

# Application of random fixed point theorems in solving nonlinear stochastic integral equation of the Hammerstein type

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## Abstract

In the present paper, we apply random analogue Kannan fixed point theorem [10] to analyze the existence of a solution of a nonlinear stochastic integral equation of the Hammerstein type of the form

$$x(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega) f(s, x(s; \omega)) d\mu(s)$$

where  $t \in S$ , a  $\sigma$ -finite measure space with certain properties,  $\omega \in \Omega$ , the supporting set of a probability measure space  $(\Omega, \beta, \mu)$  and the integral is a Bochner integral.

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## 1 Introduction

The importance of random fixed point theory lies in its vast applicability in probabilistic functional analysis and various probabilistic models. The introduction of randomness however leads to several new questions of measurability of solutions, probabilistic and statistical aspects of random solutions. It is well known that random fixed point theorems are stochastic generalization of classical fixed point theorems what we call as deterministic results. Random fixed point theorems for random contraction mappings on separable complete metric spaces were first proved by Špaček [18] and Hanš (see [6]-[7]). The survey article by Bharucha-Reid [4] in 1976 attracted the attention of several mathematicians and gave wings to this theory. Itoh [8] extended Špaček's and Hanš's theorems to multivalued contraction mappings. Random fixed point theorems with an application to Random differential equations in Banach spaces are obtained by Itoh [8]. Sehgal and Waters [17] had obtained several random fixed point theorems including random analogue of the classical results due to Rothe [13]. In recent past, several fixed point theorems including Kannan type [10] Chatterjee [5] and Zamfirescu type [20] have been generalized in stochastic version (see for detail in Joshi and Bose [9], Saha et al. ([14], [15])).

On the otherhand, Padgett [12] used the random analogue of Banach fixed point theorem [3] to analyze the existence and uniqueness of random solution of a nonlinear stochastic integral equation of the Hammerstein type of the form

$$x(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega) f(s, x(s; \omega)) d\mu(s)$$

and proved several theorems. Achari [1] and Saha et al.[16] continued to work on this application for more generalized random nonlinear contraction operators.

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In the following section, we study on application of two basic random fixed point theorems of importance, one - is Kannan fixed point theorem [10] and the other one is - Chatterjea fixed point theorem [5] to analyze the existence of solution for such integral equation.

## 2 Preliminaries

Let  $(X, \beta_X)$  be a separable Banach space where  $\beta_X$  is a  $\sigma$ -algebra of Borel subsets of  $X$ , and let  $(\Omega, \beta, \mu)$  denote a complete probability measure space with measure  $\mu$ , and  $\beta$  be a  $\sigma$ -algebra of subsets of  $\Omega$ . For more details one can see Joshi and Bose [9].

**Theorem 2.1.** (Joshi and Bose [9]) *Let  $X$  be a separable Banach space and  $(\Omega, \beta, \mu)$  be a complete probability measure space. Let  $T : \Omega \times X \rightarrow X$  be a continuous random operator satisfying*

$$\begin{aligned} \|T(\omega, x_1) - T(\omega, x_2)\| &\leq k_1(\omega) [\|x_1 - T(\omega, x_1)\| + \|x_2 - T(\omega, x_2)\|] \\ &\quad + k_2(\omega) [\|x_1 - T(\omega, x_2)\| + \|x_2 - T(\omega, x_1)\|] \\ &\quad + k_3 \|x_1 - x_2\| \end{aligned} \quad (2.1)$$

for all  $\omega \in \Omega$  and  $x_1, x_2 \in X$ ,  $k_i(\omega) \geq 0$ ;  $1 \leq i \leq 3$ . are real valued random variables with  $2k_1(\omega) + 2k_2(\omega) + k_3(\omega) < 1$  almost surely. Then there exists a unique random fixed point of  $T$ .

**Remark 2.1.** (I) *In the above theorem, setting  $k_2(\omega) = k_3(\omega) = 0$ , one can find random analogue of kannan fixed point theorem [10] and in that case the operator  $T : \Omega \times X \rightarrow X$  takes the form:*

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leq k_1(\omega) [\|x_1 - T(\omega, x_1)\| + \|x_2 - T(\omega, x_2)\|] \quad (2.2)$$

for all  $\omega \in \Omega$  and  $x_1, x_2 \in X$ ,  $k_1(\omega) \geq 0$  is real valued random variables with  $k_1(\omega) < \frac{1}{2}$  almost surely.

(II) *Setting  $k_1(\omega) = k_3(\omega) = 0$ , one can find random analogue of Chatterjea fixed point theorem [5] and in that case the operator  $T : \Omega \times X \rightarrow X$  takes the form:*

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leq k_2(\omega) [\|x_1 - T(\omega, x_2)\| + \|x_2 - T(\omega, x_1)\|] \quad (2.3)$$

for all  $\omega \in \Omega$  and  $x_1, x_2 \in X$ ,  $k_2(\omega) \geq 0$  is real valued random variables with  $k_2(\omega) < \frac{1}{2}$  almost surely.

**Remark 2.2.** *Note that neither Kannan operator nor Chatterjea operator is continuous in general. So random fixed point theorems for these two operators are slightly different from their deterministic approach.*

## 3 Application to a random nonlinear integral equation

We now show an application of stochastic version of Kannan fixed point theorem in solving nonlinear stochastic integral equation of the Hammerstein type of the form:

$$x(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega) f(s, x(s; \omega)) d\mu_0(s) \quad (3.4)$$

where

- (i)  $S$  is a locally compact metric space with metric  $d$  on  $S \times S$ ,  $\mu_0$  is a complete  $\sigma$ -finite measure defined on the collection of Borel subsets of  $S$ ;
- (ii)  $\omega \in \Omega$ , where  $\omega$  is a supporting set of probability measure space  $(\Omega, \beta, \mu)$ ;
- (iii)  $x(t; \omega)$  is the unknown vector-valued random variables for each  $t \in S$ .
- (iv)  $h(t; \omega)$  is the stochastic free term defined for  $t \in S$ ;
- (v)  $k(t, s; \omega)$  is the stochastic kernel defined for  $t$  and  $s$  in  $S$  and
- (vi)  $f(t, x)$  is vector-valued function of  $t \in S$  and  $x$

and the integral in equation (3.4) is a Bochner integral.

We will further assume that  $S$  is the union of a countable family of compact sets  $\{C_n\}$  having the properties that  $C_1 \subset C_2 \subset \dots$  and that for any other compact set  $S$  there is a  $C_i$  which contains it (see [2]).

**Definition 3.1.** We define the space  $C(S, L_2(\Omega, \beta, \mu))$  to be the space of all continuous functions from  $S$  into  $L_2(\Omega, \beta, \mu)$  with the topology of uniform convergence on compacta i.e. for each fixed  $t \in S$ ,  $x(t; \omega)$  is a vector valued random variable such that

$$\|x(t; \omega)\|_{L_2(\Omega, \beta, \mu)}^2 = \int_{\Omega} |x(t; \omega)|^2 d\mu(\omega) < \infty$$

It may be noted that  $C(S, L_2(\Omega, \beta, \mu))$  is locally convex space (see [19]) whose topology is defined by a countable family of seminorms given by

$$\|x(t; \omega)\|_n = \sup_{t \in C_n} \|x(t; \omega)\|_{L_2(\Omega, \beta, \mu)}, n = 1, 2, \dots$$

Moreover  $C(S, L_2(\Omega, \beta, \mu))$  is complete relative to this topology since  $L_2(\Omega, \beta, \mu)$  is complete.

We further define  $BC = BC(S, L_2(\Omega, \beta, \mu))$  to be the Banach space of all bounded continuous functions from  $S$  into  $L_2(\Omega, \beta, \mu)$  with norm

$$\|x(t; \omega)\|_{BC} = \sup_{t \in S} \|x(t; \omega)\|_{L_2(\Omega, \beta, \mu)}$$

The space  $BC \subset C$  is the space of all second order vector-valued stochastic process defined on  $S$  which are bounded and continuous in mean square.

We will consider the function  $h(t; \omega)$  and  $f(t, x(t; \omega))$  to be in the space  $C(S, L_2(\Omega, \beta, \mu))$  with respect to the stochastic kernel. We assume that for each pair  $(t, s)$ ,  $k(t, s; \omega) \in L_{\infty}(\Omega, \beta, \mu)$  and denote the norm by

$$\|k(t, s; \omega)\| = \|k(t, s; \omega)\|_{L_{\infty}(\Omega, \beta, \mu)} = \mu - \text{ess sup}_{\omega \in \Omega} |k(t, s; \omega)|.$$

Also we will suppose that  $k(t, s; \omega)$  is such that  $\|k(t, s; \omega)\| \cdot \|x(s; \omega)\|_{L_2(\Omega, \beta, \mu)}$  is  $\mu_0$ -integrable with respect to  $s$  for each  $t \in S$  and  $x(s; \omega)$  in  $C(S, L_2(\Omega, \beta, \mu))$  and there exists a real valued function  $G$  defined  $\mu_0$ -a.e. on  $S$ , so that  $G(S) \|x(s; \omega)\|_{L_2(\Omega, \beta, \mu)}$  is  $\mu_0$ -integrable and for each pair  $(t, s) \in S \times S$ ,

$$\|k(t, u; \omega) - k(s, u; \omega)\| \cdot \|x(u, \omega)\|_{L_2(\Omega, \beta, \mu)} \leq G(u) \|x(u, \omega)\|_{L_2(\Omega, \beta, \mu)}$$

$\mu_0$ -a.e. Further, for almost all  $s \in S$ ,  $k(t, s; \omega)$  will be continuous in  $t$  from  $S$  into  $L_{\infty}(\Omega, \beta, \mu)$ .

We now define the random integral operator  $T$  on  $C(S, L_2(\Omega, \beta, \mu))$  by

$$(Tx)(t; \omega) = \int_S k(t, s; \omega)x(s; \omega)d\mu_0(s) \tag{3.5}$$

where the integral is a Bochner integral. Moreover, we have that for each  $t \in S$ ,  $(Tx)(t; \omega) \in L_2(\Omega, \beta, \mu)$  and that  $(Tx)(t; \omega)$  is continuous in mean square by Lebesgue's dominated convergence theorem. So  $(Tx)(t; \omega) \in C(S, L_2(\Omega, \beta, \mu))$ .

**Definition 3.2.** (see [1], [11]) Let  $B$  and  $D$  be Banach spaces. The pair  $(B, D)$  is said to be admissible with respect to a random operator  $T(\omega)$  if  $T(\omega)(B) \subset D$ .

**Lemma 3.1.** (see [12]) The linear operator  $T$  defined by (3.5) is continuous from  $C(S, L_2(\Omega, \beta, \mu))$  into itself.

**Lemma 3.2.** (see [12], [11]) If  $T$  is a continuous linear operator from  $C(S, L_2(\Omega, \beta, \mu))$  into itself and  $B, D \subset C(S, L_2(\Omega, \beta, \mu))$  are Banach spaces stronger than  $C(S, L_2(\Omega, \beta, \mu))$  such that  $(B, D)$  is admissible with respect to  $T$ , then  $T$  is continuous from  $B$  into  $D$ .

**Remark 3.3.** (see [12]) The operator  $T$  defined by (3.5) is a bounded linear operator from  $B$  into  $D$ .

It is to be noted that by a random solution of the equation (3.4) we will mean a function  $x(t; \omega)$  in  $C(S, L_2(\Omega, \beta, \mu))$  which satisfies the equation (3.4)  $\mu$ -a.e.

We are now in a state to prove the following theorem.

**Theorem 3.2.** We consider the stochastic integral equation (3.4) subject to the following conditions:

(a)  $B$  and  $D$  are Banach spaces stronger than  $C(S, L_2(\Omega, \beta, \mu))$  such that  $(B, D)$  is admissible with respect to

the integral operator defined by (3.5);

(b)  $x(t; \omega) \rightarrow f(t, x(t; \omega))$  is an operator from the set

$$Q(\rho) = \{x(t; \omega) : x(t; \omega) \in D, \|x(t; \omega)\|_D \leq \rho\}$$

into the space  $B$  satisfying

$$\begin{aligned} \|f(t, x_1(t; \omega)) - f(t, x_2(t; \omega))\|_B &\leq \lambda(\omega) [\|x_1(t; \omega) - f(t, x_1(t; \omega))\|_D \\ &\quad + \|x_2(t; \omega) - f(t, x_2(t; \omega))\|_D] \end{aligned} \quad (3.6)$$

for  $x_1(t; \omega), x_2(t; \omega) \in Q(\rho)$ , where  $0 \leq \lambda(\omega) < \frac{1}{2}$  is a real valued random variable almost surely,

(c)  $h(t; \omega) \in D$ .

Then there exists a unique random solution of (3.4) in  $Q(\rho)$ , provided

$\lambda(\omega)(1 + c(\omega)) < \frac{1}{2}$  and

$$\|h(t; \omega)\|_D + \frac{1 + \lambda(\omega)}{1 - \lambda(\omega)} c(\omega) \|f(t; 0)\|_B \leq \rho \left(1 - \frac{c(\omega)\lambda(\omega)}{1 - \lambda(\omega)}\right)$$

where  $c(\omega)$  is the norm of  $T(\omega)$ .

*Proof.* Define the operator  $U(\omega)$  from  $Q(\rho)$  into  $D$  by

$$(Ux)(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega) f(s, x(s; \omega)) d\mu_0(s)$$

Now

$$\begin{aligned} \|(Ux)(t; \omega)\|_D &\leq \|h(t; \omega)\|_D + c(\omega) \|f(t, x(t; \omega))\|_B \\ &\leq \|h(t; \omega)\|_D + c(\omega) \|f(t; 0)\|_B + c(\omega) \|f(t, x(t; \omega)) - f(t; 0)\|_B \end{aligned}$$

Then from the condition (3.6) of this theorem

$$\begin{aligned} \|f(t, x(t; \omega)) - f(t; 0)\|_B &\leq \lambda(\omega) [\|x(t; \omega) - f(t, x(t; \omega))\|_D + \|f(t; 0)\|_D] \\ &\leq \lambda(\omega) [\|x(t; \omega)\|_D + \|f(t, x(t; \omega))\|_D + \|f(t; 0)\|_D] \\ &\leq \lambda(\omega) [\|x(t; \omega)\|_D + \|f(t, x(t; \omega)) - f(t; 0)\|_D + 2\|f(t; 0)\|_D] \end{aligned}$$

implies

$$\|f(t, x(t; \omega)) - f(t; 0)\|_B \leq \frac{\lambda(\omega)}{1 - \lambda(\omega)} \rho + \frac{2\lambda(\omega)}{1 - \lambda(\omega)} \|f(t; 0)\|_B \quad (3.7)$$

Therefore by (3.7), we have

$$\begin{aligned} \|(Ux)(t; \omega)\|_D &\leq \|h(t; \omega)\|_D + c(\omega) \|f(t; 0)\|_B \\ &\quad + c(\omega) \left[ \frac{\lambda(\omega)\rho}{1 - \lambda(\omega)} + \frac{2\lambda(\omega)}{1 - \lambda(\omega)} \|f(t; 0)\|_B \right] \\ &= \|h(t; \omega)\|_D + \frac{c(\omega)\lambda(\omega)}{1 - \lambda(\omega)} \rho + \frac{1 + \lambda(\omega)}{1 - \lambda(\omega)} c(\omega) \|f(t; 0)\|_B \\ &< \rho \end{aligned}$$

Hence  $(Ux)(t; \omega) \in Q(\rho)$ . Then for  $x_1(t; \omega), x_2(t; \omega) \in Q(\rho)$ , we have by condition (b)

$$\begin{aligned} \|(Ux_1)(t; \omega) - (Ux_2)(t; \omega)\|_D &= \left\| \int_S k(t, s; \omega) [f(s, x_1(s; \omega)) - f(s, x_2(s; \omega))] d\mu_0(s) \right\|_D \\ &\leq c(\omega) \|f(t, x_1(t; \omega)) - f(t, x_2(t; \omega))\|_B \\ &\leq c(\omega)\lambda(\omega) [\|x_1(t; \omega) - f(t, x_1(t; \omega))\|_D \\ &\quad + \|x_2(t; \omega) - f(t, x_2(t; \omega))\|_D] \end{aligned}$$

since  $c(\omega)\lambda(\omega) < \frac{1}{2}$ ,  $U(\omega)$  is a Kannan contraction on  $Q(\rho)$ . Hence, by Theorem 2.1 and Remark 2.1(I), there exists a unique  $x^*(t, \omega) \in Q(\rho)$ , which is a fixed point of  $U$ , that is  $x^*(t, \omega)$  is the unique random solution of the equation (3.4).  $\square$

A similar theorem can be obtained using random analogue of Chatterjea fixed point theorem [5].

**Theorem 3.3.** Assume that the stochastic integral equation (3.4) subject to the following conditions:

(a') Same as (a) of Theorem 3.2;

(b')  $x(t; \omega) \rightarrow f(t, x(t; \omega))$  is an operator from the set

$$Q(\rho) = \{x(t; \omega) : x(t; \omega) \in D, \|x(t; \omega)\|_D \leq \rho\}$$

into the space  $B$  satisfying

$$\begin{aligned} \|f(t, x_1(t; \omega)) - f(t, x_2(t; \omega))\|_B &\leq \lambda(\omega) [\|x_1(t; \omega) - f(t, x_2(t; \omega))\|_D \\ &\quad + \|x_2(t; \omega) - f(t, x_1(t; \omega))\|_D] \end{aligned} \quad (3.8)$$

for  $x_1(t; \omega), x_2(t; \omega) \in Q(\rho)$ , where  $0 \leq \lambda(\omega) < \frac{1}{2}$  is a real valued random variable almost surely,

(c')  $h(t; \omega) \in D$ .

Then there exists a unique random solution of (3.4) in  $Q(\rho)$ , provided

$\lambda(\omega)(1 + c(\omega)) < \frac{1}{2}$  and

$$\|h(t; \omega)\|_D + \frac{1 + \lambda(\omega)}{1 - \lambda(\omega)} c(\omega) \|f(t; 0)\|_B \leq \rho \left(1 - \frac{c(\omega)\lambda(\omega)}{1 - \lambda(\omega)}\right)$$

where  $c(\omega)$  is the norm of  $T(\omega)$ .

*Proof.* The proof is similar to that of Theorem 3.2. So we avoid repetition. □

The following example illustrates the strength of our main result-Theorem 3.2.

**Example 3.1.** Consider the following nonlinear stochastic integral equation:

$$x(t; \omega) = \int_0^\infty \frac{e^{-t-s}}{8(1 + |x(s; \omega)|)} ds$$

Comparing with (3.4), we see that

$$h(t, \omega) = 0, k(t, s; \omega) = \frac{1}{2} e^{-t-s}, f(s, x(s; \omega)) = \frac{1}{4(1 + |x(s; \omega)|)}$$

Then one can check that equation (3.6) is satisfied with  $\lambda(\omega) = \frac{1}{3}$ .

Comparing with integral operator equation (3.5), we see that the norm of  $T(\omega)$  is  $c(\omega) = \frac{1}{4}$  satisfying  $\lambda(\omega)(1 + c(\omega)) < \frac{1}{2}$ . So, all the conditions of Theorem 3.2 are satisfied and hence there exists a random fixed point of the integral operator  $T$  satisfying (3.5).

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