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Compactly Coinvariant Subspaces

Batool Rostamzadeh Kasrineh[a,*]

[a]Department of Mathematics, Islamic Azad University, Rafsanjan Branch, Rafsanjan, Iran.

Abstract

First, we obtain some results on compactly invariant subspaces in normed spaces. Then we introduce the notion of compactly coinvariant pair. Also, a sufficient condition for a normed space X has a compactly coinvariant pair, is given.

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1 Introduction

The notion of an invariant subspace is fundamental to the subject of operator theory (see [1]-[3]). Given a linear operator T on a Banach space X, a closed subspace M of X is said to be a non-trivial invariant subspace for T if $T(M) \subset M$ and $M \neq \emptyset$ and X. This generalizes the idea of eigenspaces of $n \times n$ matrices. A famous unsolved problem, called the invariant subspace problem, "asks whether every bounded linear operator on a Hilbert space (more generally, a Banach space) admits a non-trivial invariant subspace"?

Definition 1.1. *[5]. Let X and Y be normed spaces and $T : X \to Y$ a linear operator. T is compact if for every bounded set A of X, $\overline{T(A)}$ is compact.*
The space of all compact operators of X into Y is denoted by $K(X, Y)$.

Uniformly invariant normed spaces are an another class of normed spaces which introduced by A. M. Forouzanfar et.al (see [4]). Authors of [4] have also introduced the useful notion compactly invariant subspace.

Definition 1.2. *A normed space X is called to be compactly invariant when for each closed subspace Y, there exists nonzero $T \in K(X)$ such that $T(Y) \subseteq Y$.*

In this paper we give more details on compactly invariant subspaces and then introduce the concept of compactly coinvariant subspaces.

2 Main Results

The purpose of this section mainly consists in proving some results on compactly invariant subspaces. To do so, we first give the following proposition. It shows that an isometrically isomorphisms preserve compactly invariant subspaces.

Proposition 2.1. *Let X and Y be normed spaces such that $X \cong Y$. If $Z \subset X$ is a compactly invariant subspace, then its isomorphic image is a compactly invariant subspace in Y.*

*Corresponding author.
: f.rostamzadeh_88@yahoo.com (Batool Rostamzadeh Kasrineh).

Proof. Let $f : X \to Y$ be an isometrically isomorphism and T a nonzero compact operator on X such that $T(Z) \subset Z$. Define $S : Y \to Y$ by $S(y) =: (f \circ T \circ f^{-1})(y)$. S is compact, since T is compact. Moreover, $S(f(Z)) \subset f(Z)$. □

The next theorem is deal with the existence of a compactly invariant subspace in the dual space of a normed space.

Theorem 2.1. *If Y is a compactly invariant subspace in X, then $Y^{\perp}$ is a compactly invariant subspace in X^*, where $Y^{\perp} = \{f \in X^* : \ f(y) = 0 \ \forall y \in Y\}$.*

Proof. By definition, there exist a nonzero compact operator T on X such that $T(Y) \subset Y$. It is well-known and easy to show that the linear operator $T^* : X^* \to X^*$ defined by $T^*(f) = f \circ T$ is compact [5]. Also, it can be seen easily that $T^*(Y^{\perp}) \subset Y^{\perp}$, as desired. □

Corollary 2.1. *If X has a compactly invariant subspace, then X^* has also.*

The previous corollary can be generalized for finite-dimensional spaces as follow:

Theorem 2.2. *Let X be a finite-dimensional space. Then $X^{(n)}$, nth dual of X, has a compactly invariant subspace, for every $n \in \mathbb{N}$.*

Proof. X has a compactly invariant subspace [4]. Since X is reflexive, so the canonical embedding $J : X \to X^{**}$ is surjective and so by Proposition 2.1, X^{**} has a compactly invariant subspace. In exactly the same way, since X^* has a compactly invariant subspace (Corollary 2.1), so X^{***} has also. By continuing the process, we get the desired conclusion. □

A natural question which arises here is that "is the intersection of two compactly invariant subspace, compactly invariant"?
This question inspire us to define the following notion.

Definition 2.3. *A pair (Y, Z) of subspaces of a normed space X is called compactly coinvariant, when there is a nonzero compact operator T on X such that $T(Y) \subset Y$ and $T(Z) \subset Z$.*

It is clear that each of the component of a pair (Y, Z) is compactly invariant.

Remark 2.1. *If Y is a compactly invariant subspace in X, then $(Y, \{0\})$ and (Y, X) are compactly coinvariant pairs. These pairs are called trivial. Note that if (Y, Z) is a compactly coinvariant pair, then $Y \cap Z$ is a compactly invariant subspace and so this can be a partial answer to the question which presented before of Definition 2.3.*

Proposition 2.2. *Let X has a compactly coinvariant pair, then X^* has also.*

Proof. Suppose that (Y, Z) is a compactly coinvariant pair of X. It can be seen easily that $(Y^{\perp}, Z^{\perp})$ is a compactly coinvariant pair of X^*. □

Corollary 2.2. *Let X be a reflexive space which has a compactly coinvariant pair. Then $X^{(n)}$ has a compactly coinvariant pair.*

Proof. Is similar to the proof of Corollary 2.1. □

Finally, we give a sufficient conditions for a normed space X which has a compactly coinvariant pair.

Theorem 2.3. *Let X be a normed space with a compactly invariant and complemented subspace Y such that $Y \cap Z = \{0\}$, for every subspace Z in X. Then X has a compactly coinvariant pair.*

Proof. Let $X = Y \oplus W$, where W is a subspace of X. There exist a nonzero operator T on X such that $T(Y) \subset Y$. On the other hand,

$$T(Y) \oplus T(W) \subset Y \oplus T(W)$$

Now since $T(W) \cap Y = \{0\}$, then $T(W) \subset W$. □

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