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Third Hankel Determinant for Certain Subclass of Analytic Functions

M. A. Ganiyu[a,*], K.O. Babalola[b]

[a]Department of Physical Sciences, College of Natural Sciences, Al-Hikmah University, Ilorin, Nigeria.

[b]Department of Mathematics, College of science, University of Ilorin, Ilorin, Nigeria.

Abstract

The third Hankel determinant, $H_3(1)$ for subclass of analytic functions satisfying geometric condition

$$Re\,\frac{zf'(z)}{f(z)}\frac{f(z)^{\alpha-1}f'(z)}{z^{\alpha-1}} > 0$$

for nonnegative real number α, in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ is derived in line with a method of classical analysis devised by Libera and Zlotkiewicz [9].

Keywords: Hankel determinant, caratheodory functions, product of geometric expression, analytic functions.

2010 MSC: 30C45.

1 Introduction

Let A denote the class of functions

$$f(z) = z + a_2z^2 + a_3z^3 + ... \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the condition $f(0) = f'(0) - 1 = 0$. By S, S^*, C and R, we mean the well known subclasses of A which consist of univalent, starlike, convex and bounded turning functions respectively. In [8], Jimoh et-al introduced a subclass of analytic functions denoted by $\mathcal{J}_\alpha$ which satisfy the geometric condition:

$$Re\,\frac{zf'(z)}{f(z)}\frac{f(z)^{\alpha-1}f'(z)}{z^{\alpha-1}} > 0. \tag{1.2}$$

for non negative real number α,where estimates on the bounds of some coefficients were investigated. Also in [6], Ganiyu et-al obtained the bound on the second Hankel determinant, $H_2(2)$ for this same subclass of analytic functions, $\mathcal{J}_\alpha$. In [10], Noonan and Thomas defined the qth Hankel determinant of f for $q \geq 1$, $n \geq 0$ by:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2(q-1)} \end{vmatrix}$$

This determinant has been considered for specific choices of q and n by several authors with subject of inquiry ranging from rate of growth of $H_q(n)$ as $n \to \infty$ to the determination of precise bounds on $H_q(n)$ for some subclasses of analytic functions. It is well known that the Fekete-Szego functional is $|a_3 - a_2^2|$=$H_2(1)$. The

*Corresponding author.
bidex1425@gmail.com : bidex1425@gmail.com (M.A. Ganiyu) and kobabalola@gmail.com (K.O. Babalola).

second Hankel determinant defined by $H_2(2) = |a_2a_4 - a_3^2|$ also received a lot of attention by researchers among which is the notable work of Janteng, et-al, [7] where they obtained the second Hankel determinant for some subclasses of analytic functions. Other contributors in this regard include Abubaker [1], Al-Refai [2], Norlyda et-al [11], Vamshee [13].

Babalola [4], Shanmungam et-al [12], Vamshee et-al [14] have studied the third Hankel determinant, $H_3(1)$ for various classes of analytic and univalent functions. In the present investigation, our focus is on the third Hankel determinant, $H_3(1)$ for the subclass $\mathcal{J}_\alpha$ given by:

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

For $f \in A$, $a_1 = 1$ so that

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$$

and by using the triangle inequality, we have

$$|H_3(1)| \le |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2| \tag{1.3}$$

In this paper, we seek to find the sharp upper bound on $|a_2a_3 - a_4|$, $|a_3 - a_2^2|$ and $|H_3(1)|$ respectively for the functions belonging to the subclass $\mathcal{J}_\alpha$. We shall make use of our earlier results on the bounds on each of the coefficients and the functional $|a_2a_4 - a_3^2|$=$H_2(2)$.

2 Preliminary Lemmas.

To prove the main results in the next section, we need the following lemmas. Let P denote the class of Caratheodory functions $p(z) = 1 + c_1z + c_2z^2 + \cdots$ which are analytic and satisfy $p(0) = 1$, Re $p(z) > 0$ in open unit disk U.

Lemma 2.1. *[5] Let $p \in P$. Then $|c_k| \le 2, k = 1, 2, 3....$ Equality is attained by the moebius function*

$$L_0(z) = \frac{1+z}{1-z}.$$

Lemma 2.2. *[9] Let $p \in P$, then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{2.1}$$

and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \tag{2.2}$$

for some value of x,z such that $|x| \le 1$ and $|z| \le 1$.

Lemma 2.3. *[3] Let $p \in P$. Then we have sharp inequalities*

$$\left|c_2 - \sigma\frac{c_1^2}{2}\right| \le \begin{cases} 2(1-\sigma), & \text{if } \sigma \le 0, \\ 2, & \text{if } 0 \le \sigma \le 2, \\ 2(\sigma - 1), & \text{if } \sigma \ge 2. \end{cases}$$

Lemma 2.4. *[6] Let $f \in \mathcal{J}_\alpha$. Then*

$$|H_2(2)| \le \frac{4}{(\alpha+4)^2}$$

Lemma 2.5. *[8] Let $f \in J_\alpha$. Then*

$$|a_2| \le \frac{2}{\alpha+2}$$

$$|a_3| \le \begin{cases} \frac{2(\alpha+6)}{(\alpha+2)^2(\alpha+4)} & \text{if } 0 < \alpha \le \frac{-3+\sqrt{17}}{2}, \\ \frac{2}{\alpha+4} & \text{if } \alpha \ge \frac{-3+\sqrt{17}}{2}. \end{cases}$$

$$|a_4| \le \begin{cases} \frac{52\alpha^4+472\alpha^3+1208\alpha^2+896\alpha+288}{6(\alpha+2)^3(\alpha+4)(\alpha+6)} & \text{if } \alpha \le \frac{-5+\sqrt{33}}{2}, \\ \frac{14\alpha^2+96\alpha+232}{3(\alpha+2)(\alpha+4)(\alpha+6)} & \text{if } \alpha \ge \frac{-5+\sqrt{33}}{2}. \end{cases}$$

$$|a_5| \le \begin{cases} \frac{14\alpha^5+236\alpha^4+1348\alpha^3+2976\alpha^2+2160\alpha+1024}{(\alpha+2)^2(\alpha+4)^2(\alpha+6)(\alpha+8)} & \text{if } \alpha \le \frac{-7+\sqrt{57}}{2}, \\ \frac{4\alpha^4+74\alpha^3+584\alpha^2+2152\alpha+3072}{(\alpha+2)(\alpha+4)^2(\alpha+6)(\alpha+8)} & \text{if } \alpha \ge \frac{-7+\sqrt{57}}{2}. \end{cases}$$

3 Main Results.

Theorem 3.1. *let $f \in J_\alpha$. Then we have*

$$|a_2a_3 - a_4| \leq \frac{2(\alpha^2+6\alpha+16)}{3(\alpha+2)(\alpha+4)(\alpha+6)}\sqrt{\frac{\alpha^3+8\alpha^2+28\alpha+32}{\alpha^3+8\alpha^2+25\alpha+18}}, \quad 0 \leq \alpha < 1$$

Proof. Using the results obtained earlier in [8], we have that if $f \in \mathcal{J}_\alpha$, then

$$a_2 = \frac{c_1}{\alpha+2} \tag{3.1}$$

$$a_3 = \frac{c_2}{\alpha+4} - \frac{\alpha^2+3\alpha-2}{2(\alpha+2)^2(\alpha+4)}c_1^2 \tag{3.2}$$

$$a_4 = \frac{c_3}{\alpha+6} + \frac{2-5\alpha-\alpha^2}{(\alpha+2)(\alpha+4)(\alpha+6)}c_1c_2 + \frac{2\alpha^4+17\alpha^3+31\alpha^2-8\alpha+12}{6(\alpha+2)^3(\alpha+4)(\alpha+6)}c_1^3 \tag{3.3}$$

so that

$$|a_2a_3 - a_4| = \left|\frac{\alpha^2+6\alpha+4}{(\alpha+2)(\alpha+4)(\alpha+6)}c_1c_2 - \frac{\alpha^4+10\alpha^3+29\alpha^2+20\alpha-12}{3(\alpha+2)^3(\alpha+4)(\alpha+6)}c_1^3 - \frac{c_3}{\alpha+6}\right| \tag{3.4}$$

substituting c_2 and c_3 in Lemma 2.2 into equation (3.4), we have

$$\begin{aligned}|a_2a_3 - a_4| = &\left|\frac{48-8\alpha-32\alpha^2-10\alpha^3-\alpha^4}{12(\alpha+2)^3(\alpha+4)(\alpha+6)}c_1^3 - \frac{2(4-c_1^2)}{(\alpha+2)(\alpha+4)(\alpha+6)}c_1x\right.\\ &\left.+\frac{(4-c_1^2)}{4(\alpha+6)}c_1x^2 - \frac{(4-c_1^2)(1-|x|^2)z}{2(\alpha+6)}\right|\end{aligned}$$

By Lemma 2.1, $|c_1| \leq 2$. Suppose that $c_1 = c$, we may assume without restriction that $c \in [0,2]$. By the use of triangle inequality with $\xi = |x|$ and noting that $48-8\alpha-32\alpha^2-10\alpha^3-\alpha^4 \geq 0$ for $0 \leq \alpha < 1$, we obtain

$$\begin{aligned}|a_2a_3 - a_4| &\leq \frac{48-8\alpha-32\alpha^2-10\alpha^3-\alpha^4}{12(\alpha+2)^3(\alpha+4)(\alpha+6)}c^3 + \frac{2(4-c^2)}{(\alpha+2)(\alpha+)(\alpha+6)}c\xi\\ &+\frac{(c-2)(4-c^2)}{4(\alpha+6)}\xi^2 + \frac{4-c^2}{2(\alpha+6)}\\ &= F(c,\xi)\end{aligned} \tag{3.5}$$

we assume the upper bound for equation (3.5) occurs at an interior point of the set $\{(\xi,c) : \xi \in [0,1]$ and $c \in [0,2]\}$. Differentiating $F(c,\xi)$ partially with respect to ξ, we get

$$F'(c,\xi) = \frac{2(4-c^2)c}{(\alpha+2)(\alpha+4)(\alpha+6)} + \frac{(c-2)(4-c^2)\xi}{2(\alpha+6)}$$

For $0 < \xi < 1$ and for fixed c with $0 < c < 2$, we observe that $F'(c,\xi) > 0$. Therefore, $F'(c,\xi)$ is an increasing function of ξ, which contracdicts our assumption that the maximum value of it occurs at an interior point of the set $\{(\xi,c) : \xi \in [0,1]$ and $c \in [0,2]\}$. Also for fixed $c \in [0,2]$, we have

$$\max_{0\leq\xi\leq1} F(c,\xi) = F(c,1) = G(c), \ \ say$$

replacing ξ by 1 in equation (3.5), we obtain

$$G(c) = F(c,1) = \frac{\alpha^2+6\alpha+16}{(\alpha+2)(\alpha+4)(\alpha+6)}c - \frac{\alpha^4+10\alpha^3+41\alpha^2+68\alpha+36}{3(\alpha+2)^3(\alpha+4)(\alpha+6)}c^3$$

so that

$$G'(c) = \frac{\alpha^2+6\alpha+16}{(\alpha+2)(\alpha+4)(\alpha+6)} - \frac{\alpha^4+10\alpha^3+41\alpha^2+68\alpha+36}{(\alpha+2)^3(\alpha+4)(\alpha+6)}c^2$$

$G'(c) = 0$ implies

$$c = \pm\sqrt{\frac{(\alpha+2)(\alpha^2+6\alpha+16)}{\alpha^3+8\alpha^2+25\alpha+18}}$$

since $c \in [0,2]$, we have $c = \sqrt{\frac{\alpha^3+8\alpha^2+28\alpha+32}{\alpha^3+8\alpha^2+25\alpha+18}}$ as the maximum point of $G(c)$. Therefore

$$G(c) \le \frac{2(\alpha^2+6\alpha+16)}{3(\alpha+2)(\alpha+4)(\alpha+6)}\sqrt{\frac{\alpha^3+8\alpha^2+28\alpha+32}{\alpha^3+8\alpha^2+25\alpha+18}}$$

That is the upper bound of equation (3.5) correspounds to $\xi = 1$ and $c = \sqrt{\frac{\alpha^3+8\alpha^2+28\alpha+32}{\alpha^3+8\alpha^2+25\alpha+18}}$. ☐

Theorem 3.2. *Let $f \in \mathcal{J}_\alpha$.Then*

$$|a_3 - a_2^2| \le \frac{2}{\alpha+4}$$

Proof. Using equations (3.1) and (3.2),

$$|a_3 - a_2^2| = \frac{1}{\alpha+4}\left|c_2 - \left(\frac{\alpha+3}{\alpha+2}\right)\frac{c_1^2}{2}\right|$$

Applying Lemma 2.3, with $\sigma = \frac{\alpha+3}{\alpha+2}$, we obtain

$$\left|c_2 - \left(\frac{\alpha+3}{\alpha+2}\right)\frac{c_1^2}{2}\right| \le 2$$

hence the result. ☐

Corollary 3.1. *Let $f \in \mathcal{J}_\alpha$.Then*

$$|H_3(1)| \le \begin{cases} \frac{J_1+J_2\sqrt{J_5}}{(\alpha+2)^2 J_6}, & \text{if } 0 \le \alpha \le \frac{-3+\sqrt{17}}{2}, \\ \frac{J_3+J_4\sqrt{J_5}}{J_6}, & \text{if } \alpha \ge \frac{-3+\sqrt{17}}{2}. \end{cases}$$

where,

$$J_1 = 252\alpha^{11} + 8784\alpha^{10} + 134316\alpha^9 + 1188072\alpha^8 + 6737328\alpha^7 + 25615584\alpha^6 + 66411072\alpha^5 + 117846144\alpha^4 + 143325504\alpha^3 + 41925888\alpha^2 + 64143360\alpha + 16920576,$$

$$J_2 = 52\alpha^8 + 1408\alpha^7 + 15944\alpha^6 + 99248\alpha^5 + 369248\alpha^4 + 818240\alpha^3 + 997120\alpha^2 + 569344\alpha + 147456,$$

$$J_3 = 72\alpha^8 + 2304\alpha^7 + 31176\alpha^6 + 233442\alpha^5 + 1057248\alpha^4 + 2945178\alpha^3 + 4854420\alpha^2 + 4258872\alpha + 1496880,$$

$$J_4 = 28\alpha^6 + 696\alpha^5 + 7280\alpha^4 + 42144\alpha^3 + 143744\alpha^2 + 276480\alpha + 237568,$$

$$J_5 = \alpha^6 + 16\alpha^5 + 117\alpha^4 + 474\alpha^3 + 1100\alpha^2 + 1304\alpha + 576,$$

$$J_6 = 9(\alpha+2)^2(\alpha+4)^3(\alpha+6)^2(\alpha+8)(\alpha^3+8\alpha^2+25\alpha+18).$$

Proof. By equation (1.3), we have

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|$$

using Lemma 4, the first inequality of the result in Lemma 5 together with the results obtained in Theorems 3.1 and 3.2,

$$\begin{aligned} |H_3(1)| \leq & \frac{8(\alpha+6)}{(\alpha+2)^2(\alpha+4)^3} + \frac{2(14\alpha^5+236\alpha^4+1348\alpha^3+2976\alpha^2+2160\alpha+1024)}{(\alpha+2)^2(\alpha+4)^3(\alpha+6)(\alpha+8)} \\ & + \left(\frac{(\alpha^2+6\alpha+16)(52\alpha^4+472\alpha^3+1208\alpha^2+896\alpha+288)}{9(\alpha+2)^4(\alpha+4)^2(\alpha+6)^2(\alpha^3+8\alpha^2+25\alpha+18)}\right) \\ & \left(\sqrt{(\alpha^3+8\alpha^2+28\alpha+32)(\alpha^3+8\alpha^2+25\alpha+18)}\right) \end{aligned}$$

simplifying, we have the first inequality.
Also by using Lemma 2.4, the second inequality of the result in Lemma 2.5 together with the results obtained in Theorems 3.1 and 3.2,

$$\begin{aligned} |H_3(1)| \leq & \frac{8}{(\alpha+4)^3} + \frac{2(4\alpha^4+74\alpha^3+584\alpha^2+2152\alpha+3072)}{(\alpha+2)(\alpha+4)^3(\alpha+6)(\alpha+8)} \\ & + \frac{2(\alpha^2+6\alpha+16)(14\alpha^2+96\alpha+232)}{9(\alpha+2)^2(\alpha+4)^2(\alpha+6)^2}\sqrt{\frac{\alpha^3+8\alpha^2+28\alpha+32}{\alpha^3+8\alpha^2+25\alpha+18}} \end{aligned}$$

By simplification, we obtain the other inequality.

□

4 Conclusion

We have been able to find the sharp upper bound on functionals $|a_2a_3 - a_4|$, $|a_3 - a_2^2|$ and the third Hankel determinant, $|H_3(1)|$ for the functions belonging to the subclass $\mathcal{J}_\alpha$.

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