Chaos and bifurcation of discontinuous dynamical systems with piecewise constant arguments

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Abstract

In this paper we are concerned with the definition and some properties of the discontinuous dynamical systems generated by piecewise constant arguments. Then we study a discontinuous dynamical system of the Riccati type equation as an example. The local stability at the fixed points is studied. The bifurcation analysis and chaos are discussed. In addition, we compare our results with the discrete dynamical system of the Riccati type equation.

Keywords: Discontinuous dynamical systems, piecewise constant arguments, Riccati type equation, fixed points, bifurcation, chaos.

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1 Introduction

The discontinuous dynamical systems generated by the retarded functional equations has been defined in [1]-[4]. The dynamical systems with piecewise constant arguments has been studied in [5]-[7] and references therein. In this work we define the discontinuous dynamical systems generated by functional equations with piecewise constant arguments. The dynamical properties of the discontinuous dynamical system of the Riccati type equation will be discussed. Comparison with the corresponding discrete dynamical system of the Riccati type equation

$$x_n = 1 - \rho x_{n-1}^2, \quad n = 1, 2, 3, ...,$$

will be given.

2 Piecewise constant arguments

Consider the problem of functional equation with piecewise constant arguments

$$x(t) = f(x(r[\frac{t}{r}])), \quad t > 0, r > 0. \tag{2.1}$$

$$x(0) = x_0, \tag{2.2}$$

where [.] denotes the greatest integer function.

Let $n = 1, 2, 3, ...$ and $t \in [nr, (n+1)r)$, then

$$x(t) = f(x_n(nr)), \quad t \in [nr, (n+1)r).$$

Let $r = 1$ and take the limit as $t \to n + 1$, we get

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, ...$$

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This shows that the discrete dynamical system

$$x_n = f(n, x_{n-1}), \quad n = 1, 2, 3, ..., T.$$

$$x(0) = x_o,$$

is a special case of the problem of functional equation with piecewise constant arguments (2.1)-(2.2). Now let $t \in [0, r)$, then $\frac{t}{r} \in [0, 1)$, $x(r[\frac{t}{r}]) = x(0)$ and the solution of (2.1)-(2.2) is given by

$$x(t) = x_1(r) = f(x(0)), \quad t \in [0, r),$$

with

$$x_1(r) = \lim_{t \to r^-} x(t) = f(x(0)).$$

For $t \in [r, 2r)$, then $\frac{t}{r} \in [1, 2)$, $x(r[\frac{t}{r}]) = x(r)$ and the solution of (2.1)-(2.2) is given by

$$x(t) = x_2(t) = f(x_1(r)), \quad t \in [r, 2r).$$

Repeating the process we can easily deduce that the solution of (2.1)-(2.2) is given by

$$x(t) = x_{(n+1)}(t) = f(x_n(nr)), \quad t \in [nr, (n+1)r),$$

which is continuous on each subinterval $(k, (k+1))$, $k = 1, 2, 3, ..., n$, but

$$\lim_{t \to kr^+} x_{(k+1)}(t) = f(x_k(kr)) \neq x_k(kr).$$

Hence the problem (2.1)-(2.2) is a discontinuous and we have proved the following theorem.

Theorem 2.1. *The solution of the problem of functional equation with piecewise constant arguments (2.1)-(2.2) is discontinuous (sectionally continuous) even if the function f is continuous.*

Now let $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $r \in \mathbb{R}^+$. Then, the following definition can be given.

Definition 2.1. *The discontinuous dynamical system generated by piecewise constant arguments is the problem*

$$x(t) = f(t, x(r[\frac{t}{r}]), x(r[\frac{t-1}{r}]), ..., x(r[\frac{t-n}{r}])), \quad t \in [0, T], \tag{2.3}$$

$$x(t) = x_0, \quad t \leq 0. \tag{2.4}$$

Definition 2.2. *The fixed points of the discontinuous dynamical system (2.3) and (2.4) are the solution of the equation*

$$x(t) = f(t, x, x, ..., x).$$

3 Main Problem

Consider the discontinuous dynamical system generated by piecewise constant arguments of Riccati type equation

$$x(t) = 1 - \rho x^2(r[\frac{t}{r}]), \quad t, r > 0, \text{ and } x(0) = x_0. \tag{3.1}$$

Here we study the stability at the fixed points. In order to study bifurcation and chaos we take firstly $r = 1$ and we compare the results with the results of the discrete dynamical system of Riccati type difference equation

$$x_{n+1} = 1 - \rho x_n^2, \quad n = 1, 2, 3, ..., \text{ and } x_0 = x_o. \tag{3.2}$$

Secondly, we take some other values of r and T and study some examples.

3.1   Fixed points and stability

As in the case of discrete dynamical systems, the fixed points of the dynamical system (3.1) are the solution of the equation $f(x) = x$. Thus there are two fixed points which are

$$(x_{fixed})_1 = \frac{-1 + \sqrt{1 + 4\rho}}{2\rho},$$

$$(x_{fixed})_2 = \frac{-1 - \sqrt{1 + 4\rho}}{2\rho}.$$

To study the stability of these fixed points, we take into account the following theorem.

Theorem 3.1. *[8] Let f be a smooth map on $\mathbb{R}$, and assume that x_0 is a fixed point of f.*
1. If $|f'(x_0)| < 1$, then x_0 is stable.
2. If $|f'(x_0)| > 1$, then x_0 is unstable.

Now since in our case $f(x) = 1 - \rho x^2$, the first fixed point $(x_{fixed})_1 = \frac{-1 + \sqrt{1+4\rho}}{2\rho}$ is stable if

$$|1 - \sqrt{1 + 4\rho}| < 1,$$

that is, $\frac{-1}{4} < \rho < \frac{3}{4}$.
The second fixed point $(x_{fixed})_2 = \frac{-1 - \sqrt{1+4\rho}}{2\rho}$ is stable if

$$|1 + \sqrt{1 + 4\rho}| < 1,$$

which can never happen since $1 + \sqrt{1 + 4\rho}$ is always > 1. So, the second fixed point is unstable.
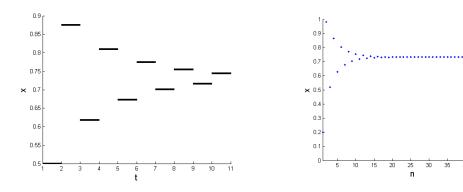Figure (1) shows the trajectories of (3.1) when $r = 1$, while Figure (2) shows the trajectories of (3.2).

Figure 1: Trajectories of (3.1), r=1.

Figure 2: Trajectories of (3.2).

4   Bifurcation and Chaos

In this section, the numerical experiments show that the dynamical behaviors of the discontinuous dynamical system (3.1) depends completely on both r and T as follows:

1. Take $r = 1$ and $t \in [0, 30]$, in this case the dynamical behaviors of the two dynamical systems (3.1) and (3.2) are identical (Figure 4).

2. Take $r = 0.25$ and $t \in [0, 2]$ in the dynamical system (3.1) (Figure 5).

3. Take $r = 0.5$ and $t \in [0, 2]$ in the dynamical system (3.1) (Figure 6).

4. Take $r = 0.25$ and $T = N = 13$ in the dynamical system (3.1) (Figure 7).

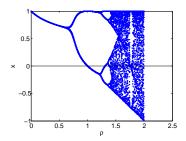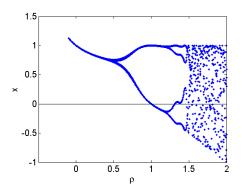5. Take $r = 0.5$ and $T = N = 35$ in the dynamical system (3.1) (Figure 3).

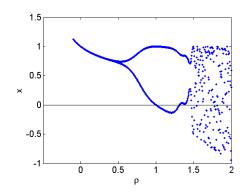Figure 3: Bifurcation diagram of the dynamical systems (3.1) with $r = 1$ and (3.2) where $N = T = 70$.
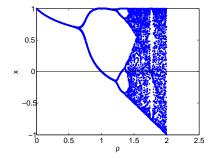

Figure 4: Bifurcation diagram for (3.1), $r = 0.5$, $t = [0, 3]$.

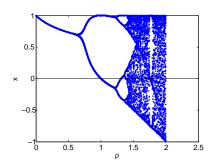Figure 5: Bifurcation diagram for (3.1), $r = 0.25$, $t = [0, 3]$.

Figure 6: Bifurcation diagram for (3.1), $r = 0.5$, $T = N = 13$.

Figure 7: Bifurcation diagram for (3.1), $r = 0.25$, $T = N = 13$.

5   Conclusion

The discontinuous dynamical system models generated by piecewise constant arguments have the same behavior as its discrete version when $r = 1$.

On the other hand, changing the parameter r together with the time $t \in [0, T]$ affects the chaos behavior of the dynamical system generated by the piecewise constant arguments model as it is shown clearly in the above figures.

$\square$

References

[1] A. M. A. El-Sayed, A. El-Mesiry and H. EL-Saka, On the fractional-order logistic equation, *Applied Mathematics Letters*, 20(2007), 817-823.

[2] A. M. A. El-Sayed and M. E. Nasr, Existence of uniformly stable solutions of nonau- tonomous discontinuous dynamical systems, *J. Egypt Math. Soc.*, 19(1) (2011), 10-16

[3] A. M. A. El-Sayed and M. E. Nasr, On some dynamical properties of discontinuous dynamical systems, *American Academic and Scholarly Research Journal*, 2(1)(2012), 28-32.

[4] A. M. A. El-Sayed and M. E. Nasr, On some dynamical properties of the discontinuous dynamical system represents the Logistic equation with different delays, *I-manager's Journal on Mathematics*, 'accepted manuscript'.

[5] D. Altintan, Extension of the Logistic equation with piecewise constant arguments and population dynamics, *Master dissertation*, Turkey 2006 .

[6] M. U. Akhmet, Stability of differential equations with piecewise constant arguments of generalized type, 'accepted manuscript'.

[7] M.U. Akhmet, D. Altntana and T. Ergen, Chaos of the logistic equation with piecewise constant arguments, *Applied Mathematis Letters*, preprint (2010), 2-5.

[8] S. El Aidy, *An Introducation to Difference Equations*, Springer, Third Edition, 2005.