-t_i)}.$$

Integrating both sides of the above inequality from $t-\tau$ to t, we obtain

$$\frac{x(t-\tau)}{x(t)} > \left(\frac{t-t_i-\tau}{t-t_i}\right),\ t \in (t_i+\tau, t_{i+1}]. \tag{2.8}$$

Case(b): $t \in (t_i, t_i + \tau)$.
Then $t - \tau \in (t_i - \tau, t_i)$. ie, $t_i - \tau < t - \tau < t_i < t < t_i + \tau$. Then there is an impulsive moment t_i in $(t-\tau, t)$. Then we have,

$$x(t) - x(t_i^+) = x'(\xi_2)(t-t_i),\ \xi_2 \in (t_i, t).$$

Using the impulsive condition of equation(1.1) and the monotone properties of $r(t)$, $\phi_\alpha(t)$ and $r(t)\phi_\alpha(x'(t))$, we get

$$\begin{aligned}\phi_\alpha(x(t) - a_i x(t_i)) &\le \frac{r(t_i^+)\phi_\alpha(x'(t_i^+))}{r(\xi_2)}(t-t_i)^\alpha \\ &= \phi_\alpha(b_i x'(t_i))(t-t_i)^\alpha \\ \Rightarrow \phi_\alpha\left(\frac{x(t)}{x(t_i)} - a_i\right) &\le \phi_\alpha\left(b_i\frac{x'(t_i)}{x(t_i)}(t-t_i)\right)\end{aligned} \tag{2.9}$$

In addition, by mean value theorem on $[t_i - \tau, t_i]$, we have

$$x(t_i) - x(t_i - \tau) = x'(\xi_3)\tau,\ \xi_3 \in (t_i - \tau, t_i)$$
$$\text{and hence,}\quad \phi_\alpha(x(t_i)) > \phi_\alpha(x'(\xi_3)\tau)$$

By using the monotone properties of $r(t), \phi_\alpha(t)$ and $r(t)\phi_\alpha(x'(t))$, we have

$$\begin{aligned} &\phi_\alpha(x(t_i)) \ge \phi_\alpha(x'(t_i)\tau) \\ \Rightarrow\quad &\frac{x'(t_i)}{x(t_i)} < \frac{1}{\tau}\end{aligned} \tag{2.10}$$

From (2.9) and (2.10), we have,

$$\begin{aligned} &\phi_\alpha\left(\frac{x(t)}{x(t_i)} - a_i\right) \le \phi_\alpha\left(\frac{b_i(t-t_i)}{\tau}\right) \\ \Rightarrow\quad &\frac{x(t)}{x(t_i)} \le \frac{b_i(t-t_i+\tau)}{\tau}\end{aligned} \tag{2.11}$$

For some $s \in (t_i - \tau, t_i)$, we have

$$\begin{aligned} &x(s) - x(t_i - \tau) = x'(\xi_4)(s - t_i + \tau),\ \xi_4 \in (t_i - \tau, s) \\ \Rightarrow\quad &\phi_\alpha(x(s)) > \frac{r(\xi_4)\phi_\alpha(x'(\xi_4))}{r(\xi_4)}(s - t_i + \tau)^\alpha.\end{aligned}$$

Again by using the monotone properties of $r(t)$, $\phi_\alpha(t)$ and $r(t)\phi_\alpha(x'(t))$, we have

$$\phi_\alpha(x(s)) \geq \phi_\alpha(x'(s)(s-t_i+\tau))$$
$$\Rightarrow \quad \frac{x'(s)}{x(s)} < \frac{1}{(s-t_i+\tau)}.$$

Integrating both sides of the above inequality from $t-\tau$ to t_i where $t \in (t_i, t_i+\tau)$, we have

$$\frac{x(t-\tau)}{x(t_i)} > \frac{t-t_i}{\tau}, \ t \in (t_i, t_i+\tau). \tag{2.12}$$

Hence, from (2.11) and (2.12), we have

$$\frac{x(t-\tau)}{x(t)} > \left(\frac{t-t_i}{b_i(t+\tau-t_i)}\right), \ t \in (t_i, t_i+\tau).$$

Case(c): $t \in [c_1, t_{k(c_1)+1}]$.
Then $t-\tau \in [c_1-\tau, t_{k(c_1)+1}-\tau]$ and hence there is no impulsive moment in $(t-\tau, t)$.
For any $s \in (t-\tau, t)$ as in Case(a), we have

$$\phi_\alpha(x(s)) > \phi_\alpha(x'(\xi_5)(s-t_{k(c_1)}))$$

By the monotone properties of $\phi_\alpha(*)$ and $r(s)\Phi_\alpha(x'(s))$, we have

$$\Phi_\alpha(x(s)) \geq \frac{r(s)\Phi_\alpha(x'(s))}{r(\xi_5)}(s-t_{k(c_1)})^\alpha.$$

Since $r(s)$ is positive and non decreasing, the above inequality becomes

$$\phi_\alpha(x(s)) \geq \phi_\alpha(x'(s)(s-t_{k(c_1)})), \ \xi_5 \in (t_{k(c_1)}, s)$$
$$\Rightarrow \qquad \frac{x'(s)}{x(s)} < \frac{1}{(s-t_{k(c_1)})}$$

Integrating both sides of the above inequality from $t-\tau$ to t, we obtain

$$\frac{x(t-\tau)}{x(t)} > \left(\frac{t-t_{k(c_1)}-\tau}{t-t_{k(c_1)}}\right), \ t \in [c_1, t_{k(c_1)+1}].$$

Case(d): $t \in (t_{k(d_1)}, d_1]$.
Then $t-\tau \in (t_{k(d_1)}-\tau, d_1-\tau]$. ie, $t_{k(d_1)}-\tau < t-\tau < t_{k(d_1)} < t < t_{k(d_1)}+\tau$. Then there is an impulsive moment $t_{k(d_1)}$ in $(t-\tau, t)$. Making a similar analysis of Case(b), we obtain

$$\frac{x(t-\tau)}{x(t)} > \left(\frac{t-t_{k(d_1)}}{b_{k(d_1)}(t+\tau-t_{k(d_1)})}\right), \ t \in (t_{k(d_1)}, d_1].$$

When $x(t) < 0$, we can choose interval $[c_2, d_2]$ to study equation(1.1). The proof is similar and hence omitted. This completes the proof. □

Theorem 2.1. *Assume that for any $T \geq t_0$, there exists $c_j, d_j \notin \{t_k\}$, $j=1,2$, such that $T < c_1 < d_1 \leq c_2 < d_2$ and* (2.5) *holds. If there exists $\omega_j(t) \in \Omega_j(c_j, d_j)$, $j=1,2$ such that, for $k(c_j) < k(d_j)$,*

$$\begin{aligned}
&\int_{c_j}^{t_{k(c_j)+1}} W_j(t)\left(\frac{t-t_{k(c_j)}-\tau}{t-t_{k(c_j)}}\right)^\alpha dt \\
&+\sum_{i=k(c_j)+1}^{k(d_j)-1}\left[\int_{t_i}^{t_i+\tau} W_j(t)\left(\frac{t-t_i}{b_i(t+\tau-t_i)}\right)^\alpha dt + \int_{t_i+\tau}^{t_{i+1}} W_j(t)\left(\frac{t-t_i-\tau}{t-t_i}\right)^\alpha\right] \\
&+\int_{t_{k(d_j)}}^{d_j} W_j(t)\left(\frac{t-t_{k(d_j)}}{b_{k(d_j)}(t+\tau-t_{k(d_j)})}\right)^\alpha dt - \int_{c_j}^{d_j}(r(t)\left|\omega_j'(t)\right|^{\alpha+1})dt \\
&\geq M_j\Psi_{c_j}^{d_j}[\omega_j^{\alpha+1}],
\end{aligned} \tag{2.13}$$

and for $k(c_j) = k(d_j)$,

$$\int_{c_j}^{d_j} \left(W_j(t) \left(\frac{t - c_j}{t - c_j + \tau} \right)^{\alpha} - r(t) \left| \omega_j'(t) \right|^{\alpha+1} \right) dt \geq 0, \tag{2.14}$$

where, $W_j(t) = Q(t)\omega_j^{\alpha+1}$, $j = 1, 2.$, *and*

$$Q(t) = \left(p(t) + \eta_0^{-\eta_0} \prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) |e(t)|^{\eta_0} \right),$$

then equation (1.1) *is oscillatory.*

Proof. To arrive at a contradiction, let us suppose that $x(t)$ is a non-oscillatory solution of equation(1.1). Without loss of generality, we assume that $x(t) > 0$ and $x(t - \tau) > 0$ for $t \geq t_0$. In this case the interval of t selected for the following discussion is $[c_1, d_1]$. We define

$$u(t) = r(t) \frac{\phi_\alpha(x'(t))}{x^\alpha(t)}, \ t \in [c_1, d_1]. \tag{2.15}$$

It follows that for $t \neq t_k$,

$$u'(t) = -\left(p(t) \frac{x^\alpha(t - \tau)}{x^\alpha(t)} + \frac{\sum_{i=1}^{n} q_i(t) \phi_{\beta_i}(x(t - \tau))}{x^\alpha(t)} + \frac{|e(t)|}{x^\alpha(t)} \right) - \alpha u(t) \frac{x'(t)}{x(t)} \tag{2.16}$$

for all $t \neq t_k$, $t \geq t_0$, and $u(t_k^+) = \frac{b_k}{a_k} u(t_k)$ for all $k \in \mathbb{N}$.
From the assumptions, we can choose $c_1, d_1 \geq t_0$ such that $p(t) \geq 0$ and $q_i(t) \geq 0$ for $t \in [c_1 - \tau, d_1]$, $i = 1, 2, \ldots, n$, and $e(t) \leq 0$ for $t \in [c_1 - \tau, d_1]$. By Lemma 2.1, there exist $\eta_i > 0$, $i = 1, \ldots, n$, such that $\sum_{i=1}^{n} \beta_i \eta_i = \alpha$ and $\sum_{i=1}^{n} \eta_i < 1$.
Define $\eta_0 := 1 - \sum_{i=1}^{n} \eta_i$ and let

$$u_0 := \eta_0^{-1} \left| \frac{e(t) x(t - \tau)}{x^\alpha(t)} \right| x^{-1}(t - \tau),$$

$$u_i := \eta_i^{-1} q_i(t) \frac{x(t - \tau)}{x^\alpha(t)} x^{\beta_i - 1}(t - \tau), \quad i = 1, 2, \ldots, n.$$

Then by the arithmetic-geometric mean inequality (see Beckenbach and Bellman [3])

$$\sum_{i=0}^{n} \eta_i u_i \geq \prod_{i=0}^{n} u_i^{\eta_i}, \ u_i \geq 0, \text{ and } \eta_i > 0$$

we have

$$\begin{aligned} u'(t) \leq & - p(t) \frac{x^\alpha(t - \tau)}{x^\alpha(t)} - \eta_0^{-\eta_0} \prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) \frac{x^{\eta_i}(t - \tau)}{(x^{\eta_i}(t))^\alpha} x^{(\beta_i - 1)\eta_i}(t - \tau) |e(t)|^{\eta_0} \times \frac{x^{\eta_0}(t - \tau)}{(x^{\eta_0}(t))^\alpha} x^{-\eta_0}(t - \tau) \\ & - \frac{\alpha}{r^{1/\alpha}} u(t) \left(\frac{r(t)\phi_\alpha(x'(t))}{x^\alpha(t)} \right)^{1/\alpha} \frac{x'(t)}{\phi_\alpha(x'(t))^{1/\alpha}}, \ t \neq t_k. \end{aligned} \tag{2.17}$$

Since, by using Lemma(2.2), we have

$$\prod_{i=0}^{n} \frac{x^{\eta_i}(t - \tau)}{(x^{\eta_i}(t))^\alpha} = \frac{x^{\eta_0 + \eta_1 + \cdots + \eta_n}(t - \tau)}{(x^{\eta_0 + \eta_1 + \cdots + \eta_n}(t))^\alpha} = \frac{x(t - \tau)}{x^\alpha(t)}$$

and

$$\prod_{i=1}^{n} x^{(\beta_i - 1)\eta_i}(t - \tau) x^{-\eta_0}(t - \tau) = x^{\alpha - 1}(t - \tau),$$

the inequality (2.17) becomes

$$\begin{aligned} u'(t) \quad & \leq \quad - \left[p(t) + \eta_0^{-\eta_0} \prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) |e(t)|^{\eta_0} \right] \times \frac{x^\alpha(t - \tau)}{x^\alpha(t)} - \frac{\alpha}{r^{1/\alpha}(t)} u^{\frac{1+\alpha}{\alpha}}(t), \\ & = \quad -Q(t) \left(\frac{x(t - \tau)}{x(t)} \right)^\alpha - \frac{\alpha}{r^{1/\alpha}(t)} u^{1 + \alpha/\alpha}(t), \ t \neq t_k \end{aligned} \tag{2.18}$$

where

$$Q(t) = \left(p(t) + \eta_0^{-\eta_0} \prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) |e(t)|^{\eta_0} \right).$$

First we consider the case $k(c_1) < k(d_1)$. In this case the impulsive moments in $[c_1, d_1]$ are $t_{k(c_1)+1}, t_{k(c_1)+2}, \ldots, t_{k(d_1)}$. Choosing a $\omega_1(t) \in \Omega_1(c_1, d_1)$, multiplying both sides of (2.18) by $\omega_1^{\alpha+1}(t)$, and then integrating it from c_1 to d_1, we have

$$\begin{aligned}
&\sum_{i=k(c_1)+1}^{k(d_1)} \omega_1^{\alpha+1}(t_i)[u(t_i) - u(t_i^+)] \\
&\quad \le \int_{c_1}^{t_{k(c_1)+1}} \left[(\alpha+1) \left| \omega_1^{\alpha}(t)\omega_1'(t) \right| |u(t)| - \frac{\alpha}{r^{1/\alpha}(t)} |u(t)|^{(1+\alpha)/\alpha} \omega_1^{\alpha+1}(t) \right] dt \\
&\quad + \sum_{i=k(c_1)+1}^{k(d_1)-1} \int_{t_i}^{t_{i+1}} \left[(\alpha+1) \left| \omega_1^{\alpha}(t)\omega_1'(t) \right| |u(t)| - \frac{\alpha}{r^{1/\alpha}(t)} |u(t)|^{(1+\alpha)/\alpha} \omega_1^{\alpha+1}(t) \right] dt \\
&\quad + \int_{t_{k(d_1)}}^{d_1} \left[(\alpha+1) \left| \omega_1^{\alpha}(t)\omega_1'(t) \right| |u(t)| - \frac{\alpha}{r^{1/\alpha}(t)} |u(t)|^{(1+\alpha)/\alpha} \omega_1^{\alpha+1}(t) \right] dt \\
&\quad - \int_{c_1}^{t_{k(c_1)+1}} \left(\frac{x(t-\tau)}{x(t)} \right)^{\alpha} W_1(t) dt \\
&\quad - \sum_{i=k(c_1)+1}^{k(d_1)-1} \left[\int_{t_i}^{t_i+\tau} \left(\frac{x(t-\tau)}{x(t)} \right)^{\alpha} W_1(t) dt + \int_{t_i+\tau}^{t_{i+1}} \left(\frac{x(t-\tau)}{x(t)} \right)^{\alpha} W_1(t) dt \right] \\
&\quad - \int_{t_{k(d_1)}}^{d_1} \left(\frac{x(t-\tau)}{x(t)} \right)^{\alpha} W_1(t) dt.
\end{aligned} \tag{2.19}$$

where $W_1(t) = Q(t)\omega_1^{\alpha+1}$.
Letting

$$\lambda = 1 + \frac{1}{\alpha},\ X = \left(\frac{\alpha}{r^{1/\alpha}(t)} \right)^{\alpha/\alpha+1} |\omega_1^{\alpha}(t)|\, |u(t)| \text{ and } Y = [\alpha r(t)]^{\alpha/\alpha+1} \left| \omega_1'(t) \right|^{\alpha},$$

and then by using Lemma(2.3), we get

$$(\alpha+1) \left| \omega_1^{\alpha}(t)\omega_1'(t) \right| |u(t)| - \frac{\alpha}{r^{1/\alpha}(t)} |u(t)|^{(1+\alpha)/\alpha} \omega_1^{\alpha+1}(t) \le r(t) \left| \omega_1'(t) \right|^{\alpha+1}. \tag{2.20}$$

Meanwhile, for $t = t_k$, $k = 1, 2, \ldots$

$$u(t_k^+) = \left(\frac{b_k}{a_k} \right)^{\alpha} u(t_k). \tag{2.21}$$

Then the left hand side of the inequality(2.19) becomes

$$\sum_{i=k(c_1)+1}^{k(d_1)} \omega_1^{\alpha+1}(t_i)[u(t_i) - u(t_i^+)] = \sum_{i=k(c_1)+1}^{k(d_1)} \frac{a_i^{\alpha} - b_i^{\alpha}}{a_i^{\alpha}} \omega_1^{\alpha+1}(t_i) u(t_i). \tag{2.22}$$

Substituting (2.20) and (2.22) in (2.19), we get

$$\begin{aligned}
&\sum_{i=k(c_1)+1}^{k(d_1)} \frac{a_i^{\alpha} - b_i^{\alpha}}{a_i^{\alpha}} \omega_1^{\alpha+1}(t_i) u(t_i) \\
&\quad \le \int_{c_1}^{d_1} r(t) \left| \omega_1'(t) \right|^{\alpha+1} dt - \int_{c_1}^{t_{k(c_1)+1}} \left(\frac{x(t-\tau)}{x(t)} \right)^{\alpha} W_1(t) dt \\
&\quad - \sum_{i=k(c_1)+1}^{k(d_1)-1} \left[\int_{t_i}^{t_i+\tau} \left(\frac{x(t-\tau)}{x(t)} \right)^{\alpha} W_1(t) dt + \int_{t_i+\tau}^{t_{i+1}} \left(\frac{x(t-\tau)}{x(t)} \right)^{\alpha} W_1(t) dt \right] \\
&\quad - \int_{t_{k(d_1)}}^{d_1} \left(\frac{x(t-\tau)}{x(t)} \right)^{\alpha} W_1(t) dt.
\end{aligned} \tag{2.23}$$

On the other hand, for $t \in (t_{i-1}, t_i] \subset [c_1, d_1]$, $i = k(c_1)+2, \dots, k(d_1)$, we have

$$x(t) - x(t_{i-1}) = x'(\xi)(t - t_{i-1}),\ \xi \in (t_{i-1}, t).$$

In view of $x(t_{i-1}) > 0$ and the monotone properties of $\phi_\alpha(t)$, $r(t)\phi_\alpha(x'(t))$ and $r(t)$ we obtain

$$\phi_\alpha(x(t)) > \phi_\alpha x'(\xi)\phi_\alpha(t - t_{i-1}) \geq \frac{r(t)}{r(\xi)}\phi_\alpha x'(t)\phi_\alpha(t - t_{i-1})$$
$$\Longrightarrow \qquad \frac{r(t)\phi_\alpha(x'(t))}{\phi_\alpha(x(t))} < \frac{r(\xi)}{(t - t_{i-1})^\alpha}.$$

Let $t \to t_i^-$, it follows that

$$u(t_i) = \frac{r(t_i)\phi_\alpha(x'(t_i))}{\phi_\alpha(x(t_i))} < \frac{M_1}{(t_i - t_{i-1})^\alpha},\ i = k(c_1)+2, \dots, k(d_1). \tag{2.24}$$

Making a similar analysis on $(c_1, t_{k(c_1)+1}]$, we get

$$u(t_{k(c_1)+1}) = \frac{r(t_{k(c_1)+1})\phi_\alpha(x'(t_{k(c_1)+1}))}{\phi_\alpha(x(t_{k(c_1)+1}))} < \frac{M_1}{(t_{k(c_1)+1} - c_1)^\alpha}. \tag{2.25}$$

Then from (2.24), (2.25) and (A_3), we have

$$\begin{aligned}\sum_{i=k(c_1)+1}^{k(d_1)} \frac{b_i^\alpha - a_i^\alpha}{a_i^\alpha}\omega_1^{\alpha+1}(t_i)u(t_i) &< M_1\left[\omega_1^{\alpha+1}(t_{k(c_1)+1})\theta(c_1) + \sum_{i=k(c_1)+2}^{k(d_1)}\omega_1^{\alpha+1}(t_i)\varepsilon(t_i)\right] \\ &= M_1\Psi_{c_1}^{d_1}\left[\omega_1^{\alpha+1}\right].\end{aligned} \tag{2.26}$$

Hence, from (2.23) and (2.26) and applying Lemma (2.4), we obtain

$$\begin{aligned}&\int_{c_1}^{t_{k(c_1)+1}} W_1(t)\left(\frac{t - t_{k(c_1)} - \tau}{t - t_{k(c_1)}}\right)^\alpha dt \\ &+ \sum_{i=k(c_1)+1}^{k(d_1)-1}\left[\int_{t_i}^{t_i+\tau} W_1(t)\left(\frac{t - t_i}{b_i(t + \tau - t_i)}\right)^\alpha dt + \int_{t_i+\tau}^{\tau_{i+1}} W_1(t)\left(\frac{t - t_i - \tau}{t - t_i}dt\right)^\alpha\right] \\ &+ \int_{t_{k(d_1)}}^{d_1} W_1(t)\left(\frac{t - t_{k(d_1)}}{b_{k(d_1)}(t + \tau - t_{k(d_1)})}\right)^\alpha dt - \int_{c_1}^{d_1} r(t)\left|\omega_1'(t)\right|^{\alpha+1} dt \\ &< M_1\Psi_{c_1}^{d_1}\left[\omega_1^{\alpha+1}\right].\end{aligned} \tag{2.27}$$

This contradicts (2.13).

Next we consider the case $k(c_1) = k(d_1)$. By the condition $(\tilde{s_1})$ we know there is no impulse moments in $[c_1, d_1]$. Multipling both sides of (2.18) by $\omega_1^{\alpha+1}(t)$, with ω as prescribed in the hypothesis of the theorem, and then integrating it from c_1 to d_1, we obtain

$$\int_{c_1}^{d_1} u'(t)\omega_1^{\alpha+1}dt \leq -\int_{c_1}^{d_1}\frac{\alpha}{r^{1/\alpha}(t)}|u(t)|^{(\alpha+1)/\alpha}\omega_1^{\alpha+1}(t)dt - \int_{c_1}^{d_1}\left(\frac{x(t-\tau)}{x(t)}\right)^\alpha W_1(t)dt. \tag{2.28}$$

Using integration by parts on the left hand side and noting the condition $\omega_1(c_1) = \omega_1(d_1) = 0$, we obtain

$$\int_{c_1}^{d_1}\left[(\alpha+1)\omega_1^\alpha\omega_1'(t)u(t) - \frac{\alpha}{r^{1/\alpha}(t)}|u(t)|^{(\alpha+1)/\alpha}\omega_1^{\alpha+1}(t)\right]dt - \int_{c_1}^{d_1}\left(\frac{x(t-\tau)}{x(t)}\right)^\alpha W_1(t)dt \geq 0. \tag{2.29}$$

It follows that

$$\int_{c_1}^{d_1}\left[(\alpha+1)\left|\omega_1^\alpha\omega_1'(t)\right||u(t)| - \frac{\alpha}{r^{1/\alpha}(t)}\omega_1^{\alpha+1}(t)|u(t)|^{(\alpha+1)/\alpha}\right]dt - \int_{c_1}^{d_1}\left(\frac{x(t-\tau)}{x(t)}\right)^\alpha W_1(t)dt \geq 0. \tag{2.30}$$

Letting

$$\lambda = 1 + \frac{1}{\alpha},\ X = \left(\frac{\alpha}{r^{1/\alpha}(t)}\right)^{\alpha/\alpha+1}|\omega_1^\alpha(t)||u(t)| \text{ and } Y = [\alpha r(t)]^{\alpha/\alpha+1}|\omega_1'(t)|^\alpha$$

and applying the Lemma(2.3), we get

$$\int_{c_1}^{d_1} \left[r(t) \left|\omega_1'(t)\right|^{\alpha+1} - \left(\frac{x(t-\tau)}{x(t)} \right)^{\alpha} W_1(t) \right] dt \geq 0. \tag{2.31}$$

Now to estimate $\frac{x(t-\tau)}{x(t)}$ on $[c_1, d_1]$.
If $t \in [c_1, d_1]$ then $t - \tau \in [c_1 - \tau, d_1 - \tau]$ and then there is no impulsive moment in $(t - \tau, t)$. For any $t \in (t - \tau, t)$, we have

$$x(t) - x(c_1 - \tau) = x'(\xi)(t - c_1 + \tau),\ \xi \in (c_1 - \tau, t).$$

By using the monotone properties of $r(t), \phi_\alpha(*)$ and $r(t)\Phi_\alpha(x'(t))$, we get

$$\begin{aligned} \phi_\alpha(x(t)) &> \phi_\alpha(x'(\xi))(t - c_1 + \tau) = \frac{r(\xi)}{r(\xi)} \phi_\alpha(x'(\xi))(t - c_1 + \tau)^\alpha \\ &\geq \frac{r(t)\Phi_\alpha(x'(t))}{r(t)}(t - c_1 + \tau)^\alpha = \phi_\alpha(x'(t))(t - c_1 + \tau). \end{aligned}$$

Therefore,

$$\frac{x'(t)}{x(t)} < \frac{1}{(t - c_1 + \tau)}.$$

Integrating both sides of the above inequality from $t - \tau$ to t, we obtain

$$\frac{x(t-\tau)}{x(t)} > \left(\frac{t - c_1}{t - c_1 + \tau} \right),\ t \in [c_1, d_1]. \tag{2.32}$$

From (2.31) and (2.32) we obtain

$$\int_{c_1}^{d_1} \left[W_1(t) \left(\frac{t - c_1}{t - c_1 + \tau} \right)^{\alpha} - r(t) \left|\omega_1'(t)\right|^{\alpha+1} \right] dt < 0. \tag{2.33}$$

This again contradicts our assumption.

When $x(t)$ is eventually negative, we can consider the interval $[c_2, d_2]$ and reach a similar contradiction.Thus the proof is complete. □

Following Kong [9] and Philos [16], we introduce a class of functions:
Let $D = \{(t, s) : t_0 \leq s \leq t\}$, $H_1, H_2 \in C^1(D, \mathbb{R})$. A pair of functions (H_1, H_2) is said to belong to a function class $\mathcal{H}$, if $H_1(t, t) = H_2(t, t) = 0$, $H_1(t, s) > 0$, $H_2(t, s) > 0$ for $t > s$ and there exist $h_1, h_2 \in L_{\text{loc}}(D, \mathbb{R})$ such that

$$\frac{\partial H_1(t, s)}{\partial t} = h_1(t, s) H_1(t, s), \quad \frac{\partial H_2(t, s)}{\partial s} = -h_2(t, s) H_2(t, s).$$

We assume there exists c_j, d_j, $\delta_j \notin \{t_k\}$, $k = 1, 2, \ldots,$ $(j = 1, 2)$ which satisfy $T < c_1 < \delta_1 < d_1 \leq c_2 < \delta_2 < d_2$ for any $T \geq t_0$. Noticing whether or not there are impulse moments of $x(t)$ in $[c_j, \delta_j]$ and $[\delta_j, d_j]$, we should consider the following four cases,

$$\begin{aligned} &(S_1)\ k(c_j) < k(\delta_j) < k(d_j); \quad (S_2)\ k(c_j) = k(\delta_j) < k(d_j); \\ &(S_3)\ k(c_j) < k(\delta_j) = k(d_j); \quad (S_4)\ k(c_j) = k(\delta_j) = k(d_j),\ j = 1, 2. \end{aligned}$$

Moreover in the discussion of impulse moments of $x(t - \tau)$, it is necessary to consider the following two cases,

$$(\bar{S}_1)\ t_{k(\delta_j)} + \tau > \delta_j; \quad (\bar{S}_2)\ t_{k(\delta_j)} + \tau \leq \delta_j,\ j = 1, 2.$$

In the following theorem, we only consider the case of combination of (S_1) with $(\bar{S_1})$. For the other cases, similar conclusions can be given and hence their proof is omitted.

For our convenience, we define

$$\begin{aligned}\Pi_{1,j} =: \frac{1}{H_1(\delta_j, c_j)} \Bigg\{ & \int_{c_j}^{t_{k(c_j)+1}} \tilde{H_1}(t, c_j) \left(\frac{t - t_{k(c_j)} - \tau}{t - t_{k(c_j)}} \right)^{\alpha} dt \\ & + \sum_{i=k(c_j)+1}^{k(\delta_1)-1} \left[\int_{t_i}^{t_i+\tau} \tilde{H_1}(t, c_j) \left(\frac{t - t_i}{b_i(t + \tau - t_i)} \right)^{\alpha} dt + \int_{t_i+\tau}^{t_{i+1}} \tilde{H_1}(t, c_j) \left(\frac{t - t_i - \tau}{t - t_i} \right)^{\alpha} dt \right] \\ & + \int_{t_{k(\delta_j)}}^{\delta_j} \tilde{H_1}(t, c_j) \left(\frac{t - t_{k(\delta_j)}}{b_{k(\delta_j)}(t + \tau - t_{k(\delta_j)})} \right)^{\alpha} dt \\ & - \frac{1}{(\alpha+1)^{\alpha+1}} \int_{c_j}^{\delta_j} r(t) H_1(t, c_j) \left| h_1(t, c_j) \right|^{\alpha+1} dt \Bigg\}\end{aligned} \tag{2.34}$$

and

$$\begin{aligned}\Pi_{2,j} =: \frac{1}{H_2(d_j, \delta_j)} \Bigg\{ & \int_{\delta_j}^{t_{k(\delta_j)}+\tau} \tilde{H_2}(d_j, t) \left(\frac{t - t_{k(\delta_j)}}{b_{k(\delta_j)}(t + \tau - t_{k(\delta_j)})} \right)^{\alpha} dt + \int_{t_{k(\delta_j)}+\tau}^{t_{k(\delta_j)+1}} \tilde{H_2}(d_j, t) \left(\frac{t - t_{k(\delta_j)} - \tau}{t - t_{k(\delta_j)}} \right)^{\alpha} dt \\ & + \sum_{i=k(\delta_j)+1}^{k(d_j)-1} \left[\int_{t_i}^{t_i+\tau} \tilde{H_2}(d_j, t) \left(\frac{t - t_i}{b_i(t + \tau - t_i)} \right)^{\alpha} dt + \int_{t_i+\tau}^{t_{i+1}} \tilde{H_2}(d_j, t) \left(\frac{t - t_i - \tau}{t - t_i} \right)^{\alpha} dt \right] \\ & + \int_{t_{k(d_j)}}^{d_j} \tilde{H_2}(d_j, t) \left(\frac{t - t_{k(d_j)}}{b_{k(d_j)}(t + \tau - t_{k(d_j)})} \right)^{\alpha} dt \\ & - \frac{1}{(\alpha+1)^{\alpha+1}} \int_{\delta_j}^{d_j} r(t) H_2(d_j, t) \left| h_2(d_j, t) \right|^{\alpha+1} dt \Bigg\},\end{aligned} \tag{2.35}$$

where $\tilde{H_1}(t, c_j) = H_1(t, c_j) Q(t)$, $\quad \tilde{H_2}(d_j, t) = H_2(d_j, t) Q(t)$, $(j = 1, 2)$ and

$$Q(t) = \left(p(t) + \eta_0^{-\eta_0} \prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) |e(t)|^{\eta_0} \right).$$

Theorem 2.2. *Assume that for any $T \geq t_0$, there exist c_j, d_j, $\delta_j \notin \{t_k\}$, $j = 1, 2$ such that $c_1 < \delta_1 < d_1 \leq c_2 < \delta_2 < d_2$, and* (2.5) *holds. If there exists $(H_1, H_2) \in \mathcal{H}$ such that*

$$\Pi_{1,j} + \Pi_{2,j} > \frac{M_j}{H_1(\delta_j, c_j)} \Psi_{c_j}^{\delta_j}[H_1(., c_j)] + \frac{M_j}{H_2(d_j, \delta_j)} \Psi_{\delta_j}^{d_j}[H_2(d_j, .)],\ j = 1, 2, \tag{2.36}$$

then equation(1.1) *is oscillatory.*

Proof. To arrive at a contradiction, let us suppose that $x(t)$ is a non-oscillatory solution of equation(1.1). Without loss of generality, we assume that $x(t) > 0$ and $x(t - \tau) > 0$ for $t \geq t_0$. In this case the interval of t selected for the following discussion is $[c_1, d_1]$. Continuing as in Theorem(2.5), we can get (2.18). Multiplying both sides of (2.18) by $H_1(t, c_1)$ and integrating it from c_1 to δ_1, we have

$$\begin{aligned}\int_{c_1}^{\delta_1} H_1(t, c_1) u'(t) dt \leq & - \int_{c_1}^{\delta_1} H_1(t, c_1) \frac{\alpha}{r^{1/\alpha}(t)} \left| u(t) \right|^{(1+\alpha)/\alpha} dt \\ & - \int_{c_1}^{\delta_1} \tilde{H_1}(t, c_1) \left(\frac{x(t - \tau)}{x(t)} \right)^{\alpha} dt\end{aligned} \tag{2.37}$$

Since the impulsive moments $t_{k(c_1)+1}, t_{k(c_1)+2}, \ldots, t_{k(\delta_1)}$ are in $[c_1, \delta_1]$, using the integration by parts on the left-hand side of the above inequality, we obtain

$$\begin{aligned}\int_{c_1}^{\delta_1} H_1(t,c_1)u'(t)dt &= \left(\int_{c_1}^{t_{k(c_1)+1}} + \int_{t_{k(c_1)+1}}^{t_{k(c_1)+2}} +\ldots+ \int_{t_{k(\delta_1)}}^{\delta_1}\right) H_1(t,c_1)du(t)\\ &= \sum_{i=k(c_1)+1}^{k(\delta_1)} [u(t_i)-u(t_i^+)]H_1(t_i,c_1) + u(\delta_1)H(\delta_1,c_1)\\ &\quad - \left(\int_{c_1}^{t_{k(c_1)+1}} + \int_{t_{k(c_1)+1}}^{t_{k(c_1)+2}} +\ldots+ \int_{t_{k(\delta_1)}}^{\delta_1}\right) u(t)h_1(t,c_1)H_1(t,c_1)dt\\ &= \sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{a_i^\alpha - b_i^\alpha}{a_i^\alpha} H_1(t_i,c_1)u(t_i) + H_1(\delta_1,c_1)u(\delta_1)\\ &\quad - \left(\int_{c_1}^{t_{k(c_1)+1}} + \int_{t_{k(c_1)+1}}^{t_{k(c_1)+2}} \ldots+ \int_{t_{k(\delta_1)}}^{\delta_1}\right) u(t)h_1(t,c_1)H_1(t,c_1)dt.\end{aligned} \tag{2.38}$$

Substituting (2.38) into (2.37), we have

$$\begin{aligned}\int_{c_1}^{\delta_1} \tilde{H}_1(t,c_1)\left(\frac{x(t-\tau)}{x(t)}\right)^\alpha dt \le& \sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{b_i^\alpha - a_i^\alpha}{a_i^\alpha} H_1(t_i,c_1)u(t_i) - H_1(\delta_1,c_1)u(\delta_1)\\ &+ \int_{c_1}^{\delta_1} H_1(t,c_1)\left[|h_1(t,c_1)|\,|u(t)| - \frac{\alpha}{r^{1/\alpha}(t)}\,|u(t)|^{(1+\alpha)/\alpha}\right]dt.\end{aligned} \tag{2.39}$$

Letting

$$\lambda = 1+\frac{1}{\alpha},\ X = \frac{\alpha^{\alpha/\alpha+1}\,|u(t)|}{[r(t)]^{1/\alpha+1}} \text{ and } Y = \left[\alpha(\alpha+1)^{-(\alpha+1)}r(t)\right]^{\alpha/\alpha+1} |h_1(t,c_1)|^\alpha,$$

and then by using Lemma(2.3), the above inequality becomes

$$\begin{aligned}\int_{c_1}^{\delta_1} \tilde{H}_1(t,c_1)\left(\frac{x(t-\tau)}{x(t)}\right)^\alpha dt \le& \sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{b_i^\alpha - a_i^\alpha}{a_i^\alpha} H_1(t_i,c_1)u(t_i) - H_1(\delta_1,c_1)u(\delta_1)\\ &+ \frac{1}{(1+\alpha)^{1+\alpha}}\int_{c_1}^{\delta_1} r(t)H_1(t,c_1)\,|h_1(t,c_1)|^{\alpha+1}dt.\end{aligned} \tag{2.40}$$

To estimate $\frac{x(t-\tau)}{x(t)}$, we have to divide the interval $[c_1, \delta_1]$ into several sub intervals and by using Lemma(2.4), we get estimation for the left hand side of the above inequality as follows,

$$\begin{aligned}&\int_{c_1}^{\delta_1} \tilde{H}_1(t,c_1)\left(\frac{x(t-\tau)}{x(t)}\right)^\alpha dt\\ &> \int_{c_1}^{t_{k(c_1)+1}} \tilde{H}_1(t,c_1)\left(\frac{t-t_{k(c_1)}-\tau}{t-t_{k(c_1)}}\right)^\alpha dt\\ &\quad + \sum_{i=k(c_1)+1}^{k(\delta_1)-1}\left[\int_{t_i}^{t_i+\tau} \tilde{H}_1(t,c_1)\left(\frac{t-t_i}{b_i(t+\tau-t_i)}\right)^\alpha dt + \int_{t_i+\tau}^{t_{i+1}} \tilde{H}_1(t,c_1)\left(\frac{t-t_i-\tau}{t-t_i}\right)^\alpha dt\right]\\ &\quad + \int_{t_{k(\delta_1)}}^{\delta_1} \tilde{H}_1(t,c_1)\left(\frac{t-t_{k(\delta_1)}}{b_{k(\delta_1)}(t+\tau-t_{k(\delta_1)})}\right)^\alpha dt.\end{aligned} \tag{2.41}$$

From (2.40) and (2.41), we have

$$\begin{aligned}
&\int_{c_1}^{t_{k(c_1)+1}} \tilde{H}_1(t,c_1)\left(\frac{t-t_{k(c_1)}-\tau}{t-t_{k(c_1)}}\right)^{\alpha} dt \\
&+\sum_{i=k(c_1)+1}^{k(\delta_1)-1}\left[\int_{t_i}^{t_i+\tau} \tilde{H}_1(t,c_1)\left(\frac{t-t_i}{b_i(t+\tau-t_i)}\right)^{\alpha} dt+\int_{t_i+\tau}^{t_{i+1}} \tilde{H}_1(t,c_1)\left(\frac{t-t_i-\tau}{t-t_i}\right)^{\alpha} dt\right] \\
&+\int_{t_{k(\delta_1)}}^{\delta_1} \tilde{H}_1(t,c_1)\left(\frac{t-t_{k(\delta_1)}}{b_{k(\delta_1)}(t+\tau-t_{k(\delta_1)})}\right)^{\alpha} dt-\frac{1}{(1+\alpha)^{1+\alpha}}\int_{c_1}^{\delta_1} r(t)H_1(t,c_1)\left|h_1(t,c_1)\right|^{\alpha+1} dt \\
&<\sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{b_i^{\alpha}-a_i^{\alpha}}{a_i^{\alpha}} H_1(t_i,c_1)u(t_i)-H_1(\delta_1,c_1)u(\delta_1).
\end{aligned} \tag{2.42}$$

Multiplying both sides of (2.18) by $H_2(d_1,t)$ and using similar analysis as above, we can obtain

$$\begin{aligned}
&\int_{\delta_1}^{t_{k(\delta_1)}+\tau} \tilde{H}_2(d_1,t)\left(\frac{t-t_{k(\delta_1)}}{b_{k(\delta_1)}(t+\tau-t_{k(\delta_1)})}\right)^{\alpha} dt+\int_{t_{k(\delta_1)}+\tau}^{t_{k(\delta_1)+1}} \tilde{H}_2(d_1,t)\left(\frac{t-t_{k(\delta_1)}-\tau}{t-t_{k(\delta_1)}}\right)^{\alpha} dt \\
&+\sum_{i=k(\delta_1)+1}^{k(d_1)-1}\left[\int_{t_i}^{t_i+\tau} \tilde{H}_2(d_1,t)\left(\frac{t-t_i}{b_i(t+\tau-t_i)}\right)^{\alpha} dt+\int_{t_i+\tau}^{t_{i+1}} \tilde{H}_2(d_1,t)\left(\frac{t-t_i-\tau}{t-t_i}\right)^{\alpha} dt\right] \\
&+\int_{t_{k(d_1)}}^{d_1} \tilde{H}_2(d_1,t)\left(\frac{t-t_{k(d_1)}}{b_{k(d_1)}(t+\tau-t_{k(d_1)})}\right)^{\alpha} dt-\frac{1}{(\alpha+1)^{\alpha+1}}\int_{\delta_1}^{d_1} r(t)H_2(d_1,t)\left|h_2(d_1,t)\right|^{\alpha+1} dt \\
&<\sum_{i=k(\delta_1)+1}^{k(d_1)} \frac{b_i^{\alpha}-a_i^{\alpha}}{a_i^{\alpha}} H_2(d_1,t_i)u(t_i)+H_2(d_1,\delta_1)u(\delta_1).
\end{aligned} \tag{2.43}$$

Dividing (2.42) and (2.43) by $H_1(\delta_1,c_1)$ and $H_2(d_1,\delta_1)$ respectively, and adding them, we get

$$\begin{aligned}
\Pi_{1,1}+\Pi_{2,1} <\ & \frac{1}{H_1(\delta_1,c_1)}\sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{b_i^{\alpha}-a_i^{\alpha}}{a_i^{\alpha}} H_1(t_i,c_1)u(t_i) \\
&+\frac{1}{H_2(d_1,\delta_1)}\sum_{i=k(\delta_1)+1}^{k(d_1)} \frac{b_i^{\alpha}-a_i^{\alpha}}{a_i^{\alpha}} H_2(d_1,t_i)u(t_i).
\end{aligned} \tag{2.44}$$

On the other hand, similar to (2.26), we have

$$\sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{b_i^{\alpha}-a_i^{\alpha}}{a_i^{\alpha}} H_1(t_i,c_1)u(t_i) \leq M_1\Psi_{c_1}^{\delta_1}[H_1(.,c_1)] \tag{2.45}$$

and

$$\sum_{i=k(\delta_1)+1}^{k(d_1)} \frac{b_i^{\alpha}-a_i^{\alpha}}{a_i^{\alpha}} H_2(d_1,t_i)u(t_i) \leq M_1\Psi_{\delta_1}^{d_1}[H_2(d_1,.)]. \tag{2.46}$$

Substituting (2.45)and (2.46) in (2.44), we obtain a contradiction to the condition (2.36).

When $x(t)$ is eventually negative, we can consider $[c_2,d_2]$ and reach a similar contradiction. Hence the proof is complete. □

Remark 2.1. *When $\alpha=1$, our results reduces to Theorem(2.2) and Theorem(2.4) of [11].*

Remark 2.2. *When $\tau=0$ and $\alpha=1$, Theorem(2.5) reduces to Theorem(2.1) of [13].*

Remark 2.3. *When $a_k=b_k=1$ for all $k=1,2,3,...$, $\tau=0$ and $\alpha=1$, our results reduces to Theorem(1) of [17] for the case $\rho(t)=1$.*

3 Examples

In this section we give two examples to illustrate our main results.

Example 3.1. Consider the impulsive differential equation

$$\begin{aligned}
&\left(\Phi_\alpha(x'(t))\right)' + \gamma_0 \sin t \Phi_\alpha\left(x(t-\frac{\pi}{12})\right) + \gamma_1 e^{-t/2}\Phi_{\beta_1}\left(x(t-\frac{\pi}{12})\right)\\
&\quad + \gamma_2 \cos^2 t \Phi_{\beta_2}\left(x(t-\frac{\pi}{12})\right) = \sin 2t,\ t \geq t_0,\ t \neq t_{k,i},\\
&x(t_{k,i}^+) = a_k x(t_{k,i}), \quad x'(t_{k,i}^+) = b_k x'(t_{k,i}),\\
&\text{where} \quad t_{k,i} = 2k\pi + \frac{3\pi}{8} + (-1)^{i-2}\left(\frac{\pi}{4}\right), \quad i = 1,2 \text{ and } k = 1,2,...
\end{aligned} \tag{3.47}$$

Here,

$$r(t) = 1, p(t) = \gamma_0 \sin t, q_1(t) = \gamma_1 e^{-t/2}, q_2(t) = \gamma_2 \cos^2 t \text{ and } e(t) = \sin 2t,\ t \geq t_0 > 0,$$

where γ_0, γ_1 and γ_2 are positive constants. If we choose $\eta_0 = 1/2$, $\beta_1 = 19/2$, $\beta_2 = 5/2$ and $\alpha = 3$, then by Lemma (2.1), we can easily find $\eta_1 = \eta_2 = 1/4$. For any $T > 0$, we can choose n large enough such that $T < c_1 = 2n\pi + \frac{\pi}{12} < d_1 = 2n\pi + \frac{\pi}{6}$ and $c_2 = 2n\pi + \frac{\pi}{4} < d_2 = 2n\pi + \frac{2\pi}{3}$, then there are impulsive moments $t_{n,1} = 2n\pi + \frac{\pi}{8}$ in $[c_1, d_1]$ and $t_{n,2} = 2n\pi + \frac{5\pi}{8}$ in $[c_2, d_2]$.
Let

$$\omega_j(t) = \sin 12t \in \Omega_j(c_j, d_j),\ j = 1,2.$$

Then we have,

$$Q(t) = \gamma_0 \sin t + (1/2)^{-1/2}(1/4)^{-1/4}(1/4)^{-1/4}\gamma_1^{1/4}(e^{-t/2})^{1/4}\gamma_2^{1/4}(\cos t)^{1/2}|\sin 2t|^{1/2},$$

and

$$W_j(t) = Q(t)\omega_j^{\alpha+1}(t),\ j = 1,2.$$

In view of $\sum_{i=k(c_j)+1}^{k(d_j)-1} = 0$ as $k(c_j)+1 > k(d_j)-1$, $j = 1,2$, the left hand side of (2.13) is the following

$$\begin{aligned}
&\int_{c_1}^{t_{k(c_1)+1}} W_1(t)\left(\frac{t - t_{k(c_1)} - \tau}{t - t_{k(c_1)}}\right)^\alpha dt\\
&+ \sum_{i=k(c_1)+1}^{k(d_1)-1}\left[\int_{t_i}^{t_i+\tau} W_1(t)\left(\frac{t - t_i}{b_i(t+\tau-t_i)}\right)^\alpha dt + \int_{t_i+\tau}^{t_{i+1}} W_1(t)\left(\frac{t-t_i-\tau}{t-t_i}\right)^\alpha\right]\\
&+ \int_{t_{k(d_1)}}^{d_1} W_1(t)\left(\frac{t - t_{k(d_1)}}{b_{k(d_1)}(t+\tau-t_{k(d_1)})}\right)^\alpha dt - \int_{c_1}^{d_1}(r(t)\left|\omega_1'(t)\right|^{\alpha+1})dt\\
&= \int_{2n\pi+\pi/12}^{2n\pi+\pi/8} W_1(t)\left(\frac{t-(2(n-1)\pi+5\pi/8)-\pi/12}{t-(2(n-1)\pi+5\pi/8))}\right)^3 dt\\
&+ \int_{2n\pi+\pi/8}^{2n\pi+\pi/6} W_1(t)\left(\frac{t-(2n\pi+\pi/8)}{b_{n,1}(t+\pi/12-(2n\pi+\pi/8))}\right)^3 dt - 12^4\int_{2n\pi+\pi/12}^{2n\pi+\pi/6}\left(\cos^4 12t\right)dt\\
&= \int_{\pi/12}^{\pi/8} W_1(t)\left(\frac{t+31\pi/24}{t+11\pi/8)}\right)^3 dt + \int_{\pi/8}^{\pi/6} W_1(t)\left(\frac{t-\pi/8}{b_{n,1}(t-\pi/24)}\right)^3 dt - 12^4\int_{\pi/12}^{\pi/6}\left(\cos^4 12t\right)dt.\\
&\approx \left[0.01464\gamma_0 + 0.0878\gamma_1^{1/4}\gamma_2^{1/4}\right] + b_{n,1}^{-3}\left[0.00004889\gamma_0 + 0.0002811\gamma_1^{1/4}\gamma_2^{1/4}\right] - 648\pi.
\end{aligned} \tag{3.48}$$

On the other hand , the right hand side of (2.13)

$$\begin{aligned}
\Psi_{c_1}^{d_1}[\omega_1^{\alpha+1}] &= \omega_1^{\alpha+1}(t_{k(c_1)+1})\frac{b_{k(c_1)+1}^\alpha - a_{k(c_1)+1}^\alpha}{(a_{k(c_1)+1}^\alpha(t_{k(c_1)+1} - c_1)^\alpha)} + \sum_{i=k(c_1)+2}^{k(d_1)}\omega_1^{\alpha+1}(t_i)\frac{b_i^\alpha - a_i^\alpha}{(a_i^\alpha(t_i - t_{i-1})^\alpha)}.\\
&= \sin^4 12(2n\pi+\pi/8)\left(\frac{b_{n,1}-a_{n,1}}{a_{n,1}(2n\pi+\pi/8-(2n\pi+\pi/12))}\right)^3\\
&= \left(\frac{24}{\pi}\right)^3\left(\frac{b_{n,1}-a_{n,1}}{a_{n,1}}\right)^3.
\end{aligned} \tag{3.49}$$

Thus for $t \in [c_1, d_1]$, if we choose γ_0, γ_1 and γ_2 large enough so that

$$\begin{aligned} &0.01464\gamma_0 + 0.0878\gamma_1^{1/4}\gamma_2^{1/4} + b_{n,1}^{-3}\left(0.00004889\gamma_0 + 0.0002811\gamma_1^{1/4}\gamma_2^{1/4}\right) - 648\pi \\ &\geq \left(\frac{24}{\pi}\right)^3 \left(\frac{b_{n,1} - a_{n,1}}{a_{n,1}}\right)^3, \end{aligned} \tag{3.50}$$

then (2.13) will be satisfied.

Similarly for $t \in [c_2, d_2]$, we can get the following condition

$$\begin{aligned} &0.153651\gamma_0 + 0.02648\gamma_1^{1/4}\gamma_2^{1/4} + b_{n,2}^{-3}\left(0.00010044\gamma_0 - 0.000143\gamma_1^{1/4}\gamma_2^{1/4}\right) - 3240\pi \\ &\geq \left(\frac{8}{3\pi}\right)^3 \left(\frac{b_{n,2} - a_{n,2}}{a_{n,2}}\right)^3. \end{aligned} \tag{3.51}$$

Hence by Theorem (2.1) for suitable γ_0, γ_1 and γ_2, equation(3.47) becomes oscillatory.

Example 3.2. Consider the impulsive differential equation

$$\begin{aligned} &(\Phi_\alpha(x'(t)))' + \kappa_0 p(t)\Phi_\alpha\left(x(t - \frac{\pi}{12})\right) + \kappa_1 q_1(t)\Phi_{\beta_1}\left(x(t - \frac{\pi}{12})\right) \\ &\quad + \kappa_2 q_2(t)\Phi_{\beta_2}\left(x(t - \frac{\pi}{12})\right) = e(t),\ t \geq t_0,\ t \neq t_{k,i}, \\ &x(t_{k,i}^+) = a_k x(t_{k,i}), \quad x'(t_{k,i}^+) = b_k x'(t_{k,i}) \end{aligned} \tag{3.52}$$

where κ_0, κ_1, and κ_2 are positive constants, and

$$t_{n,1} = 2n\pi + \pi/8,\ t_{n,2} = 2n\pi + 3\pi/8,\ t_{n,3} = 2n\pi + 13\pi/8 \text{ and } t_{n,4} = 2n\pi + 17\pi/8.$$

In addition let, $q_1(t) = e^{t/2}$, $q_2(t) = e^{t/4}$,

$$p(t) = \begin{cases} e^4 t, & t \in [2n\pi + \pi/12, 2n\pi + \pi/2], \\ \sin^2 t, & t \in [2n\pi + 3\pi/2, 2n\pi + 5\pi/2] \end{cases}$$

and

$$e(t) = \begin{cases} -\sin 2t, & t \in [2n\pi + \pi/12, 2n\pi + \pi/2], \\ \cos^2 t, & t \in [2n\pi + 3\pi/2, 2n\pi + 5\pi/2]. \end{cases}$$

For any $t_0 > 0$, we choose n large enough such that $t_0 < 2n\pi + \pi/12$ and let $[c_1, d_1] = [2n\pi + \pi/12, 2n\pi + \pi/2]$, $[c_2, d_2] = [2n\pi + 3\pi/2, 2n\pi + 5\pi/2]$, $\delta_1 = 2n\pi + \pi/6$, $\delta_2 = 2n\pi + 5\pi/3$. Then $p(t)$, $q(t)$ and $e(t)$ satisfy (2.5) on $[c_1, d_1]$ and $[c_2, d_2]$. Let $H_1(t,s) = H_2(t,s) = (t-s)^3$ then $h_1(t,s) = -h_2(t,s) = 3/(t-s)$. Now choose $\eta_0 = 1/2$, $\beta_1 = 5/2$, $\beta_2 = 1/2$, and $\alpha = 1$.
Then one can easily find $\eta_1 = 3/8$, $\eta_2 = 1/8$.

$$Q(t) = p(t) + (1/2)^{-1/2}(3/8)^{-3/8}(1/8)^{-1/8} q_1^{3/8}(t) q_2^{1/8}(t) |e(t)|^{1/2}.$$

Also by a simple calculation, we get

$$\begin{aligned} \Pi_{1,1} = &\frac{1}{H_1(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{\pi}{12})} \\ &\left\{ \int_{2n\pi+\pi/12}^{2n\pi+\pi/8} H_1(t, 2n\pi + \pi/12) Q(t) \left(\frac{t - (2(n-1)\pi + 3\pi/8) - \pi/12}{t - (2(n-1)\pi + 3\pi/8)}\right) dt \right. \\ &\quad + \int_{2n\pi+\pi/8}^{2n\pi+\pi/6} H_1(t, 2n\pi + \pi/12) Q(t) \left(\frac{t - (2n\pi + \pi/8)}{b_{n,1}(t + \pi/12 - (2n\pi + \pi/8))}\right) dt \\ &\quad \left. - \frac{1}{2^2} \int_{2n\pi+\pi/12}^{2n\pi+\pi/6} H_1(t, 2n\pi + \pi/12) \, |h_1(t, 2n\pi + \pi/12)|^2 \, dt \right\} \\ &\approx \kappa_0 \left(0.0169 + \frac{0.1042}{b_{n,1}}\right) + {\kappa_1}^{3/8}{\kappa_2}^{1/8}\left((0.0101) + \frac{0.0411}{b_{n,1}}\right) - 4.2971 \end{aligned} \tag{3.53}$$

and

$$\begin{aligned}\Pi_{2,1} =&\frac{1}{H_2(2n\pi+\frac{\pi}{2},2n\pi+\frac{\pi}{6})}\\ &\left\{\int_{2n\pi+\pi/6}^{2n\pi+\pi/8+\pi/12}\tilde{H}_2(2n\pi+\pi/2,t)\left(\frac{t-(2n\pi+\pi/8)}{b_{n,1}(t+\pi/12-(2n\pi+\pi/8))}\right)dt\right.\\ &+\int_{2n\pi+\pi/8+\pi/12}^{2n\pi+3\pi/8}\tilde{H}_2(2n\pi+\pi/2,t)\left(\frac{t-(2n\pi+\pi/8)-\pi/12}{t-(2n\pi+\pi/8)}\right)dt\\ &+\int_{2n\pi+3\pi/8}^{2n\pi+\pi/2}\tilde{H}_2(2n\pi+\pi/2,t)\left(\frac{t-(2n\pi+3\pi/8)}{b_{n,2}(t+\pi/12-(2n\pi+3\pi/8))}\right)dt\\ &\left.-\frac{1}{(2)^2}\int_{2n\pi+\pi/6}^{2n\pi+\pi/2}H_2(2n\pi+\pi/2,t)\,|h_2(2n\pi+\pi/2,t)|^2\,dt\right\}.\\ \approx&\ \kappa_0\left(2.0198+\frac{0.4843}{b_{n,1}}+\frac{0.1987}{b_{n,2}}\right)+{\kappa_1}^{3/8}{\kappa_2}^{1/8}\left(0.1597+\frac{0.1340}{b_{n,1}}+\frac{0.0031}{b_{n,2}}\right)-1.0742.\end{aligned}\tag{3.54}$$

From (3.53) and (3.54), we get

$$\Pi_{1,1}+\Pi_{2,1}\approx\kappa_0\left(2.0367+\frac{0.5885}{b_{n,1}}+\frac{0.1987}{b_{n,2}}\right)+{\kappa_1}^{3/8}{\kappa_2}^{1/8}\left(0.1698+\frac{0.1751}{b_{n,1}}+\frac{0.0031}{b_{n,2}}\right)-5.3713.\tag{3.55}$$

which gives the left hand side of (2.36).
On the other hand, the right hand side of the inequality (2.36) is

$$\begin{aligned}\frac{M_1}{H_1(\delta_1,c_1)}\Psi_{c_1}^{\delta_1}[H_1(.,c_1)]=&\frac{1}{H_1(2n\pi+\pi/6,2n\pi+\pi/12)}H_1(2n\pi+\pi/8,2n\pi+\pi/12)\\ &\times\left(\frac{b_{n,1}-a_{n,1}}{a_{n,1}(2n\pi+\pi/8-(2n\pi+\pi/12))}\right)\\ \approx&(0.9549)\left(\frac{b_{n,1}-a_{n,1}}{a_{n,1}}\right),\end{aligned}\tag{3.56}$$

and

$$\begin{aligned}\frac{M_1}{H_2(d_1,\delta_1)}\Psi_{\delta_1}^{d_1}[H_2(d_1,.)]=&\frac{1}{(2n\pi+\pi/2-2n\pi-\pi/6)^3}(2n\pi+\pi/2-2n\pi-3\pi/8)^3\\ &\times\left(\frac{b_{n,2}-a_{n,2}}{a_{n,2}(2n\pi+3\pi/8-2n\pi-\pi/6))}\right)\\ \approx&(0.0805)\left(\frac{b_{n,2}-a_{n,2}}{a_{n,2}}\right).\end{aligned}\tag{3.57}$$

From (3.56) and (3.57), we have the right hand side of (2.36) as

$$\begin{aligned}&\frac{M_1}{H_1(\delta_1,c_1)}\Psi_{c_1}^{\delta_1}[H_1(.,c_1)]+\frac{M_1}{H_2(d_1,\delta_1)}\Psi_{\delta_1}^{d_1}[H_2(d_1,.)]\\ &\qquad\approx(0.9549)\left(\frac{b_{n,1}-a_{n,1}}{a_{n,1}}\right)+(0.0805)\left(\frac{b_{n,2}-a_{n,2}}{a_{n,2}}\right).\end{aligned}\tag{3.58}$$

Thus (2.36) is satisfied for $j=1$ if

$$\begin{aligned}&\kappa_0\left(2.0367+\frac{0.5885}{b_{n,1}}+\frac{0.1987}{b_{n,2}}\right)+{\kappa_1}^{3/8}{\kappa_2}^{1/8}\left(0.1698+\frac{0.1751}{b_{n,1}}+\frac{0.0031}{b_{n,2}}\right)\\ &\qquad>5.3713+(0.9549)\left(\frac{b_{n,1}-a_{n,1}}{a_{n,1}}\right)+(0.0805)\left(\frac{b_{n,2}-a_{n,2}}{a_{n,2}}\right).\end{aligned}\tag{3.59}$$

Similarly for $[c_2,d_2]$, we have

$$\Pi_{1,2}+\Pi_{2,2}\approx\kappa_0\left(0.0887+\frac{0.0501}{b_{n,3}}+\frac{0.0046}{b_{n,4}}\right)+{\kappa_1}^{3/8}{\kappa_2}^{1/8}\left(2.6583+\frac{0.4302}{b_{n,3}}+\frac{0.1122}{b_{n,4}}\right)-2.5782.\tag{3.60}$$

and

$$\frac{M_2}{H_1(\delta_2, c_2)} \Psi_{c_2}^{\delta_2}[H_1(., c_2)] + \frac{M_2}{H_2(d_2, \delta_2)} \Psi_{\delta_2}^{d_2}[H_2(d_2, .)] \\ \approx (1.0742) \left(\frac{b_{n,3} - a_{n,3}}{a_{n,3}} \right) + (0.0632) \left(\frac{b_{n,4} - a_{n,4}}{a_{n,4}} \right). \tag{3.61}$$

Thus (2.36) is satisfied for $j = 2$ if

$$\kappa_0 \left(0.0887 + \frac{0.0501}{b_{n,3}} + \frac{0.0046}{b_{n,4}} \right) + \kappa_1{}^{3/8} \kappa_2{}^{1/8} \left(2.6583 + \frac{0.4302}{b_{n,3}} + \frac{0.1122}{b_{n,4}} \right) \\ > 2.5782 + (1.0742) \left(\frac{b_{n,3} - a_{n,3}}{a_{n,3}} \right) + (0.0632) \left(\frac{b_{n,4} - a_{n,4}}{a_{n,4}} \right). \tag{3.62}$$

Hence, by Theorem (2.2), equation(3.52) is oscillatory if (3.59) and (3.62) hold.

4 Conclusion

In this paper, we have established interval oscillation results for equation (1.1) using Riccati transformation, some classical inequalities and Kong's technique. These results extend some well-known results in [11, 13, 17].

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