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# 3 Dimensional Additive Quadratic Functional Equation

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**Abstract**

In this paper, the authors established the general solution and generalized Ulam - Hyers stability of an 3 dimensional additive quadratic functional equation

$$\begin{aligned} h(x+2y+3z)+h(x+2y-3z)+h(x-2y+3z)+h(-x+2y+3z) \\ = h(x+y+z)+h(x+y-z)+h(x-y+z)+h(-x+y+z) \\ +2h(y)+4h(z)+5[h(y)+h(-y)]+14[h(z)+h(-z)] \end{aligned}$$

via Banach space and non-Archimedean fuzzy Banach Space using direct and fixed point methods.

*Keywords:* Additive functional equation, quadratic functional equation, mixed additive-quadratic functional equations, generalized Ulam - Hyers stability, Banach space, non-Archimedean fuzzy Banach space, fixed point

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## 1 Introduction

The study of stability problems for functional equations is related to a question of Ulam [51] concerning the stability of group homomorphisms was affirmatively answered for Banach spaces by Hyers [18]. Subsequently, this result of Hyers was generalized by Aoki [2] for additive mappings and by Rassias [42] for linear mappings by considering an unbounded Cauchy difference. The article of Rassias [42] has provided a lot of influence in the development of what we now call generalized Ulam-Hyers stability of functional It was further generalized via excellent results obtained by a number of authors [2, 17, 40, 42, 44]. The terminology generalized Ulam - Hyers stability originates from these historical backgrounds. These terminologies are also applied to the case of other functional equations. For more detailed definitions of such terminologies, one can refer to [1, 14, 19, 23, 25, 41, 43, 45] .

K.W. Jun and H.M. Kim [21] introduced the following generalized quadratic and additive type functional equation

$$f\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} f(x_i) = \sum_{1\le i<j\le n} f(x_i+x_j) \tag{1.1}$$

in real vector spaces. For $n = 3$, Pl.Kannappan proved that a function $f$ satisfies the functional equation (1.1) if and only if there exists a symmetric bi-additive function $B$ and an additive function $A$ such that $f(x) = B(x,x) + A(x)$ (see [24]). The Hyers-Ulam stability for the equation (1.1) when $n = 3$ was proved by S.M.

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Jung [22]. The Hyers-Ulam-Rassias stability for the equation (1.1) when $n = 4$ was also investigated by I.S. Chang et al., [13].

The general solution and the generalized Hyers-Ulam stability for the **quadratic and additive type functional equation**

$$f(x+ay) + af(x-y) = f(x-ay) + af(x+y) \tag{1.2}$$

for any positive integer $a$ with $a \neq -1, 0, 1$ was discussed by K.W. Jun and H.M. Kim [20]. Also, A. Najati and M.B. Moghimi [33] investigated the generalized Hyers-Ulam-Rassias stability for the **quadratic and additive type functional equation** of the form

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 2f(2x) - 4f(x) \tag{1.3}$$

Recently, the general solution and generalized Ulam - Hyers stability of a mixed type Additive Quadratic(AQ)-functional equation

$$g(x+y) + g(x-y) = 2g(x) + g(y) + g(-y). \tag{1.4}$$

was investigated by M. Arunkumar and J.M. Rassias [5].

In this paper, the authors established the general solution and generalized Ulam - Hyers stability of an 3 dimensional additive quadratic functional equation

$$\begin{aligned} &h(x+2y+3z) + h(x+2y-3z) + h(x-2y+3z) + h(-x+2y+3z) \\ &= h(x+y+z) + h(x+y-z) + h(x-y+z) + h(-x+y+z) \\ &\quad + 2h(y) + 4h(z) + 5[h(y)+h(-y)] + 14[h(z)+h(-z)] \end{aligned} \tag{1.5}$$

via Banach space and non-Archimedean fuzzy Banach Space using direct and fixed point methods. The above functional equation having

$$h(x) = ax + bx^2 \tag{1.6}$$

In Section 2 the the general solution of (1.5) is provided.

In Section 3 the generalized Ulam - Hyers stability of (1.5) is proved via Banach spaces using direct and fixed point methods.

In section 4 the generalized Ulam - Hyers stability of (1.5) is given in non-Archimedean fuzzy Banach Space using direct and fixed point methods.

## 2 General Solution of (1.5)

In this section, the general solution of the 3 dimensional additive quadratic functional equation (1.5) is given. For this, let us consider $A$ and $B$ be real vector spaces.

**Theorem 2.1.** *If $h : A \to B$ be an odd mapping satisfying (1.5) for all $x, y, z \in A$ if and only if $h : A \to B$ satisfying*

$$h(x+y) = h(x) + h(y) \tag{2.1}$$

*for all $x, y \in A$.*

*Proof.* Assume $h : A \to B$ be an odd mapping satisfying (1.5). Setting $(x, y, z)$ by $(0, 0, 0)$ in (1.5), we get $h(0) = 0$. Letting $z$ by 0 in (1.5) and using oddness of $h$, we obtain

$$h(x+2y) = h(x+y) + h(y) \tag{2.2}$$

for all $x, y \in A$. Setting $x$ by 0 and $x$ by $y$ in (2.2) respectively, we reach

$$h(2y) = 2h(y) \qquad h(3y) = 3h(y) \tag{2.3}$$

for all $y \in A$. In general, for any positive integer $a$, we have

$$h(ay) = ah(y) \tag{2.4}$$

for all $y \in A$. Replacing $x$ by $x-y$ in (2.2), we arrive (2.1) as desired.

Conversely, assume $h: A \to B$ be an odd mapping satisfying (2.1). Setting $(x,y)$ by $(0,0)$ in (2.1), we get $h(0)=0$. Setting $x$ by $y$ and $x$ by $2y$ in (2.1) respectively, we reach

$$h(2y) = 2h(y) \qquad h(3y) = 3h(y) \tag{2.5}$$

for all $y \in A$. In general, for any positive integer $b$, we have

$$h(by) = bh(y) \tag{2.6}$$

for all $y \in A$. Replacing $(x,y)$ by $(x+2y, 3z)$ in (2.1), using (2.1) and (2.5), we have

$$h(x+2y+3z) = h(x+2y) + h(3z) = h(x) + 2h(y) + 3h(z) \tag{2.7}$$

for all $x,y,z \in A$. Again replacing $(x,y)$ by $(x-2y, 3z)$ in (2.1), using (2.1), oddness of $h$ and (2.5), we get

$$h(x-2y+3z) = h(x-2y) + h(3z) = h(x) - 2h(y) + 3h(z) \tag{2.8}$$

for all $x,y,z \in A$. Setting $(x,y)$ by $(x+2y, -3z)$ in (2.1), using (2.1), oddness of $h$ and (2.5), we get

$$h(x+2y-3z) = h(x+2y) + h(-3z) = h(x) + 2h(y) - 3h(z) \tag{2.9}$$

for all $x,y,z \in A$. Again setting $(x,y)$ by $(-x+2y, 3z)$ in (2.1), using (2.1), oddness of $h$ and (2.5), we get

$$h(-x+2y+3z) = h(-x+2y) + h(3z) = -h(x) + 2h(y) + 3h(z) \tag{2.10}$$

for all $x,y,z \in A$. Adding (2.7), (2.8), (2.9) and (2.10), we reach

$$\begin{aligned} h(x+2y+3z) + h(x+2y-3z) + h(x-2y+3z) + h(-x+2y+3z) \\ = 2h(x) + 4h(y) + 6h(z) \end{aligned} \tag{2.11}$$

for all $x,y,z \in A$. Replacing $(x,y)$ by $(x+y, z)$ in (2.1), using (2.1) and (2.5), we have

$$h(x+y+z) = h(x+y) + h(z) = h(x) + h(y) + h(z) \tag{2.12}$$

for all $x,y,z \in A$. Again replacing $(x,y)$ by $(x-y, z)$ in (2.1), using (2.1), oddness of $h$ and (2.5), we get

$$h(x-y+z) = h(x-y) + h(z) = h(x) - h(y) + h(z) \tag{2.13}$$

for all $x,y,z \in A$. Setting $(x,y)$ by $(x+y, -z)$ in (2.1), using (2.1), oddness of $h$ and (2.5), we get

$$h(x+y-z) = h(x+y) + h(-z) = h(x) + h(y) - h(z) \tag{2.14}$$

for all $x,y,z \in A$. Again setting $(x,y)$ by $(-x+y, z)$ in (2.1), using (2.1), oddness of $h$ and (2.5), we get

$$h(-x+y+z) = h(-x+y) + h(z) = -h(x) + h(y) + h(z) \tag{2.15}$$

for all $x,y,z \in A$. Adding (2.12), (2.13), (2.14) and (2.15), we reach

$$\begin{aligned} h(x+y+z) + h(x+y-z) + h(x-y+z) + h(-x+y+z) \\ = 2h(x) + 2h(y) + 2h(z) \end{aligned} \tag{2.16}$$

for all $x,y,z \in A$. Using (2.16) in (2.11), we arrive

$$\begin{aligned} &h(x+2y+3z) + h(x+2y-3z) + h(x-2y+3z) + h(-x+2y+3z) \\ &= h(x+y+z) + h(x+y-z) + h(x-y+z) + h(-x+y+z) + 2h(y) + 4h(z) \end{aligned} \tag{2.17}$$

for all $x,y,z \in A$. Adding $5f(y) + 14f(z)$ on both sides of (2.17), we get

$$\begin{aligned} &h(x+2y+3z) + h(x+2y-3z) + h(x-2y+3z) + h(-x+2y+3z) + 5f(y) + 14f(z) \\ &= h(x+y+z) + h(x+y-z) + h(x-y+z) + h(-x+y+z) \\ &\qquad + 2h(y) + 4h(z) + 5f(y) + 14f(z) \end{aligned} \tag{2.18}$$

for all $x, y, z \in A$. It follows from (2.18) that

$$\begin{aligned} h(x+2y+3z) + h(x+2y-3z) + h(x-2y+3z) + h(-x+2y+3z) \\ = h(x+y+z) + h(x+y-z) + h(x-y+z) + h(-x+y+z) \\ + 2h(y) + 4h(z) + 5[f(y) - f(y)] + 14[f(z) - f(z)] \end{aligned} \tag{2.19}$$

for all $x, y, z \in A$. Using oddness of $h$ in (2.19), we desired our result. □

**Theorem 2.2.** *If $h : A \to B$ be an even mapping satisfying (1.5) for all $x, y, z \in A$ if and only if $h : A \to B$ satisfying*

$$h(x+y) + h(x-y) = 2h(x) + 2h(y) \tag{2.20}$$

*for all $x, y \in A$.*

*Proof.* Assume $h : A \to B$ be an even mapping satisfying (1.5). Setting $(x, y, z)$ by $(0, 0, 0)$ in (1.5), we get $h(0) = 0$. Letting $z$ by 0 in (1.5) and using evenness of $h$, we obtain

$$h(x+2y) + h(x-2y) = h(x+y) + h(x-y) + 6h(y) \tag{2.21}$$

for all $x, y \in A$. Setting $x$ by 0 and $x$ by $y$ in (2.21) respectively, we reach

$$h(2y) = 4h(y) \qquad h(3y) = 9h(y) \tag{2.22}$$

for all $y \in A$. In general, for any positive integer $c$, we have

$$h(cy) = c^2 h(y) \tag{2.23}$$

for all $y \in A$. Interchanging $x$ and $y$ in (2.21) and using evenness of $h$, we arrive

$$h(2x+y) + h(2x-y) = h(x+y) + h(x-y) + 6h(x) \tag{2.24}$$

for all $x, y \in A$. By Theorem 2.1 of [12], we desired our result.

Conversely, assume $h : A \to B$ be an even mapping satisfying (2.20). Setting $(x, y)$ by $(0, 0)$ in (2.20), we get $h(0) = 0$. Setting $x$ by $y$ and $x$ by $2y$ in (2.20) respectively, we reach

$$h(2y) = 4h(y) \qquad h(3y) = 9h(y) \tag{2.25}$$

for all $y \in A$. In general, for any positive integer $d$, we have

$$h(dy) = d^2 h(y) \tag{2.26}$$

for all $y \in A$. Replacing $(x, y)$ by $(x+2y, 3z)$ in (2.20) and using (2.25), we have

$$h(x+2y+3z) + h(x+2y-3z) = 2h(x+2y) + 2h(3z) = 2h(x+2y) + 18h(z) \tag{2.27}$$

for all $x, y, z \in A$. Again replacing $(x, y)$ by $(x-2y, 3z)$ in (2.20)and using (2.25), we get

$$h(x-2y+3z) + h(x-2y-3z) = 2h(x-2y) + 2h(3z) = 2h(x-2y) + 18h(z) \tag{2.28}$$

for all $x, y, z \in A$. Adding (2.27) and (2.28) and using (2.24), we reach

$$\begin{aligned} h(x+2y+3z) + h(x+2y-3z) + h(x-2y+3z) + h(-x+2y+3z) \\ = 2h(x+2y) + 2h(x-2y) + 36h(z) \\ = 2h(x+y) + 2h(x-y) + 12h(y) + 36h(z) \end{aligned} \tag{2.29}$$

for all $x, y, z \in A$. Replacing $(x, y)$ by $(x+y, z)$ in (2.20), we have

$$h(x+y+z) + h(x+y-z) = 2h(x+y) + 2h(z) \tag{2.30}$$

for all $x, y, z \in A$. Again replacing $(x, y)$ by $(x-y, z)$ in (2.20), we get

$$h(x-y+z) + h(x-y-z) = 2h(x-y) + 2h(z) \tag{2.31}$$

for all $x, y, z \in A$. Adding (2.30) and (2.31), we reach

$$\begin{aligned} & h(x+y+z)+h(x+y-z)+h(x-y+z)+h(-x+y+z) \\ & \quad = 2h(x+y)+2h(x-y)+4h(z) \end{aligned} \tag{2.32}$$

for all $x, y, z \in A$. Using (2.32) in (2.29) and using evenness of $h$, we arrive

$$\begin{aligned} & h(x+2y+3z)+h(x+2y-3z)+h(x-2y+3z)+h(-x+2y+3z) \\ & = h(x+y+z)+h(x+y-z)+h(x-y+z)+h(-x+y+z)+12h(y)+32h(z) \\ & = h(x+y+z)+h(x+y-z)+h(x-y+z)+h(-x+y+z)+2h(y)+4h(z) \\ & \quad + 5[h(y)+h(y)]+14[h(z)+h(z)] \end{aligned} \tag{2.33}$$

for all $x, y, z \in A$. Using evenness of $h$ in (2.33), we desired our result. □

# 3 Stability Results for (1.5) in Banach Space

In this section, the generalized Ulam - Hyers stability of the functional equation (1.5) is provided. Throughout this section, let us consider $X$ and $Y$ to be a normed space and a Banach space, respectively. Define a mapping $Dh : X \to Y$ by

$$\begin{aligned} Dh(x,y,z) = & h(x+2y+3z)+h(x+2y-3z)+h(x-2y+3z)+h(-x+2y+3z) \\ & -[h(x+y+z)+h(x+y-z)+h(x-y+z)+h(-x+y+z) \\ & +2h(y)+4h(z)+5[h(y)+h(-y)]+14[h(z)+h(-z)]] \end{aligned}$$

for all $x, y, z \in X$.

## 3.1 Banach Spaces: Stability Results: Hyers Direct Method

**Theorem 3.1.** *Let $j \in \{-1, 1\}$ and $\psi : X^3 \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \frac{\psi\left(6^{nj}x, 6^{nj}y, 6^{nj}z\right)}{6^{nj}} \text{ converges in } \mathbb{R} \quad \text{and} \quad \lim_{n\to\infty} \frac{\psi\left(6^{nj}x, 6^{nj}y, 6^{nj}z\right)}{6^{nj}} = 0 \tag{3.1}$$

*for all $x, y, z \in X$. Let $h_a : X \to Y$ be an odd function satisfying the inequality*

$$\|Dh_a(x,y,z)\| \leq \psi(x,y,z) \tag{3.2}$$

*for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ which satisfies (1.5) and*

$$\|h_a(x) - A(x)\| \leq \frac{1}{6} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Psi_A(6^{kj}x)}{6^{kj}} \tag{3.3}$$

*where $\Psi_A$ and $A(x)$ are respectively defined by*

$$\Psi_A(6^{kj}x) = \psi\left(6^{kj}x, 6^{kj}x, 6^{kj}x\right) + \frac{1}{2}\psi\left(6^{kj}2x, 6^{kj}x, 0\right) + \frac{1}{2}\psi\left(0, 6^{kj}x, 0\right) \tag{3.4}$$

$$A(x) = \lim_{n\to\infty} \frac{h_a(6^{nj}x)}{6^{nj}} \tag{3.5}$$

*for all $x \in X$.*

*Proof.* Assume $j = 1$. Replacing $(x, y, z)$ by $(x, x, x)$ in (3.2) and using oddness of $h_a$, we obtain

$$\|h_a(6x) + h_a(2x) + h_a(4x) - h_a(3x) - 9h_a(x)\| \leq \psi(x,x,x) \tag{3.6}$$

for all $x \in X$. Again replacing $(x, y, z)$ by $(2x, x, 0)$ in (3.2) and using oddness of $h_a$, we have

$$\|h_a(4x) - h_a(3x) - h_a(x)\| \leq \frac{1}{2}\psi(2x, x, 0) \tag{3.7}$$

for all $x \in X$. Finally replacing $(x, y, z)$ by $(0, x, 0)$ in (3.2) and using oddness of $h_a$, we get

$$\|h_a(2x) - 2h_a(x)\| \leq \frac{1}{2}\psi(0, x, 0) \tag{3.8}$$

for all $x \in X$. Now from (3.6), (3.7) and (3.8), we arrive

$$\begin{aligned} \|h_a(6x) - 6h_a(x)\| &= \|h_a(6x) + h_a(2x) + h_a(4x) - h_a(3x) - 9h_a(x) \\ &\qquad -h_a(4x) + h_a(3x) + h_a(x) - h_a(2x) + 2h_a(x)\| \\ &\leq \|h_a(6x) + h_a(2x) + h_a(4x) - h_a(3x) - 9h_a(x)\| \\ &\qquad \|h_a(4x) - h_a(3x) - h_a(x)\| + \|h_a(2x) - 2h_a(x)\| \\ &\leq \psi(x, x, x) + \frac{1}{2}\psi(2x, x, 0) + \frac{1}{2}\psi(0, x, 0) \end{aligned} \tag{3.9}$$

for all $x \in X$. It follows from (3.9) that

$$\|h_a(6x) - 6h_a(x)\| \leq \Psi_A(x) \tag{3.10}$$

where

$$\Psi_A(x) = \psi(x, x, x) + \frac{1}{2}\psi(2x, x, 0) + \frac{1}{2}\psi(0, x, 0) \tag{3.11}$$

for all $x \in X$. Divide (3.10) by 6, we get

$$\left\|h_a(x) - \frac{h_a(x)}{6}\right\| \leq \frac{1}{6}\Psi_A(x) \tag{3.12}$$

for all $x \in X$. Now replacing $x$ by $6x$ and dividing by 6 in (3.12), we obtain

$$\left\|\frac{h_a(6x)}{6} - \frac{h_a(6^2x)}{6^2}\right\| \leq \frac{\Psi_A(6x)}{6^2} \tag{3.13}$$

for all $x \in X$. It follows from (3.12) and (3.13) that

$$\begin{aligned} \left\|h_a(x) - \frac{h_a(6^2x)}{6^2}\right\| &\leq \left\|h_a(x) - \frac{h_a(6x)}{6}\right\| + \left\|\frac{h_a(6x)}{6} - \frac{h_a(6^2x)}{6^2}\right\| \\ &\leq \frac{1}{6}\left[\Psi_A(x) + \frac{\Psi_A(6x)}{6}\right] \end{aligned} \tag{3.14}$$

for all $x \in X$. Proceeding further and using induction on a positive integer $n$, we get

$$\begin{aligned} \left\|h_a(x) - \frac{h_a(6^nx)}{6^n}\right\| &\leq \frac{1}{6}\sum_{k=0}^{n-1}\frac{\Psi_A(6^kx)}{6^k} \\ &\leq \frac{1}{6}\sum_{k=0}^{\infty}\frac{\psi(6^kx)}{6^k} \end{aligned} \tag{3.15}$$

for all $x \in X$. In order to prove the convergence of the sequence

$$\left\{\frac{h_a(6^nx)}{6^n}\right\},$$

replace $x$ by $6^mx$ and divide by $6^m$ in (3.15), for any $m, n > 0$ , we arrive the sequence $\left\{\frac{h_a(6^nx)}{6^n}\right\}$ is a Cauchy sequence. Since $Y$ is complete, there exists a mapping $A : X \to Y$ such that

$$A(x) = \lim_{n\to\infty}\frac{h_a(6^nx)}{6^n}, \quad \forall\ x \in X.$$

Letting $n \to \infty$ in (3.15) we see that (3.3) holds for all $x \in X$. To prove $A$ satisfies (1.5), replacing $(x, y, z)$ by $(6^nx, 6^ny, 6^nz)$ and dividing by $6^n$ in (3.2), we obtain

$$\frac{1}{6^n}\left\|Dh_a(6^nx, 6^ny, 6^nz)\right\| \leq \frac{1}{6^n}\psi(6^nx, 6^ny, 6^nz)$$

for all $x, y, z \in X$. Letting $n \to \infty$ in the above inequality and using the definition of $A(x)$, we see that

$$DA(x,y,z) = 0.$$

for all $x, y, z \in X$. Hence $A$ satisfies (1.5) for all $x, y, z \in X$. To show $A$ is unique, let $B(x)$ be another additive mapping satisfying (1.5) and (3.3), then

$$\begin{aligned}\|A(x) - B(x)\| &= \frac{1}{6^n}\|A(6^n x) - B(6^n x)\| \\ &\leq \frac{1}{6^n}\left\{\|A(6^n x) - h_a(6^n x)\| + \|h_a(6^n x) - B(6^n x)\|\right\} \\ &\leq \frac{1}{3}\sum_{k=0}^{\infty}\frac{\Psi_A(6^{k+n}x)}{6^{(k+n)}} \\ &\to 0 \quad as \;\; n \to \infty\end{aligned}$$

for all $x \in X$. Hence $A$ is unique.

For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.5).

**Corollary 3.1.** *Let $\theta$ and $s$ be non negative real numbers. Let an odd function $h_a : X \to Y$ satisfy the inequality*

$$\|Dh_a(x,y,z)\| \leq \begin{cases} \theta, & \\ \theta\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 1; \\ \theta\left\{||x||^s||y||^s||z||^s\right\}, & s \neq 1; \\ \theta\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 1; \end{cases} \tag{3.16}$$

*for all $x, y, z \in X$. Then there exists a unique additive function $A : X \to Y$ such that*

$$\|h_a(x) - A(x)\| \leq \begin{cases} \dfrac{2\theta}{5}, \\ \dfrac{(8+2^s)\theta||x||^s}{2|6-6^s|}, \\ \dfrac{\theta||x||^{3s}}{|6-6^{3s}|}, \\ \dfrac{(10+2^s)\theta||x||^s}{2|6-6^{3s}|}, \end{cases} \tag{3.17}$$

*for all $x \in X$.*

**Theorem 3.2.** *Let $j \in \{-1, 1\}$ and $\psi : X^3 \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty}\frac{\psi\left(6^{nj}x, 6^{nj}y, 6^{nj}z\right)}{36^{nj}} \;\; converges \;\; in \;\; \mathbb{R} \quad and \quad \lim_{n\to\infty}\frac{\psi\left(6^{nj}x, 6^{nj}y, 6^{nj}z\right)}{36^{nj}} = 0 \tag{3.18}$$

*for all $x, y, z \in X$. Let $h_q : X \to Y$ be an even function satisfying the inequality*

$$\left\|Dh_q(x,y,z)\right\| \leq \psi(x,y,z) \tag{3.19}$$

*for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies (1.5) and*

$$\left\|h_q(x) - Q(x)\right\| \leq \frac{1}{36}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Psi_Q(6^{kj}x)}{36^{kj}} \tag{3.20}$$

*where $\Psi_Q$ and $A(x)$ are respectively defined by*

$$\Psi_Q(6^{kj}x) = \psi\left(6^{kj}x, 6^{kj}x, 6^{kj}x\right) + \frac{1}{2}\psi\left(6^{kj}2x, 6^{kj}x, 0\right) + \frac{1}{4}\psi\left(0, 6^{kj}x, 0\right) \tag{3.21}$$

$$Q(x) = \lim_{n\to\infty}\frac{h_q(6^{nj}x)}{36^{nj}} \tag{3.22}$$

*for all $x \in X$.*

*Proof.* Assume $j = 1$. Replacing $(x, y, z)$ by $(x, x, x)$ in (3.19) and using evenness of $h_q$, we get

$$\left\|h_q(6x) + h_q(2x) + h_q(4x) - h_q(3x) - 47h_q(x)\right\| \leq \psi\left(x, x, x\right) \tag{3.23}$$

for all $x \in X$. Again replacing $(x, y, z)$ by $(2x, x, 0)$ in (3.19) and using evenness of $h_q$, we get

$$\left\|h_q(4x) - h_q(3x) - 7h_q(x)\right\| \leq \frac{1}{2}\psi\left(2x, x, 0\right) \tag{3.24}$$

for all $x \in X$. Finally replacing $(x, y, z)$ by $(0, x, 0)$ in (3.19) and using oddness of $h_q$, we get

$$\left\|h_q(2x) - 4h_q(x)\right\| \leq \frac{1}{4}\psi\left(0, x, 0\right) \tag{3.25}$$

for all $x \in X$. Now from (3.23), (3.24) and (3.25), we arrive

$$\begin{aligned}
\left\|h_q(6x) - 36h_q(x)\right\| &= \left\|h_q(6x) + h_q(2x) + h_q(4x) - h_q(3x) - 47h_q(x)\right. \\
&\qquad \left. -h_q(4x) + h_q(3x) + 7h_q(x) - h_q(2x) + 4h_q(x)\right\| \\
&\leq \left\|h_q(6x) + h_q(2x) + h_q(4x) - h_q(3x) - 47h_q(x)\right\| \\
&\qquad \left\|h_q(4x) - h_q(3x) - 7h_q(x)\right\| + \left\|h_q(2x) - 4h_q(x)\right\| \\
&\leq \psi\left(x, x, x\right) + \frac{1}{2}\psi\left(2x, x, 0\right) + \frac{1}{4}\psi\left(0, x, 0\right)
\end{aligned} \tag{3.26}$$

for all $x \in X$. It follows from (3.26) that

$$\left\|h_q(6x) - 36h_q(x)\right\| \leq \Psi_Q(x) \tag{3.27}$$

where

$$\Psi_Q(x) = \psi\left(x, x, x\right) + \frac{1}{2}\psi\left(2x, x, 0\right) + \frac{1}{4}\psi\left(0, x, 0\right) \tag{3.28}$$

for all $x \in X$. Divide (3.27) by 6, we get

$$\left\|h_q(x) - \frac{h_q(6x)}{36}\right\| \leq \frac{1}{36}\Psi_Q(x) \tag{3.29}$$

for all $x \in X$. The rest of the proof is similar lines to that of Theorem 3.1. □

The following Corollary is an immediate consequence of Theorem 3.2 concerning the stability of (1.5).

**Corollary 3.2.** *Let $\theta$ and $s$ be non negative real numbers. Let an even function $h_q : X \to Y$ satisfy the inequality*

$$\left\|Dh_q(x, y, z)\right\| \leq \begin{cases} \theta, & \\ \theta\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 2; \\ \theta\left\{||x||^s||y||^s||z||^s\right\}, & 3s \neq 2; \\ \theta\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 2; \end{cases} \tag{3.30}$$

*for all $x, y, z \in X$. Then there exists a unique quadratic function $Q : X \to Y$ such that*

$$\left\|h_q(x) - Q(x)\right\| \leq \begin{cases} \dfrac{9\theta}{35}, \\ \dfrac{(15 + 2^{s+1})\theta||x||^s}{4|36 - 6^s|}, \\ \dfrac{\theta||x||^{3s}}{|36 - 6^{3s}|}, \\ \dfrac{(19 + 2^{s+1})\theta||x||^{3s}}{4|36 - 6^{3s}|}, \end{cases} \tag{3.31}$$

*for all $x \in X$.*

Now we are ready to prove our main theorem.

**Theorem 3.3.** *Let $j \in \{-1,1\}$ and $\psi : X^3 \to [0,\infty)$ be a function satisfying (3.1) and (3.18) for all $x,y,z \in X$. Let $h : X \to Y$ be a function satisfying the inequality*

$$\|Dh(x,y,z)\| \leq \psi(x,y,z) \tag{3.32}$$

*for all $x,y,z \in X$. Then there exists a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ which satisfies (1.5) and*

$$\|h(x) - A(x) - Q(x)\| \leq \frac{1}{2}\left[\frac{1}{6}\sum_{k=\frac{1-j}{6}}^{\infty}\left(\frac{\Psi_A(6^{kj}x)}{6^{kj}} + \frac{\Psi_A(-6^{kj}x)}{6^{kj}}\right)\right.$$
$$\left. + \frac{1}{36}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\Psi_Q(6^{kj}x)}{36^{kj}} + \frac{\Psi_Q(-6^{kj}x)}{36^{kj}}\right)\right] \tag{3.33}$$

*for all $x \in X$. The mapping $A(x)$ and $Q(x)$ are defined in (3.4) and (3.21) respectively for all $x \in X$.*

*Proof.* Let $h_o(x) = \dfrac{h_a(x) - h_a(-x)}{2}$ for all $x \in X$. Then $h_o(0) = 0$ and $h_o(-x) = -h_o(x)$ for all $x \in X$. Hence

$$\|Dh_o(x,y,z)\| \leq \frac{\psi(x,y,z)}{2} + \frac{\psi(-x,-y,-z)}{2} \tag{3.34}$$

for all $x,y,z \in X$. By Theorem 3.1, we have

$$\|h_o(x) - A(x)\| \leq \frac{1}{12}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\Psi_A(6^{kj}x)}{6^{kj}} + \frac{\Psi_A(-6^{kj}x)}{6^{kj}}\right) \tag{3.35}$$

for all $x \in X$. Also, let $h_e(x) = \dfrac{h_q(x) + h_q(-x)}{2}$ for all $x \in X$. Then $h_e(0) = 0$ and $h_e(-x) = h_e(x)$ for all $x \in X$. Hence

$$\|Dh_e(x,y,z)\| \leq \frac{\psi(x,y,z)}{2} + \frac{\psi(-x,-y,-z)}{2} \tag{3.36}$$

for all $x,y,z \in X$. By Theorem 3.2, we have

$$\|h_e(x) - Q(x)\| \leq \frac{1}{72}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\Psi_Q(6^{kj}x)}{36^{kj}} + \frac{\Psi_Q(-6^{kj}x)}{36^{kj}}\right) \tag{3.37}$$

for all $x \in X$. Define

$$h(x) = h_e(x) + h_o(x) \tag{3.38}$$

for all $x \in X$. From (3.35),(3.37) and (3.38), we arrive

$$\begin{aligned}\|h(x) - A(x) - Q(x)\| &= \|h_e(x) + h_o(x) - A(x) - Q(x)\| \\ &\leq \|h_o(x) - A(x)\| + \|h_e(x) - Q(x)\| \\ &\leq \frac{1}{12}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\Psi_A(6^{kj}x)}{6^{kj}} + \frac{\Psi_A(-6^{kj}x)}{6^{kj}}\right) \\ &\quad + \frac{1}{72}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\Psi_Q(6^{kj}x)}{36^{kj}} + \frac{\Psi_Q(-6^{kj}x)}{36^{kj}}\right)\end{aligned}$$

for all $x \in X$. Hence the theorem is proved. □

Using Corollaries 3.1 and 3.2, we have the following Corollary concerning the stability of (1.5).

**Corollary 3.3.** *Let $\theta$ and $s$ be non negative real numbers. Let a function $h : X \to Y$ satisfy the inequality*

$$\|Dh(x,y,z)\| \le \begin{cases} \theta, & \\ \theta\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 1,2; \\ \theta\left\{||x||^s||y||^s||z||^s\right\}, & 3s \neq 1,2; \\ \theta\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 1,2; \end{cases} \tag{3.39}$$

*for all $x,y,z \in X$. Then there exists a unique additive function $\mathcal{A} : X \to Y$ and a unique quadratic function $\mathcal{Q} : X \to Y$ such that*

$$\|h(x) - \mathcal{A}(x) - \mathcal{Q}(x)\| \le \begin{cases} \theta\left(\frac{2}{5} + \frac{9\theta}{35}\right), \\ \theta||x||^s\left(\frac{(8+2^s)}{2|6-6^s|} + \frac{(15+2^{s+1})}{4|36-6^s|}\right), \\ \theta||x||^{3s}\left(\frac{1}{|6-6^{3s}|} + \frac{1}{|36-6^{3s}|}\right), \\ \theta||x||^{3s}\left(\frac{(10+2^s)}{2|6-6^{3s}|} + \frac{(19+2^{s+1})}{4|36-6^{3s}|}\right) \end{cases} \tag{3.40}$$

*for all $x \in X$.*

## 3.2 Banach Space : Stability Results : Fixed Point Method

In this section, the generalized Ulam - Hyers stability of the functional equation (1.5) is proved using fixed point method provided.

Throughout this section let $\mathcal{X}$ be a normed space and $\mathcal{Y}$ be a Banach space Define a mapping $Dh : \mathcal{X} \to \mathcal{Y}$ by

$$\begin{aligned} Dh(x,y,z) =& h(x+2y+3z) + h(x+2y-3z) + h(x-2y+3z) + h(-x+2y+3z) \\ & - h(x+y+z) - h(x+y-z) - h(x-y+z) - h(-x+y+z) \\ & - 2h(y) - 4h(z) - 5[h(y)+h(-y)] - 14[h(z)+h(-z)] \end{aligned}$$

for all $x,y,z \in \mathcal{X}$.

Now, we present the following theorem due to B. Margolis and J.B. Diaz [27] for fixed point theory.

**Theorem 3.4.** *[27] Suppose that for a complete generalized metric space $(\Omega, d)$ and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$ , either*

$$d(T^n x, T^{n+1}x) = \infty, \qquad \forall \quad n \ge 0,$$

*or there exists a natural number $n_0$ such that the properties hold:*

*(FP1)* $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \ge n_0$ ;

*(FP2) The sequence $(T^n x)$ is convergent to a fixed to a fixed point $y^*$ of $T$;*

*(FP3) $y^*$ is the unique fixed point of $T$ in the set $\Delta = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;*

*(FP4) $d(y^*, y) \le \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.*

Using the above theorem, we obtain the Hyers - Ulam stability of (1.5).

**Theorem 3.5.** *Let $Dh_a : \mathcal{X} \to \mathcal{Y}$ be a mapping for which there exists a function $\psi : \mathcal{X}^3 \to [0,\infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\rho_i^n} \psi(\rho_i^n x, \rho_i^n y, , \rho_i^n z) = 0 \tag{3.41}$$

*where*

$$\rho_i = \begin{cases} 6 & if \quad i = 0, \\ \frac{1}{6} & if \quad i = 1 \end{cases} \tag{3.42}$$

*such that the functional inequality*

$$\|Dh_a(x,y,z)\| \leq \psi(x,y,z) \tag{3.43}$$

*holds for all $x, y \in \mathcal{X}$. Assume that there exists $L = L(i)$ such that the function*

$$x \to \Phi_{AQ}(x) = \Psi_A\left(\frac{x}{6}\right)$$

*where $\Psi_A(x)$ is defined in (3.11) with the property*

$$\frac{1}{\rho_i}\Phi_{AQ}(\rho_i x) = L\,\Phi_{AQ}(x) \tag{3.44}$$

*for all $x \in \mathcal{X}$. Then there exists a unique additive mapping $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ satisfying the functional equation (1.5) and*

$$\| h_a(x) - \mathcal{A}(x) \| \leq \left(\frac{L^{1-i}}{1-L}\right)\Phi_{AQ}(x) \tag{3.45}$$

*for all $x \in \mathcal{X}$.*

*Proof.* Consider the set

$$\Omega = \{h/h : \mathcal{X} \to \mathcal{Y},\ h(0) = 0\}$$

and introduce the generalized metric on $\Omega$,

$$\inf\{M \in (0,\infty) : \| h(x) - g(x) \| \leq M\,\Phi_{AQ}(x), x \in \mathcal{X}\}. \tag{3.46}$$

It is easy to see that (3.46) is complete with respect to the defined metric. Define $J : \Omega \to \Omega$ by

$$Jh(x) = \frac{1}{\rho_i}h(\rho_i x),$$

for all $x \in \mathcal{X}$. Now, from (3.46) and $h, g \in \Omega$

$$\begin{aligned}
&\inf\left\{M \in (0,\infty) : \| h(x) - g(x) \| \leq M\,\Phi_{AQ}(x), x \in \mathcal{X}\right\} \quad or\\
&\inf\left\{M \in (0,\infty) : \left\|\frac{1}{\rho_i}h(\rho_i x) - \frac{1}{\rho_i}g(\rho_i x)\right\| \leq \frac{M}{\rho_i}\Phi_{AQ}(\rho_i x), x \in \mathcal{X}\right\} \quad or\\
&\inf\left\{LM \in (0,\infty) : \left\|\frac{1}{\rho_i}h(\rho_i x) - \frac{1}{\rho_i}g(\rho_i x)\right\| \leq LM\Phi_{AQ}(x), x \in \mathcal{X}\right\} \quad or\\
&\inf\left\{LM \in (0,\infty) : \| Jh(x) - Jg(x) \|_{\mathcal{Y}} \leq LM\Phi_{AQ}(x), x \in \mathcal{X}\right\}.
\end{aligned}$$

This implies $J$ is a strictly contractive mapping on $\Omega$ with Lipschitz constant $L$. It follows from (3.46),(3.10) and (3.44) for the case $i = 0$, we reach

$$\begin{aligned}
&\inf\{1 \in (0,\infty) : \|h_a(6x) - 6h_a(x)\| \leq \Psi_A(x), x \in \mathcal{X}\} \quad or &(3.47)\\
&\inf\left\{1 \in (0,\infty) : \left\|\frac{h_a(x)}{6} - h_a(x)\right\| \leq \frac{1}{6}\Psi_A(x), x \in \mathcal{X}\right\} \quad or\\
&\inf\left\{L \in (0,\infty) : \|Jh_a f(x) - h_a(x)\| \leq L\,\Phi_{AQ}(x), x \in \mathcal{X}\right\} \quad or\\
&\inf\left\{L^1 \in (0,\infty) : \|Jh_a f(x) - h_a(x)\| \leq L\,\Phi_{AQ}(x), x \in \mathcal{X}\right\} \quad or\\
&\inf\left\{L^{1-0} \in (0,\infty) : \|Jh_a f(x) - h_a(x)\| \leq L\,\Phi_{AQ}(x), x \in \mathcal{X}\right\}. &(3.48)
\end{aligned}$$

Again replacing $x = \frac{x}{6}$ in (3.47) and (3.44) for the case $i = 1$ we get

$$\begin{aligned}
&\inf\left\{1 \in (0,\infty) : \left\|h_a(x) - 6h_a\left(\frac{x}{6}\right)\right\| \leq \Psi_A\left(\frac{x}{6}\right), x \in \mathcal{X}\right\} \quad or\\
&\inf\left\{1 \in (0,\infty) : \|h_a f(x) - Jh_a(x)\| \leq \Phi_{AQ}(x), x \in \mathcal{X}\right\} \quad or\\
&\inf\left\{L^0 \in (0,\infty) : \|h_a f(x) - Jh_a(x)\| \leq \Phi_{AQ}(x), x \in \mathcal{X}\right\} \quad or\\
&\inf\left\{L^{1-1} \in (0,\infty) : \|h_a f(x) - Jh_a(x)\| \leq \Phi_{AQ}(x), x \in \mathcal{X}\right\}. &(3.49)
\end{aligned}$$

Thus, from (4.8) and (3.49), we arrive

$$\inf\left\{L^{1-i} \in (0,\infty) : \|h_a f(x) - Jh_a(x)\| \le L^{1-i}\Phi_{AQ}(x), x \in \mathcal{X}\right\}. \tag{3.50}$$

Hence property (FP1) holds. It follows from property (FP2) that there exists a fixed point $\mathcal{A}$ of $J$ in $\Omega$ such that

$$\mathcal{A}(x) = \lim_{n\to\infty} \frac{1}{\rho_i^n} h_a(\rho_i^n x) \tag{3.51}$$

for all $x \in \mathcal{X}$. In order to show that $\mathcal{A}$ satisfies (1.5), replacing $(x,y,z)$ by $(\rho_i^n x, \rho_i^n y, \rho_i^n z)$ and dividing by $\rho_i^n$ in (3.43), we have

$$\|\mathcal{A}(x,y,z)\| = \lim_{n\to\infty} \frac{1}{\rho_i^n} \|Dh_a(\rho_i^n x, \rho_i^n y, \rho_i^n z)\| \le \lim_{n\to\infty} \frac{1}{\rho_i^n} \psi(\rho_i^n x, \rho_i^n y, \rho_i^n z) = 0$$

for all $x,y,z \in \mathcal{X}$, and so the mapping $\mathcal{A}$ is Additive. i.e., $\mathcal{A}$ satisfies the functional equation (1.5).

By property (FP3), $\mathcal{A}$ is the unique fixed point of $J$ in the set $\Delta = \{\mathcal{A} \in \Omega : d(h_a, \mathcal{A}) < \infty\}$, $\mathcal{A}$ is the unique function such that

$$\inf\left\{M \in (0,\infty) : \|h_a(x) - \mathcal{A}(x)\| \le M\Phi_{AQ}(x), x \in \mathcal{X}\right\}.$$

Finally by property (FP4), we obtain

$$\|h_a(x) - \mathcal{A}(x)\| \le \|h_a(x) - Jh_a(x)\|$$

this implies

$$\|h_a(x) - \mathcal{A}(x)\| \le \frac{L^{1-i}}{1-L}$$

which yields

$$\inf\left\{\frac{L^{1-i}}{1-L} \in (0,\infty) : \|h_a(x) - \mathcal{A}(x)\| \le \left(\frac{L^{1-i}}{1-L}\right)\Phi_{AQ}(x), x \in \mathcal{X}\right\}$$

this completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.5 concerning the stability of (1.5).

**Corollary 3.4.** *Let $Dh_a : \mathcal{X} \to \mathcal{Y}$ be a mapping. If there exist real numbers $\lambda$ and $s$ such that*

$$\|Dh_a f(x,y,z)\| \le \begin{cases} \lambda, & \\ \lambda\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \ne 1; \\ \lambda\left\{||x||^s||y||^s||z||^s\right\}, & 3s \ne 1; \\ \lambda\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \ne 1; \end{cases} \tag{3.52}$$

*for all $x,y,z \in \mathcal{X}$, then there exists a unique additive function $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(x) - \mathcal{A}(x)\| \le \begin{cases} \frac{2\lambda}{5}, \\ \frac{(8+2^s)\lambda||x||^s}{2|6-6^s|}, \\ \frac{\lambda||x||^{3s}}{|6-6^{3s}|}, \\ \frac{(10+2^s)\lambda||x||^s}{2|6-6^{3s}|}, \end{cases} \tag{3.53}$$

*for all $x \in \mathcal{X}$.*

*Proof.* Let

$$\psi(x,y,z) = \begin{cases} \lambda, \\ \lambda\left\{||x||^s + ||y||^s + ||z||^s\right\}, \\ \lambda\left\{||x||^s||y||^s||z||^s\right\}, \\ \lambda\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\} \end{cases}$$

for all $x, y, z \in \mathcal{X}$. Now

$$\frac{1}{\rho_i^n}\psi(\rho_i^n x, \rho_i^n y, \rho_i^n z) = \begin{cases} \dfrac{\lambda}{\rho_i^n}, \\ \dfrac{\lambda}{\rho_i^n}\left\{||\rho_i^n x||^s + ||\rho_i^n y||^s + ||\rho_i^n z||^s\right\}, \\ \dfrac{\lambda}{\rho_i^n}||\rho_i^n x||^s \, ||\rho_i^n y||^s \, ||\rho_i^n z||^s, \\ \dfrac{\lambda}{\rho_i^n}\left\{||\rho_i^n x||^s \, ||\rho_i^n y||^s \, ||\rho_i^n z||^s \right. \\ \qquad \left. + \left\{||\rho_i^n x||^{3s} + ||\rho_i^n y||^{3s} + ||\rho_i^n z||^{3s}\right\}\right\}, \end{cases} = \begin{cases} \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \\ \to 0 \text{ as } n \to \infty. \end{cases}$$

Thus, (3.41) holds. But, we have

$$\Phi_{AQ}(x) = \Psi_A(\frac{x}{6})$$

has the property

$$\frac{1}{\rho_i}\Phi_{AQ}(\rho_i x) = L\,\Phi_{AQ}(x)$$

for all $x \in \mathcal{X}$. Hence

$$\begin{aligned}\Phi_{AQ}(x) = \Psi_A\left(\frac{x}{6}\right) &= \psi\left(\frac{x}{6},\frac{x}{6},\frac{x}{6}\right) + \frac{1}{2}\psi\left(\frac{2x}{6},\frac{x}{6},0\right) + \frac{1}{2}\psi\left(0,\frac{x}{6},0\right) \\ &= \begin{cases} 2\lambda, \\ \dfrac{\lambda}{6^s}\left(\dfrac{8+2^s}{2}\right)\|x\|^s, \\ \dfrac{\lambda}{6^{3s}}\|x\|^{3s}, \\ \dfrac{\lambda}{6^{3s}}\left(\dfrac{10+2^{3s}}{2}\right)\|x\|^{3s}. \end{cases}\end{aligned}$$

Now,

$$\frac{1}{\rho_i}\Phi_{AQ}(\rho_i x) = \begin{cases} \rho_i^{-1}\,2\lambda, \\ \rho_i^{s-1}\dfrac{\lambda}{6^s}\left(\dfrac{8+2^s}{2}\right)\|x\|^s, \\ \rho_i^{3s-1}\dfrac{\lambda}{6^{3s}}\|x\|^{3s} \\ \rho_i^{3s-1}\dfrac{\lambda}{6^{3s}}\left(\dfrac{10+2^{3s}}{2}\right)\|x\|^{3s}, \end{cases} = \begin{cases} \rho_i^{-1}\,\Phi_{AQ}(x), \\ \rho_i^{s-1}\Phi_{AQ}(x), \\ \rho_i^{3s-1}\Phi_{AQ}(x), \\ \rho_i^{3s-1}\Phi_{AQ}(x). \end{cases}$$

Hence, the inequality (3.45) holds for

(i). $L = \rho_i^{-1}$ if $i = 0$ and $L = \frac{1}{\rho_i^{-1}}$ if $i = 1$;

(ii). $L = \rho_i^{s-1}$ for $s < 1$ if $i = 0$ and $L = \frac{1}{\rho_i^{s-1}}$ for $s > 1$ if $i = 1$;

(iii). $L = \rho_i^{3s-1}$ for $3s > 1$ if $i = 0$ and $L = \frac{1}{\rho_i^{3s-1}}$ for $3s > 1$ if $i = 1$;

(iv). $L = \rho_i^{3s-1}$ for $3s > 1$ if $i = 0$ and $L = \frac{1}{\rho_i^{3s-1}}$ for $3s > 1$ if $i = 1$.

Now, from (3.45), we prove the following cases for condition $(i)$.

$$\begin{aligned} &L = \rho_i^{-1}, i = 0 \\ &L = 6^{-1}, i = 0 \\ &\| f(x) - \mathcal{A}(x) \| \\ &\qquad \le \left(\tfrac{L^{1-i}}{1-L}\right)\Phi_{AQ}(x) \\ &\qquad = \left(\tfrac{(6^{-1})^{1-0}}{1-6^{-1}}\right)2\lambda \\ &\qquad = \left(\tfrac{6^{-1}}{1-6^{-1}}\right)2\lambda \\ &\qquad = \left(\tfrac{2\lambda}{5}\right) \end{aligned}$$

$$\begin{aligned} &L = \tfrac{1}{\rho_i^{-1}}, i = 1 \\ &L = 6, i = 1 \\ &\| f(x) - \mathcal{A}(x) \| \\ &\qquad \le \left(\tfrac{L^{1-i}}{1-L}\right)\Phi_{AQ}(x) \\ &\qquad = \left(\tfrac{6^{1-1}}{1-6}\right)2\lambda \\ &\qquad = \left(\tfrac{1}{1-6}\right)2\lambda \\ &\qquad = \left(\tfrac{2\lambda}{-5}\right) \end{aligned}$$

Also, from (3.45), we prove the following cases for condition $(ii)$.

$$L = \rho_i^{s-1}, s < 1, i = 0 \qquad L = \frac{1}{\rho_i^{s-1}}, s > 1, i = 1$$

$$L = 6^{s-1}, s < 1, i = 0 \qquad L = 6^{1-s}, s > 1, i = 1$$

$$\begin{aligned} \| f(x) - \mathcal{A}(x) \| &\leq \left(\frac{L^{1-i}}{1-L}\right) \Phi_{AQ}(x) \\ &= \left(\frac{(6^{s-1})^{1-0}}{1-6^{s-1}}\right) \frac{\lambda}{6^s} \left(\frac{8+2^s}{2}\right) \|x\|^s \\ &= \left(\frac{6^{s-1}}{1-6^{s-1}}\right) \frac{\lambda}{6^s} \left(\frac{8+2^s}{2}\right) \|x\|^s \\ &= \left(\frac{6^s}{6-6^s}\right) \frac{\lambda}{6^s} \left(\frac{8+2^s}{2}\right) \|x\|^s \\ &= \left(\frac{\lambda}{6-6^s}\right) \left(\frac{8+2^s}{2}\right) \|x\|^s \end{aligned} \qquad \begin{aligned} \| f(x) - \mathcal{A}(x) \| &\leq \left(\frac{L^{1-i}}{1-L}\right) \Phi_{AQ}(x) \\ &= \left(\frac{(6^{1-s})^{1-1}}{1-6^{1-s}}\right) \frac{\lambda}{6^s} \left(\frac{8+2^s}{2}\right) \|x\|^s \\ &= \left(\frac{1}{1-6^{1-s}}\right) \frac{\lambda}{6^s} \left(\frac{8+2^s}{2}\right) \|x\|^s \\ &= \left(\frac{6^s}{6^s-6}\right) \frac{\lambda}{6^s} \left(\frac{8+2^s}{2}\right) \|x\|^s \\ &= \left(\frac{\lambda}{6^s-6}\right) \left(\frac{8+2^s}{2}\right) \|x\|^s \end{aligned}$$

Again, from (3.45), we prove the following cases for condition $(iii)$.

$$L = \rho_i^{3s-1}, 3s < 1, i = 0 \qquad L = \frac{1}{\rho_i^{3s-1}}, 3s > 1, i = 1$$

$$L = 6^{3s-1}, 3s < 1, i = 0 \qquad L = 6^{1-3s}, 3s > 1, i = 1$$

$$\begin{aligned} \| f(x) - \mathcal{A}(x) \| &\leq \left(\frac{L^{1-i}}{1-L}\right) \Phi_{AQ}(x) \\ &= \left(\frac{(6^{3s-1})^{1-0}}{1-6^{3s-1}}\right) \frac{\lambda}{6^{3s}} \|x\|^{3s} \\ &= \left(\frac{6^{3s-1}}{1-6^{3s-1}}\right) \frac{\lambda}{6^{3s}} \|x\|^{3s} \\ &= \left(\frac{6^{3s}}{6-6^{3s}}\right) \frac{\lambda}{6^{3s}} \|x\|^{3s} \\ &= \left(\frac{\lambda}{6-6^{3s}}\right) \end{aligned} \qquad \begin{aligned} \| f(x) - \mathcal{A}(x) \| &\leq \left(\frac{L^{1-i}}{1-L}\right) \Phi_{AQ}(x) \\ &= \left(\frac{(6^{1-3s})^{1-1}}{1-6^{1-3s}}\right) \frac{\lambda}{6^{3s}} \|x\|^{3s} \\ &= \left(\frac{1}{1-6^{1-3s}}\right) \frac{\lambda}{6^{3s}} \|x\|^{3s} \\ &= \left(\frac{6^{3s}}{6^{3s}-6}\right) \frac{\lambda}{6^{3s}} \|x\|^{3s} \\ &= \left(\frac{\lambda}{6^{3s}-6}\right) \end{aligned}$$

Finally, to prove (3.45) for condition $(iv)$, the proof is similar to that of condition $(iii)$. Hence the proof is complete. □

The proof of the following theorems and corollaries is similar to that of Theorems 3.5, 3.3 and Corollaries 3.4, 3.3.

**Theorem 3.6.** *Let $Dh_q : \mathcal{X} \to \mathcal{Y}$ be a mapping for which there exists a function $\psi : \mathcal{X}^3 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\rho_i^{2n}} \psi(\rho_i^n x, \rho_i^n y, \rho_i^n z) = 0 \tag{3.54}$$

*where*

$$\rho_i = \begin{cases} 6 & if \quad i = 0, \\ \frac{1}{6} & if \quad i = 1 \end{cases} \tag{3.55}$$

*such that the functional inequality*

$$\left\|Dh_q(x, y, z)\right\|_{\mathcal{Y}} \leq \psi(x, y, z) \tag{3.56}$$

*holds for all $x, y \in \mathcal{X}$. Assume that there exists $L = L(i) = \frac{1}{6}$ such that the function*

$$x \to \Phi_{AQ}(x) = \frac{1}{6}\Psi_Q\left(\frac{x}{6}\right)$$

*where*

$$\Psi_Q(x) = \psi\left(x, x, x\right) + \frac{1}{2}\psi\left(2x, x, 0\right) + \frac{1}{4}\psi\left(0, x, 0\right)$$

*has the property*

$$\frac{1}{\rho_i^2}\Phi_{AQ}(\rho_i x) = L\,\Phi_{AQ}(x) \tag{3.57}$$

*for all $x \in \mathcal{X}$. Then there exists a unique additive mapping $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ satisfying the functional equation (1.5) and*

$$\| h_q(x) - \mathcal{A}(x) \| \leq \left(\frac{L^{1-i}}{1-L}\right)^p \Phi_{AQ}(x, x, x) \tag{3.58}$$

*for all $x \in \mathcal{X}$.*

**Corollary 3.5.** *Let $Dh_q : \mathcal{X} \to \mathcal{Y}$ be a mapping. If there exist real numbers $\lambda$ and $s$ such that*

$$\|Dh_qf(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\} & s \neq 2; \\ \rho\left\{||x||^s||y||^s||z||^s\right\} & 3s \neq 2; \\ \rho\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\} & 3s \neq 2; \end{cases} \tag{3.59}$$

*for all $x, y, z \in \mathcal{X}$, then there exists a unique quadratic function $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|h_q(x) - \mathcal{Q}(x)\| \leq \begin{cases} \dfrac{7\lambda}{28 \cdot 6}, \\ \dfrac{(15 + 2^{s+1})\lambda||x||^s}{4 \cdot 6|36 - 6^s|}, \\ \dfrac{\lambda||x||^{3s}}{6|36 - 6^{3s}|}, \\ \dfrac{(19 + 2^s)\lambda||x||^s}{4 \cdot 6|36 - 6^{3s}|}, \end{cases} \tag{3.60}$$

*for all $x \in \mathcal{X}$.*

**Theorem 3.7.** *Let $Dh : \mathcal{X} \to \mathcal{Y}$ be a mapping for which there exists a function $\psi : \mathcal{X}^3 \to [0, \infty)$ with the conditions (3.41), (3.54), (3.42), (3.55) such that the functional inequality*

$$\|Dh(x,y,z)\|_{\mathcal{Y}} \leq \psi(x,y,z) \tag{3.61}$$

*holds for all $x, y, z \in \mathcal{X}$. Assume that there exists $L = L(i)$ such that the function*

$$\frac{1}{6}\Psi_A\left(\frac{x}{6}\right) = \Phi_{AQ}(x) = \frac{1}{6}\Psi_Q\left(\frac{x}{6}\right)$$

*where $\Psi_A(x), \Psi_Q(x)$ are defined in (3.11), (3.28) has the properties (3.44), (3.57) for all $x \in \mathcal{X}$. Then there exists a unique additive mapping $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ and a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ satisfying the functional equation (1.5) and*

$$\| h(x) - \mathcal{A}(x) - \mathcal{Q}(x) \| \leq \left(\frac{L^{1-i}}{1-L}\right)^p \Phi_{AQ}(x,x,x) \tag{3.62}$$

*for all $x \in \mathcal{X}$.*

**Corollary 3.6.** *Let $Dh : \mathcal{X} \to \mathcal{Y}$ be a mapping. If there exist real numbers $\lambda$ and $s$ such that*

$$\|Dhf(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\} & s \neq 2; \\ \rho\left\{||x||^s||y||^s||z||^s\right\} & 3s \neq 2; \\ \rho\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\} & 3s \neq 2; \end{cases} \tag{3.63}$$

*for all $x, y, z \in \mathcal{X}$, then there exists a unique additive mapping $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ and a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(x) - \mathcal{Q}(x) - \mathcal{A}(x)\| \leq \begin{cases} \theta\left(\dfrac{2}{5} + \dfrac{7}{28 \cdot 6}\right), \\ \theta||x||^s\left(\dfrac{(8 + 2^s)}{2|6 - 6^s|} + \dfrac{(15 + 2^{s+1})}{4 \cdot 6|36 - 6^s|}\right), \\ \theta||x||^{3s}\left(\dfrac{1}{|6 - 6^{3s}|} + \dfrac{1}{6|36 - 6^{3s}|}\right), \\ \theta||x||^{3s}\left(\dfrac{(10 + 2^s)}{2|6 - 6^{3s}|} + \dfrac{(19 + 2^{s+1})}{4 \cdot 6|36 - 6^{3s}|}\right) \end{cases} \tag{3.64}$$

*for all $x \in \mathcal{X}$.*

## 4 Preliminaries of Non-Archimediean Fuzzy Normed Space

It is to be noted that, Mirmostafaee and Moslehian [28] initiate a notion of a non-Archimedean fuzzy norm and studied the stability of the Cauchy equation in the context of non-Archimedean fuzzy spaces. They presented an interdisciplinary relation between the theory of fuzzy spaces, the theory of non-Archimedean spaces, and the theory of functional equations.

The most important paradigm of non-Archimedean spaces are $p-$adic numbers. A key property of $p-$adic numbers is that they do not fulfill the Archimedean axiom: for all $x, y > 0$, there exists an integer $n$, such that $x > ny$. It turned out that non-Archimedean spaces have many nice applications [26, 46, 50].

During the last three decades, theory of non-Archimedean spaces has prolonged the interest of physicists for their research, in particular, in problems coming from quantum physics, $p-$adic strings, and superstrings [26]. One may note that $|n| \leq 1$ in each valuation field, every triangle is isosceles and there may be no unit vector in a non-Archimedean normed space (cf [26]). These facts show that the non-Archimedean framework is of special interest.

Fuzzy set theory is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. It has also very useful applications in various fields, such that population dynamics, chaos control, computer programming, nonlinear dynamical systems, nonlinear operators, statistical convergence, etc. [31, 47, 50]. The fuzzy topology proves to be a very valuable tool to deal with such situations where the use of classical theories breaks down. The most fascinating application of fuzzy topology in quantum particle physics arises in string and E-infinity theory of EI Naschie [34]- [38].

The definition of non-Archimedean fuzzy normed spaces was given in [30].

**Definition 4.1.** *Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}$, such that for any $a, b \in \mathbb{K}$, we have*

*(NA1)* $|a| \geq 0$ *and equality holds if and only if* $a = 0$;

*(NA2)* $|ab| = |a|\,|b|$;

*(NA3)* $|a + b| \leq \max\{|a|, |b|\}$ .

*The condition $(NA3)$ is called the strong triangle inequality. Clearly, $|1| = |-1| = |1|$ and $n \leq 1$ for all $n \geq \mathbb{N}$. We always assume in addition that $|\cdot|$ is non trivial, i.e., that*

*(NA4)* *there is an* $a_0 \in \mathbb{K}$*, such that* $|a_0| \neq 0, 1$.

The most important examples of non-Archimedean spaces are $p-$adic numbers.

**Example 4.1.** *Let $p$ be a prime number. For any nonzero rational number $x$, there exists a unique integer $n_x$, such that $x = \frac{a}{b} p^n x$, where $a$ and $b$ are integers not divisible by $p$. Then $|x|_p = p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$ which is called the $p-$adic number field. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum\limits_{k \geq n_x}^{\infty} a_k p^k$, where $|a_k| \leq p - 1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $\left| \sum\limits_{k \geq n_x}^{\infty} a_k p^k \right|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field (see [46]). Note that if $p > 2$ then $|2^n|_p = 1$ for each integer $n$ but $|2|_2 < 1$.*

Now we give the definition of a non-Archimedean fuzzy normed space.

**Definition 4.2.** *Let $X$ be a linear space over a non-Archimedean field $\mathbb{K}$. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be a non-Archimedean fuzzy norm on $X$ if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*

*(NAF1)* $N(x, c) = 0$ *for all* $c \leq 0$;

*(NAF2)* $x = 0$ *if and only if* $N(x, c) = 1$ *for all* $c > 0$;

*(NAF3)* $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$;

*(NAF4)* $N(x + y, \max\{s, t\}) \geq min\{N(x, s), N(y, t)\}$;

*(NAF5)* $\lim_{t\to\infty} N(x,t) = 1.$

A non-Archimedean fuzzy norm is a pair $(X, N)$ where $X$ be a linear space and $N$ is non-Archimedean fuzzy norm on $X$. If $(NAF4)$ holds then

(NAF6) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}.$

Recall that a classical vector space over the complex or real field satisfying $(NAF1)-(NAF5)$ is called a fuzzy normed space in the literature. We repeatedly use the fact $N(-x,t) = N(x,t),\ \ x \in X, t > 0$, which is deduced from $(NAF3)$. It is easy to see that $(NAF4)$ is equivalent to the following condition:

(NAF7) $N(x+y,t) \geq \min\{N(x,t), N(y,t)\}.$

**Example 4.2.** *Let $(X, ||\cdot||)$ be a non-Archimedean normed space. Then*

$$N(x,t) = \begin{cases} \dfrac{t}{t+\|x\|}, & t > 0, \quad x \in X, \\ 0, & t \leq 0, \quad x \in X \end{cases}$$

*Then $(X, N)$ is a non-Archimedean fuzzy normed space.*

**Example 4.3.** *Let $(X, ||\cdot||)$ be a non-Archimedean normed space. Then*

$$N(x,t) = \begin{cases} 0, & t \leq ||x||, \\ 1, & t > ||x||. \end{cases}$$

*Then $(X, N)$ is a non-Archimedean fuzzy normed space.*

**Definition 4.3.** *Let $(X, N)$ be a non-Archimedean fuzzy normed space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence $x_n$ and we denote it by $N - \lim_{n\to\infty} x_n = x$.*

**Definition 4.4.** *A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists $n_0$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$. Due to this fact that*

$$N(x_n - x_m, t) \geq \min\{N(x_{j+1} - x_j, t)/m \leq j \leq n-1, \quad n > m\},$$

*a sequence $\{x_n\}$ is Cauchy if and only if $\lim_{n\to\infty} N(x_{n+1} - x_n, t) = 1$ for all $t > 0$.*

**Definition 4.5.** *Every convergent sequence in a non-Archimedean fuzzy normed space is a Cauchy sequence. If every Cauchy sequence is convergent, then the non-Archimedean fuzzy normed space is called a non-Archimedean fuzzy Banach space.*

Here after, throughout this paper, assume that $\mathbb{K}$ non-Archimedean field, $X$ be vector space over $\mathbb{K}$, $(Y, N')$ be a non-Archimedean fuzzy Banach space over $\mathbb{K}$ and $(Z, N')$ be an (Archimedean or non-Archimedean) fuzzy normed space. Also we use the following notation for a given mapping $Dh : X \to Y$ by such that

$$\begin{aligned} Dh(x,y,z) &= h(x+2y+3z) + h(x+2y-3z) + h(x-2y+3z) + h(-x+2y+3z) \\ &\quad - h(x+y+z) - h(x+y-z) - h(x-y+z) - h(-x+y+z) - 2h(y) \\ &\quad - 4h(z) - 5[h(y)+h(-y)] - 14[h(z)+h(-z)] \end{aligned}$$

for all $x, y, z \in X$.

In this section, the non-Archimedean fuzzy stability of a 3 dimensional additive quadratic functional equation (1.5) is provided using direct and fixed point methods.

## 4.1 NAFNS: Stability Results: Direct Method

**Theorem 4.1.** *Let $\kappa = \pm 1$ be fixed and let $\vartheta : X^3 \to Z$ be a mapping such that for some $d$ with $0 < \left(\frac{d}{2}\right)^{\kappa} < 1$*

$$N'\left(\vartheta\left(6^{\kappa n}x, 6^{\kappa n}x, 6^{\kappa n}x\right), r\right) \geq N'\left(d^{\kappa n}\vartheta\left(x, x, x\right), r\right) \tag{4.1}$$

*for all $x \in X$, all $r > 0$, and*

$$\lim_{n\to\infty} N'\left(\vartheta\left(6^{\kappa n}x, 6^{\kappa n}y, 6^{\kappa n}z\right), 6^{\kappa n}r\right) = 1 \tag{4.2}$$

*for all $x, y, z \in X$ and all $r > 0$. Suppose that an odd function $h_a : X \to Y$ satisfies the inequality*

$$N\left(D\, h_a(x, y, z), r\right) \geq N'\left(\vartheta(x, y, z), r\right) \tag{4.3}$$

*for all $x, y, z \in X$ and all $r > 0$. Then the limit*

$$A(x) = N - \lim_{n\to\infty} \frac{h_a(6^{\kappa n}x)}{6^{\kappa n}} \tag{4.4}$$

*exists for all $x \in X$ and the mapping $A : X \to Y$ is a unique additive mapping satisfying (1.5) and*

$$N\left(h_a(x) - A(x), r\right) \geq N'\left(\vartheta(2x), \frac{r|6-d|}{2}\right) \tag{4.5}$$

*for all $x \in X$ and all $r > 0$.*

*Proof.* First assume $\kappa = 1$. Replacing $(x, y, z)$ by $(x, x, x)$ in (4.3) and using oddness of $h_a$, we get

$$\begin{aligned} &N\left(h_a(6x) + h_a(2x) + h_a(4x) - h_a(3x) - 9h_a(x), r\right) \\ &\qquad \geq N'\left(\vartheta(x, x, x), r\right), \quad \forall\, x \in X, r > 0. \end{aligned} \tag{4.6}$$

It follows from (4.6) and (NAF3)that

$$\begin{aligned} &N\left(2h_a(6x) + 2h_a(2x) + 2h_a(4x) - 2h_a(3x) - 18h_a(x), 2r\right) \\ &\qquad \geq N'\left(\vartheta(x, x, x), r\right), \quad \forall\, x \in X, r > 0. \end{aligned} \tag{4.7}$$

Again replacing $(x, y, z)$ by $(2x, x, 0)$ in (4.3) and using oddness of $h_a$, we get

$$N\left(2h_a(4x) - 2h_a(3x) - 2h_a(x), r\right) \geq N'\left(\vartheta(2x, x, 0), r\right), \quad \forall\, x \in X, r > 0. \tag{4.8}$$

Finally replacing $(x, y, z)$ by $(0, x, 0)$ in (4.3) and using oddness of $h_a$, we get

$$N\left(2h_a(2x) - 4h_a(x), r\right) \geq N'\left(\vartheta(0, x, 0), r\right), \quad \forall\, x \in X, r > 0. \tag{4.9}$$

With the help of (NAF3), (NAF4), (4.7), (4.8) and (4.9) we arrive

$$\begin{aligned} &N\left(2h_a(6x) - 12h_a(x), 4r\right) \\ &= N\left(2h_a(6x) + 2h_a(2x) + 2h_a(4x) - 2h_a(3x) - 18h_a(x)\right. \\ &\qquad \left. - 2h_a(4x) + 2h_a(3x) + 2h_a(x) - 2h_a(2x) + 4h_a(x), 4r\right) \\ &\geq \min\left\{N\left(2h_a(6x) + 2h_a(2x) + 2h_a(4x) - 2h_a(3x) - 18h_a(x), 2r\right),\right. \\ &\qquad \left. N\left(2h_a(4x) - 2h_a(3x) - 2h_a(x), r\right), N\left(2h_a(2x) - 4h_a(x), r\right)\right\} \\ &\geq \min\left\{N'\left(\vartheta(x, x, x), r\right), N'\left(\vartheta(2x, x, 0), r\right), N'\left(\vartheta(0, x, 0), r\right)\right\} \quad \forall\, x \in X, r > 0. \end{aligned} \tag{4.10}$$

Using (NAF3) in (4.10), we have

$$N\left(\frac{h_a(6x)}{6} - h_a(x), \frac{r}{3}\right) \geq N_1'\left(\vartheta(x), r\right), \quad \forall\, x \in X, r > 0, \tag{4.11}$$

where

$$N_1'\left(\vartheta(x), r\right) = \min\left\{N'\left(\vartheta(x, x, x), r\right), N'\left(\vartheta(2x, x, 0), r\right), N'\left(\vartheta(0, x, 0), r\right)\right\}$$

for all $x \in X$ and $r > 0$. Replacing $x$ by $6^n x$ in (4.6), we obtain

$$N\left(\frac{h_a(6^{n+1}x)}{6} - h_a(6^n x), \frac{r}{3}\right) \geq N_1'\left(\vartheta(6^n x), r\right), \quad \forall\, x \in X, r > 0. \tag{4.12}$$

Using (4.1) and $(NAF3)$ in (4.12), we have

$$N\left(\frac{h_a(6^{n+1}x)}{6} - h_a(6^n x), \frac{r}{3}\right) \geq N_1'\left(\vartheta(x), \frac{r}{d^n}\right), \quad \forall\, x \in X, r > 0. \tag{4.13}$$

One can easy to verify from (4.13), that

$$N\left(\frac{h_a(6^{n+1}x)}{6^{n+1}} - \frac{h_a(6^n x)}{6^n}, \frac{r}{3\cdot 6^n}\right) \geq N_1'\left(\vartheta(x), \frac{r}{d^n}\right), \quad \forall\, x \in X, r > 0. \tag{4.14}$$

Replacing $r$ by $d^n r$ in (4.14), we obtain

$$N\left(\frac{h_a(6^{n+1}x)}{6^{n+1}} - \frac{h_a(6^n x)}{6^n}, \frac{d^n\, r}{3\cdot 6^n}\right) \geq N_1'\left(\vartheta(x), r\right), \quad \forall\, x \in X, r > 0. \tag{4.15}$$

It is easy to see that

$$\frac{h_a(6^n x)}{6^n} - h_a(x) = \sum_{i=0}^{n-1}\left[\frac{h_a(6^{i+1}x)}{6^{i+1}} - \frac{h_a(6^i x)}{6^i}\right], \quad \forall\, x \in X. \tag{4.16}$$

It follows from (4.15) and (4.16), we have

$$\begin{aligned} N\left(\frac{h_a(6^n x)}{6^n} - h_a(x), \sum_{i=0}^{n-1}\frac{d^i\, r}{3\cdot 6^i}\right) &\geq min \bigcup_{i=0}^{n-1}\left\{N\left(\frac{h_a(6^{i+1}x)}{6^{i+1}} - \frac{h_a(6^i x)}{6^i}, \frac{d^i\, r}{3\cdot 6^i}\right)\right\} \\ &\geq min \bigcup_{i=0}^{n-1}\left\{N_1'\left(\vartheta(x), r\right)\right\} \\ &\geq N_1'\left(\vartheta(x), r\right), \quad \forall\, x \in X, r > 0. \end{aligned} \tag{4.17}$$

Replacing $x$ by $6^m x$ in (4.17) and using (4.1), $(NAF3)$, we get

$$N\left(\frac{h_a(6^{n+m}x)}{6^{n+m}} - \frac{h_a(6^m x)}{6^m}, \sum_{i=0}^{n-1}\frac{d^{i+m}\, r}{3\cdot 6^{i+m}}\right) \geq N_1'\left(\vartheta(x), \frac{r}{d^m}\right) \tag{4.18}$$

. for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Replacing $r$ by $d^m r$ in (4.18), we get

$$N\left(\frac{h_a(6^{n+m}x)}{6^{n+m}} - \frac{h_a(6^m x)}{6^m}, \sum_{i=0}^{n-1}\frac{d^i\, r}{3\cdot 6^{i+m}}\right) \geq N_1'\left(\vartheta(x), r\right) \tag{4.19}$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. It follows from (4.19) that

$$N\left(\frac{h_a(6^{n+m}x)}{6^{n+m}} - \frac{h_a(6^m x)}{6^m}, r\right) \geq N_1'\left(\vartheta(x), \frac{r}{\sum_{i=m}^{m+n-1}\frac{d^i}{3\cdot 6^i}}\right) \tag{4.20}$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Since $0 < d < 6$ and $\sum_{i=0}^{n}\left(\frac{d}{6}\right)^i < \infty$, using $(NAF5)$ implies that $\left\{\frac{h_a(6^n x)}{6^n}\right\}$ is a Cauchy sequence in $(Y, N)$. Since $(Y, N)$ is a non-Archimedean fuzzy Banach space, this sequence converges to some point $A(x) \in Y$. So, we can define a mapping $A : X \to Y$ by

$$A(x) = N - \lim_{n\to\infty}\frac{h_a(6^n x)}{6^n}, \quad \forall\, x \in X.$$

Putting $m = 0$ in (4.20), we get

$$N\left(\frac{h_a(6^n x)}{6^n} - h_a(x), r\right) \geq N_1'\left(\vartheta(x), \frac{r}{\sum_{i=0}^{n-1}\frac{d^i}{3\cdot 6^i}}\right), \forall\, x \in X, r > 0. \tag{4.21}$$

Letting $n \to \infty$ in (4.21) and using $(NAF5)$, we arrive

$$N\left(A(x) - h_a(x), r\right) \geq N_1'\left(\vartheta(x), \frac{r(6-d)}{2}\right) \quad \forall\, x \in X, r > 0.$$

To prove $A$ satisfies (1.5), replacing $(x, y, z)$ by $(6^n x, 6^n y, 6^n z)$ in (4.3), respectively , we obtain

$$N\left(\frac{1}{6^n}(Dh_a(6^n x, 6^n y, 6^n z)), r\right) \geq N'\left(\vartheta(6^n x, 6^n y, 6^n z), 6^n r\right) \tag{4.22}$$

for all $x, y, z \in X$ and all $r > 0$. Now,

$$\begin{aligned}
&N\Big(A(x+2y+3z) + A(x+2y-3z) + A(x-2y+3z) + A(-x+2y+3z)\\
&\quad - A(x+y+z) - A(x+y-z) - A(x-y+z) - A(-x+y+z)\\
&\qquad - 2A(y) - 4A(z) - 5[A(y)+A(-y)] - 14[A(z)+A(-z)], r\Big)\\
&\geq \min\Bigg\{N\left(A(x+2y+3z) - \frac{1}{6^n}h_a(6^n(x+2y+3z)), \frac{r}{13}\right),\\
&\quad N\left(A(x+2y-3z) - \frac{1}{6^n}h_a(6^n(x+2y-3z)), \frac{r}{13}\right),\\
&\quad N\left(A(x-2y+3z) - \frac{1}{6^n}h_a(6^n(x-2y+3z)), \frac{r}{13}\right),\\
&\quad N\left(A(-x+2y+3z) - \frac{1}{6^n}h_a(6^n(-x+2y+3z)), \frac{r}{13}\right),\\
&\quad N\left(-A(x+y+z) + \frac{1}{6^n}h_a(6^n(x+y+z)), \frac{r}{13}\right),\\
&\quad N\left(-A(x+y-z) + \frac{1}{6^n}h_a(6^n(x+y-z)), \frac{r}{13}\right),\\
&\quad N\left(-A(x-y+z) + \frac{1}{6^n}h_a(6^n(x-y+z)), \frac{r}{13}\right),\\
&\quad N\left(-A(-x+y+z) + \frac{1}{6^n}h_a(6^n(-x+y+z)), \frac{r}{13}\right),\\
&\quad N\left(-2A(y) + \frac{2}{6^n}h_a(6^n y), \frac{r}{13}\right), N\left(-4A(z) + \frac{4}{6^n}h_a(6^n z), \frac{r}{13}\right),\\
&\quad N\left(-5[A(y)+A(-y)] + \frac{5}{6^n}[h_a(6^n y) + h_a(-6^n y), \frac{r}{13}\right),\\
&\quad N\left(-14[A(z)+A(-z)] + \frac{14}{6^n}[h_a(6^n z) + h_a(-6^n z)], \frac{r}{13}\right),\\
&\quad N\Bigg(\frac{1}{6^n}h_a(6^n(x+2y+3z)) + \frac{1}{6^n}h_a(6^n(x+2y-3z))\\
&\qquad + \frac{1}{6^n}h_a(6^n(x-2y+3z)) + \frac{1}{6^n}h_a(6^n(-x+2y+3z))\\
&\qquad + \frac{1}{6^n}h_a(6^n(x+y+z)) + \frac{1}{6^n}h_a(6^n(x+y-z))\\
&\qquad + \frac{1}{6^n}h_a(6^n(x-y+z)) + \frac{1}{6^n}h_a(6^n(-x+y+z))\\
&\qquad - \frac{2}{6^n}h_a(6^n y) - \frac{4}{6^n}h_a(6^n z) - \frac{5}{6^n}[h_a(6^n y) + h_a(-6^n y)]\\
&\qquad - \frac{14}{6^n}[h_a(6^n z) + h_a(-6^n z)], \frac{r}{13}\Bigg), \Bigg\}
\end{aligned} \tag{4.23}$$

for all $x, y, z \in X$ and all $r > 0$. Using (4.22) and $(NAF5)$ in (4.23), we arrive

$$\begin{aligned}
&N\Big(A(x+2y+3z)+A(x+2y-3z)+A(x-2y+3z)+A(-x+2y+3z)-A(x+y+z)\\
&\quad -A(x+y-z)-A(x-y+z)-A(-x+y+z)\\
&\qquad -2A(y)-4A(z)-5[A(y)+A(-y)]-14[A(z)+A(-z)],r\Big)\\
&\quad \geq \min\left\{1,1,1,1,1,1,1,N'\left(\vartheta(6^n x,6^n y,6^n z),6^n r\right)\right\}\\
&\quad \geq N'\left(\vartheta(6^n x,6^n y,6^n z),6^n r\right)
\end{aligned} \tag{4.24}$$

for all $x,y,z \in X$ and all $r > 0$. Letting $n \to \infty$ in (4.24) and using (4.2), we see that

$$\begin{aligned}
&N\Big(A(x+2y+3z)+A(x+2y-3z)+A(x-2y+3z)+A(-x+2y+3z)\\
&\quad -A(x+y+z)-A(x+y-z)-A(x-y+z)-A(-x+y+z)\\
&\qquad -2A(y)-4A(z)-5[A(y)+A(-y)]-14[A(z)+A(-z)],r\Big)=1
\end{aligned}$$

for all $x,y,z \in X$ and all $r > 0$. Using $(NAF2)$ in the above inequality, we get

$$\begin{aligned}
&A(x+2y+3z)+A(x+2y-3z)+A(x-2y+3z)+A(-x+2y+3z)\\
&\quad = A(x+y+z)+A(x+y-z)+A(x-y+z)+A(-x+y+z)\\
&\qquad +2A(y)+4A(z)+5[A(y)+A(-y)]+14[A(z)+A(-z)]
\end{aligned}$$

for all $x,y,z \in X$. Hence $A$ satisfies the functional equation (1.5). In order to prove $A(x)$ is unique, let $A'(x)$ be another additive function satisfying (1.5) and (4.4). Hence,

$$\begin{aligned}
N(A(x)-A'(x),r) &= N\left(\frac{A(6^n x)}{6^n}-\frac{A'(6^n x)}{6^n},r\right)\\
&= N\left(A(6^n x)-A'(6^n x),6^n r\right)\\
&\geq min\Big\{N\left(A(6^n x)-\frac{h_a(6^n x)}{6^n},\frac{6^n r}{2}\right),N\left(\frac{h_a((6^n x)}{6^n}-A'(6^n x),\frac{6^n r}{2}\right)\Big\}\\
&\geq min\Big\{N_1'\left(\vartheta(6^n x),\frac{r\,6^n(6-d)}{2\cdot 2}\right),N_1'\left(\vartheta(6^n x),\frac{r\,6^n(6-d)}{2\cdot 2}\right)\Big\}\\
&= N_1'\left(\vartheta(6^n x),\frac{r\,6^n(6-d)}{4}\right)\\
&= N_1'\left(\vartheta(x),\frac{r\,6^n(6-d)}{4d^n}\right),\ \ \forall\, x\in X, r>0.
\end{aligned}$$

Since $\lim_{n\to\infty} \dfrac{r\,6^n(6-d)}{4d^n} = \infty$, we obtain

$$\lim_{n\to\infty} N_1'\left(\vartheta(x),\frac{r\,6^n(6-d)}{4d^n}\right)=1.$$

Thus

$$N(A(x)-A'(x),r)=1,\ \ \forall\, x\in X, r>0,$$

Hence $A(x)=A'(x)$. Therefore $A(x)$ is unique.

For $\kappa=-1$, we can prove the result by a similar method. This completes the proof of the theorem. □

From Theorem 4.1, we obtain the following corollary concerning the stability for the functional equation (1.5).

**Corollary 4.1.** *Suppose that a odd function $h_a : X \to Y$ satisfies the inequality*

$$N\left(D\,h_a(x,y,z,t),r\right) \geq \begin{cases} N'\left(\epsilon,r\right),\\ N'\left(\epsilon\left(||x||^s+||y||^s+||z||^s\right),r\right),\\ N'\left(\epsilon\,||x||^s\,||y||^s\,||z||^s,r\right),\\ N'\left(\epsilon\left\{||x||^s\,||y||^s\,||z||^s+\left(||x||^{3s}+||y||^{3s}+||z||^{3s}\right)\right\},r\right),\end{cases} \tag{4.25}$$

*for all $r > 0$ and all $x, y, z \in X$, where $\epsilon, s$ are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$N\left(h_a(x) - A(x), r\right) \geq \begin{cases} N'\left(\epsilon, \frac{r|6-d|}{4}\right), & \\ N'\left((8+2^s)\epsilon||x||^s, r|6-d|\right), & s < 1 \quad or \quad s > 1; \\ N'\left(\epsilon||x||^{3s}, r|6-d|\right), & s < \frac{1}{3} \quad or \quad s > \frac{1}{3}; \\ N'\left((10+2^s)\epsilon||x||^{3s}, r|6-d|\right), & s < \frac{1}{3} \quad or \quad s > \frac{1}{3}; \end{cases} \tag{4.26}$$

*for all $x \in X$ and all $r > 0$.*

*Proof.* Setting

$$\vartheta(x,y,z) = \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon\left(||x||^s + ||y||^s + ||z||^s\right), r\right), \\ N'\left(\epsilon\ ||x||^s\ ||y||^s\ ||z||^s, r\right), \\ N'\left(\epsilon\left\{||x||^s\ ||y||^s\ ||z||^s + \left(||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right)\right\}, r\right), \end{cases}$$

then the corollary is followed from Theorem 4.1. If we define

$$d = \begin{cases} 6^0, \\ 6^s, \\ 6^{3s} \\ 6^{3s}. \end{cases}$$

□

**Example 4.4.** *Let $X$ be a normed space and $N$ and $N'$ be non-archimedian fuzzy norms on $X$ and $\mathbb{R}$ defined by*

$$N(x,r) = \begin{cases} \dfrac{r}{r + \|x\|} & r > 0, \quad x \in X, \\ 0, & r \leq 0, \quad x \in X. \end{cases} \tag{4.27}$$

$$N'(x,r) = \begin{cases} \dfrac{r}{r + \|x\|} & r > 0, \quad x \in \mathbb{R}, \\ 0, & r \leq 0, \quad x \in \mathbb{R}. \end{cases} \tag{4.28}$$

*Let $\vartheta : (0,\infty) \to (0,\infty)$ be a function such that $\vartheta(6l) < d\ \vartheta(l)$ for all $l > 0$ and $0 < d < 6$. Define*

$$\begin{aligned} \beta(x,y,z) = {} & \vartheta(\|x+2y+3z\|) + \vartheta(\|x+2y-3z\|) + \vartheta(\|x-2y+3z\|) \\ & + \vartheta(\|-x+2y+3z\|) - \vartheta(\|x+y+z\|) - \vartheta(\|x+y-z\|) \\ & - \vartheta(\|x-y+z\|) - \vartheta(\|-x+y+z\|) - 2\vartheta(\|y\|) - 4\vartheta(\|z\|) \\ & - 5\vartheta(\|y\| + \|-y\|) - 14\vartheta(\|z\| + \|-z\|) \end{aligned}$$

*for all $x, y, z \in X$ . Let $x_0 \in X$ be a unit vector and define $h_a : X \to X$ by $h_a(x) = x + \vartheta(\|x\|)\, x_0$. Now for any $x, y, z \in X$ and $r > 0$, we have*

$$\begin{aligned} N\left(Dh_a(x,y,z), r\right) &= \frac{r}{r + \|\beta(x,y,z)\| \cdot \|x_0\|} \\ &\geq \frac{r}{r + \|\beta(x,y,z)\|} \\ &= N'\left(\beta(x,y,z), r\right). \end{aligned}$$

*For any $x, y, z \in X$ and $r > 0$, we have*

$$\begin{aligned} N'\left(\beta(6x,6y,6z), r\right) &= \frac{r}{r + \beta(6x,6y,6z)} \\ &\geq \frac{r}{r + d\,\beta(x,y,z)} \\ &= N'\left(d\,\beta(x,y,z), r\right). \end{aligned}$$

*Hence the inequalities (4.1) and (4.3) are satisfied. Using Theorem 4.1, there exists a unique additive mapping $A : X \to Y$ such that*

$$N\left(A\left(x\right)-h_a\left(x\right),\ r\right) \ \geq\ N_1'\left(\frac{2\beta\left(x\right)}{|6-d|},\ r\right)$$

*$x \in X$ and $r > 0$.*

The following theorem and corollary provide the stability result of (1.5) for $h$ is even function.

**Theorem 4.2.** *Let $\kappa = \pm 1$ be fixed and let $\vartheta : X^3 \to Z$ be a mapping such that for some $d$ with $0 < \left(\frac{d}{36}\right)^{\kappa} < 1$*

$$N'\left(\vartheta\left(6^{\kappa}x, 6^{\kappa}x, 6^{\kappa}x, 6^{\kappa}x\right), r\right) \geq N'\left(d^{\kappa}\vartheta\left(x, x, x\right), r\right) \tag{4.29}$$

*for all $x \in X$ and all $d > 0$, and*

$$\lim_{n\to\infty} N'\left(\vartheta\left(6^{\kappa n}x, 6^{\kappa n}y, 6^{\kappa n}z\right), 36^{\kappa n}r\right) = 1 \tag{4.30}$$

*for all $x, y, z \in X$ and all $r > 0$. Suppose that a even function $h_q : X \to Y$ satisfies the inequality*

$$N\left(D\, h_q(x, y, z), r\right) \geq N'\left(\vartheta(x, y, z), r\right) \tag{4.31}$$

*for all $x, y, z \in X$ and all $r > 0$ . Then the limit*

$$Q(x) = N - \lim_{n\to\infty} \frac{h_q(6^{\beta n}x)}{36^{\beta n}} \tag{4.32}$$

*exists for all $x \in X$ and the mapping $Q : X \to Y$ is a unique quadratic mapping satisfying (1.5) and*

$$N\left(h_q(x) - Q(x), r\right) \geq N_2'\left(\vartheta(x), \frac{r|36-d|}{2}\right) \tag{4.33}$$

*for all $x \in X$ and all $r > 0$.*

*Proof.* First assume $\kappa = 1$. Replacing $(x, y, z)$ by $(x, x, x)$ in (4.31) and using evenness of $h_q$, we get

$$\begin{aligned} N\left(h_q(6x) + h_q(2x) + h_q(4x) - h_q(3x) - 47h_q(x), r\right) \\ \geq N'\left(\vartheta(x, x, x), r\right), \quad \forall\, x \in X, r > 0. \end{aligned} \tag{4.34}$$

It follows from (4.34) and (NAF3)that

$$\begin{aligned} N\left(2h_q(6x) + 2h_q(2x) + 2h_q(4x) - 2h_q(3x) - 94h_q(x), 2r\right) \\ \geq N'\left(\vartheta(x, x, x), r\right), \quad \forall\, x \in X, r > 0. \end{aligned} \tag{4.35}$$

Again replacing $(x, y, z)$ by $(2x, x, 0)$ in (4.31) and using oddness of $h_q$, we get

$$N\left(2h_q(4x) - 2h_q(3x) - 14h_q(x), r\right) \geq N'\left(\vartheta(2x, x, 0), r\right), \quad \forall\, x \in X, r > 0. \tag{4.36}$$

Finally replacing $(x, y, z)$ by $(0, x, 0)$ in (4.31) and using oddness of $h_q$, we get

$$N\left(4h_q(2x) - 16h_q(x), r\right) \geq N'\left(\vartheta(0, x, 0), r\right), \quad \forall\, x \in X, r > 0. \tag{4.37}$$

It follows from (4.37) and (NAF3) that

$$N\left(2h_q(2x) - 8h_q(x), r\right) \geq N'\left(\vartheta(0, x, 0), \frac{r}{2}\right), \quad \forall\, x \in X, r > 0. \tag{4.38}$$

With the help of (NAF3), (NAF4), (4.35), (4.36) and (4.38) we arrive

$$\begin{aligned} & N\left(2h_q(6x) - 72h_q(x), 4r\right) \\ & = N\left(2h_q(6x) + 2h_q(2x) + 2h_q(4x) - 2h_q(3x) - 94h_q(x)\right. \\ & \qquad \left. -2h_q(4x) + 2h_q(3x) + 14h_q(x) - 2h_q(2x) + 8h_q(x), 4r\right) \\ & \geq \min\left\{N\left(2h_q(6x) + 2h_q(2x) + 2h_q(4x) - 2h_q(3x) - 18h_q(x), 2r\right),\right. \\ & \qquad \left. N\left(2h_q(4x) - 2h_q(3x) - 2h_q(x), r\right), N\left(2h_q(2x) - 8h_q(x), r\right)\right\} \\ & \geq \min\left\{N'\left(\vartheta(x, x, x), r\right), N'\left(\vartheta(2x, x, 0), r\right), N'\left(\vartheta(0, x, 0), \frac{r}{2}\right)\right\} \quad \forall\, x \in X, r > 0. \end{aligned} \tag{4.39}$$

Using (NAF3) in (4.39), we have

$$N\left(\frac{h_q(6x)}{6} - h_q(x), \frac{r}{18}\right) \geq N_2'\left(\vartheta(x), r\right), \quad \forall\, x \in X, r > 0, \tag{4.40}$$

where

$$N_2'\left(\vartheta(x), r\right) = \min\left\{N'\left(\vartheta(x,x,x), r\right), N'\left(\vartheta(2x,x,0), r\right), N'\left(\vartheta(0,x,0), \frac{r}{2}\right)\right\}$$

for all $x \in X$ and $r > 0$. The rest of the proof is similar tracing to that of Theorem 4.1. □

From Theorem 4.2, we obtain the following corollary concerning the stability for the functional equation (1.5).

**Corollary 4.2.** *Suppose that a even function $h_q : X \to Y$ satisfies the inequality*

$$N\left(D\, h_q(x,y,z,t), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon\left(||x||^s + ||y||^s + ||z||^s\right), r\right), \\ N'\left(\epsilon\, ||x||^s\, ||y||^s\, ||z||^s, r\right), \\ N'\left(\epsilon\left\{||x||^s\, ||y||^s\, ||z||^s + \left(||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right)\right\}, r\right), \end{cases} \tag{4.41}$$

*for all $x, y, z \in X$ and all $r > 0$, where $\epsilon, s$ are constants with $\epsilon > 0$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$N\left(h_q(x) - Q(x), r\right) \geq \begin{cases} N'\left(7\epsilon, r|36-d|\right), \\ N'\left((15+2^{s+1})\epsilon||x||^s, 2r|36-d|\right), & s < 2 \quad or \quad s > 2; \\ N'\left(\epsilon||x||^{3s}, \frac{r|36-d|}{2}\right), & s < \frac{2}{3} \quad or \quad s > \frac{2}{3}; \\ N'\left((19+2^{s+1})\epsilon||x||^{3s}, 2r|36-d|\right), & s < \frac{2}{3} \quad or \quad s > \frac{2}{3}; \end{cases} \tag{4.42}$$

*for all $x \in X$ and all $r > 0$.*

**Example 4.5.** *Let $X$ be a normed space and $N$ and $N'$ be non-archimedian fuzzy norms on $X$ and $\mathbb{R}$ defined by (4.27),(4.28). Let $\vartheta : (0,\infty) \to (0,\infty)$ be a function such that $\vartheta(2l) < d\,\vartheta(l)$ for all $l > 0$ and $0 < d < 36$. Define*

$$\begin{aligned}\beta(x,y,z) = {} & \vartheta\left(\|x+2y+3z\|\right) + \vartheta\left(\|x+2y-3z\|\right) + \vartheta\left(\|x-2y+3z\|\right) \\ & + \vartheta\left(\|-x+2y+3z\|\right) - \vartheta\left(\|x+y+z\|\right) - \vartheta\left(\|x+y-z\|\right) \\ & - \vartheta\left(\|x-y+z\|\right) - \vartheta\left(\|-x+y+z\|\right) - 2\vartheta\left(\|y\|\right) - 4\vartheta\left(\|z\|\right) \\ & - 5\vartheta\left(\|y\| + \|-y\|\right) - 14\vartheta\left(\|z\| + \|-z\|\right)\end{aligned}$$

*for all $x, y, z \in X$ . Let $x_0 \in X$ be a unit vector and define $h_q : X \to X$ by $h_q(x) = x + \vartheta(\|x\|)\, x_0$. Now for any $x, y, z \in X$ and $r > 0$, we have*

$$\begin{aligned}N\left(Dh_q(x,y,z), r\right) &= \frac{r}{r + \|\beta(x,y,z)\| \cdot \|x_0\|} \\ &\geq \frac{r}{r + \|\beta(x,y,z)\|} \\ &= N'\left(\beta(x,y,z), r\right).\end{aligned}$$

*For any $x, y, z \in X$ and $r > 0$, we have*

$$\begin{aligned}N'\left(\beta(2x,2y,2z), r\right) &= \frac{r}{r + \beta(2x,2y,2z)} \\ &\geq \frac{r}{r + d\,\beta(x,y,z)} \\ &= N'\left(d\,\beta(x,y,z), r\right).\end{aligned}$$

*Hence the inequalities (4.29) and (4.31) are satisfied. Using Theorem 4.2, there exists a unique additive mapping $A$ : $X \to Y$ such that*

$$N\left(Q(x) - h_q(x),\ r\right) \ \geq\ N_2'\left(\frac{\beta(x)}{|36-d|}, r\right)$$

*$x \in X$ and $r > 0$.*

The following theorem provide the stability result of (1.5) for mixed case.

**Theorem 4.3.** *Let $\kappa = \pm 1$ be fixed and let $\vartheta : X^3 \to Z$ be a mapping such that for some $d$ with $0 < \left(\frac{d}{6}\right)^{\kappa} < 1$ and satisfying (4.1),(4.2),(4.29) and (4.30). Suppose that a function $h : X \to Y$ satisfies the inequality*

$$N\left(D\,h(x,y,z),r\right) \geq N'\left(\vartheta(x,y,z),r\right),\ \forall\ x,y,z \in X, r>0. \tag{4.43}$$

*Then there exists a unique additive mapping $A : X \to Y$ and unique quadratic mapping $Q : X \to Y$ satisfying (1.5) and*

$$N\left(h(x)-A(x)-Q(x),r\right) \geq N_3\left(\vartheta(x),r\right) \tag{4.44}$$

*where*

$$N_3\left(\vartheta(x),r\right) = \min\left\{N_1''\left(\vartheta(x),\frac{r|6-d|}{2}\right),N_2''\left(\vartheta(x),\frac{r|36-d|}{2}\right)\right\} \tag{4.45}$$

*for all $x \in X$ and all $r > 0$.*

*Proof.* Clearly

$$|36| \leq |6| \leq d.$$

Let $h_o(x) = \dfrac{h_a(x)-h_a(-x)}{2}$ for all $x \in X$. Then $h_o(0) = 0$ and $h_o(-x) = -h_o(x)$ for all $x \in X$. Hence

$$\begin{aligned} N\left(D\,h_o(x,y,z),2r\right) &= N\left(D\,h_a(x,y,z) - D\,h_a(-x,-y,-z),2r\right) \\ &\geq \min\left\{N'\left(D\,h_a(x,y,z),r\right),N'\left(D\,h_a(-x,-y,-z),r\right)\right\} \\ &\geq \min\left\{N'\left(\vartheta(x,y,z),r\right),N'\left(\vartheta(-x,-y,-z),r\right)\right\} \end{aligned} \tag{4.46}$$

for all $x,y,z \in X$ and all $r > 0$. Let

$$N_1''(\vartheta(x,y,z),r) = \min\left\{N'\left(\vartheta(x,y,z),r\right),N'\left(\vartheta(-x,-y,-z),r\right)\right\} \tag{4.47}$$

for all $x,y,z \in X$ and all $r > 0$. By Theorems 4.1, there exists a unique additive mapping $A : X \to Y$ such that

$$N\left(h_o(x)-A(x),r\right) \geq N_1''\left(\vartheta(x),\frac{r|6-d|}{2}\right) \tag{4.48}$$

for all $x \in X$ and all $r > 0$.

Also, let $h_e(x) = \dfrac{h_q(x)+h_q(-x)}{2}$ for all $x \in X$. Then $h_e(0) = 0$ and $h_e(-x) = h_e(x)$ for all $x \in X$. Hence

$$\begin{aligned} N\left(D\,h_e(x,y,z),r\right) &= N\left(D\,h_q(x,y,z) - D\,h_q(-x,-y,-z),2r\right) \\ &\geq \min\left\{N'\left(\vartheta(x,y,z),r\right),N'\left(\vartheta(-x,-y,-z),r\right)\right\} \end{aligned} \tag{4.49}$$

for all $x,y,z \in X$ and all $r > 0$. Let

$$N_2(\vartheta(x,y,z),r) = \min\left\{N'\left(\vartheta(x,y,z),r\right),N'\left(\vartheta(-x,-y,-z),r\right)\right\} \tag{4.50}$$

for all $x,y,z \in X$ and all $r > 0$. By Theorem 4.2, there exists a unique quadratic mapping $Q : X \to Y$ such that

$$N\left(h_e(x)-Q(x),r\right) \geq N_2\left(\vartheta(x),\frac{r|36-d|}{2}\right) \tag{4.51}$$

for all $x \in X$ and all $r > 0$. Define

$$h(x) = h_e(x) + h_o(x) \tag{4.52}$$

for all $x \in x$. From (4.44),(4.47) and (4.48), we arrive

$$\begin{aligned} N\left(h(x)-A(x)-Q(x),r\right) &= N\left(h_e(x)+h_o(x)-A(x)-Q(x),r\right) \\ &\geq \min\left\{N\left(h_o(x)-A(x),\frac{r}{2}\right),N\left(h_e(x)-Q(x),\frac{r}{2}\right)\right\} \\ &\geq \min\left\{N_1''\left(\vartheta(x),\frac{r|6-d|}{4}\right),N_2''\left(\vartheta(x),\frac{r|36-d|}{4}\right)\right\} \\ &= N_3\left(\vartheta(x),r\right) \end{aligned}$$

where
$$N_3\left(\vartheta(x), r\right) = \min\left\{N_1''\left(\vartheta(x), \frac{r|6-d|}{4}\right), N_2''\left(\vartheta(x), \frac{r|36-d|}{4}\right)\right\} \tag{4.53}$$
for all $x \in X$ and all $r > 0$. Hence the theorem is proved. □

The following corollary is the immediate consequence of Corollaries 4.1, 4.2 and Theorem 4.3 concerning the stability for the functional equation (1.5).

**Corollary 4.3.** *Suppose that a function $h : X \to Y$ satisfies the inequality*

$$N\left(D\,h(x,y,z), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon\left(||x||^s + ||y||^s + ||z||^s\right), r\right), \\ N'\left(\epsilon\, ||x||^s\, ||y||^s\, ||z||^s, r\right), \\ N'\left(\epsilon\left\{||x||^s\, ||y||^s\, ||z||^s + \left(||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right)\right\}, r\right), \end{cases} \tag{4.54}$$

*for all $x, y, z, t \in X$ and all $r > 0$, where $\epsilon, s$ are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - A(x) - Q(x), r\right) \geq \begin{cases} N'\left(8\epsilon, r\left[\frac{|6-d|}{4} + |36-d|\right]\right), \\ N'\left((23 + 2^s + 2^{s+1})\epsilon||x||^s, r\left[|6-d| + 2|36-d|\right]\right), & s \neq 1, 2; \\ N'\left(2\epsilon||x||^{3s}, r\left[|6-d| + \frac{|36-d|}{2}\right]\right), & s \neq \frac{1}{3}, \frac{2}{3}; \\ N'\left((29 + 2^s + 2^{s+1})\epsilon||x||^{4s}, r\left[|6-d| + 2|36-d|\right]\right), & s \neq \frac{1}{3}, \frac{2}{3}; \end{cases} \tag{4.55}$$

*for all $x \in X$ and all $r > 0$.*

## 4.2 NAFNS : Stability Results: Fixed Point Method

The following theorem provide the stability result of (1.5) for $h$ is odd function.

**Theorem 4.4.** *Let $h_a : X \to Y$ be a mapping for which there exist a function $\vartheta : X^3 \to Z$ with the condition*

$$\lim_{n\to\infty} N'\left(\vartheta\left(\mu_i^n x, \mu_i^n y, \mu_i^n z\right), \mu_i^n r\right) = 1, \;\; \forall\, x, y, z \in X, r > 0 \tag{4.56}$$

*where*
$$\mu_i = \begin{cases} 6 & if \quad i = 0 \\ \frac{1}{6} & if \quad i = 1 \end{cases} \tag{4.57}$$

*and satisfying the functional inequality*

$$N\left(D\,h_a(x,y,z), r\right) \geq N'\left(\vartheta(x,y,z), r\right), \;\forall\, x, y, z \in X, r > 0. \tag{4.58}$$

*If there exists $L = L(i)$ such that the function*

$$x \to \beta(x) = \vartheta\left(\frac{x}{6}\right),$$

*has the property*
$$N'\left(L\frac{\beta(\mu_i x)}{\mu_i}, r\right) = N'\left(\beta(x), r\right), \;\forall\, x \in X, r > 0. \tag{4.59}$$

*Then there exists a unique additive function $A : X \to Y$ satisfying the functional equation (1.5) and*

$$N\left(h_a(x) - A(x), r\right) \geq N'\left(\beta(x), \frac{L^{1-i}}{1-L}r\right), \;\forall\, x \in X, r > 0. \tag{4.60}$$

*Proof.* Let $\Omega$ is the set such that
$$\Omega = \left\{g | g : X \to Y, g(0) = 0\right\}.$$

Let $d$ be a general metric on $\Omega$, such that

$$d(g,h) = \inf\left\{K \in (0,\infty) | N\left(g(x) - h(x), r\right) \geq N'\left(\beta(x), Kr\right), x \in X\right\}.$$

It is easy to see that $(\Omega, d)$ is complete. Define $T : \Omega \to \Omega$ by $Tg(x) = \frac{1}{\mu_i} g(\mu_i x)$, for all $x \in X$. For $g, h \in \Omega$, we have

$$\begin{aligned}
&\inf \left\{K \in (0\infty) : N\left(g(x) - h(x), r\right) \geq N'\left(\beta(x), Kr\right), x \in X\right\} \quad or \\
&\inf \left\{K \in (0,\infty) : N\left(\frac{g(\mu_i x)}{\mu_i} - \frac{h(\mu_i x)}{\mu_i}, r\right) \geq N'\left(\beta(\mu_i x), K\mu_i r\right), x \in X\right\} \quad or \\
&\inf \left\{LK \in (0,\infty) : N\left(Tg(x) - Th(x), r\right) \geq N'\left(\beta(x), KLr\right), x \in X\right\}
\end{aligned}$$

This implies $J$ is a strictly contractive mapping on $\Omega$ with Lipschitz constant $L$. We already proved the result (4.10)

$$N\left(2h_a(6x) - 12h_a(x), 4r\right) \geq N_1'\left(\vartheta(x), r\right) \tag{4.61}$$

where

$$N_1'\left(\vartheta(x), r\right) = \min\left\{N'\left(\vartheta(x,x,x), r\right), N'\left(\vartheta(2x,x,0), r\right), N'\left(\vartheta(0,x,0), r\right)\right\}$$

for all $x \in X$ and $r > 0$. Using (NAF3) in (4.61), we arrive

$$N\left(\frac{h_a(6x)}{6} - h_a(x), \frac{4r}{6 \cdot 2}\right) \geq N_1'\left(\vartheta(x), r\right), \quad \forall\, x \in X, r > 0, \tag{4.62}$$

With the help of (4.59), when $i = 0$, it follows from (4.62), that

$$\begin{aligned}
&\inf \left\{1 \in (0,\infty) : N\left(\frac{h_a(6x)}{6} - h_a(x), \frac{4r}{6 \cdot 2}\right) \geq N_1'\left(\vartheta(x), r\right), x \in X\right\} \quad or \\
&\inf \left\{1 \in (0,\infty) : N\left(\frac{h_a(6x)}{6} - h_a(x), r\right) \geq N_1'\left(\vartheta(x), 3r\right), x \in X\right\} \quad or \\
&\inf \left\{L \in (0,\infty) : N\left(Jh_a f(x) - h_a(x), r\right) \geq N_1'\left(\beta(x), r\right), x \in X\right\} \quad or \\
&\inf \left\{L^1 \in (0,\infty) : N\left(Jh_a f(x) - h_a(x), r\right) \geq N_1'\left(\beta(x), r\right), x \in X\right\} \quad or \\
&\inf \left\{L^{1-0} \in (0,\infty) : N\left(Jh_a f(x) - h_a(x), r\right) \geq N_1'\left(\beta(x), r\right), x \in X\right\}
\end{aligned} \tag{4.63}$$

Again replacing $x = \frac{x}{6}$ in (4.61), we obtain

$$N\left(h_a(x) - 6h_a\left(\frac{x}{6}\right), 2r\right) \geq N'\left(\vartheta\left(\frac{x}{6}\right), r\right), \quad \forall \;\; x \in X, r > 0. \tag{4.64}$$

When $i = 1$, it follows from (4.64), that

$$\begin{aligned}
&\inf \left\{1 \in (0,\infty) : N\left(h_a(x) - 6h_a\left(\frac{x}{6}\right), 2r\right), \geq N'\left(\vartheta\left(\frac{x}{6}\right), r\right), x \in X\right\} \quad or \\
&\inf \left\{1 \in (0,\infty) : N\left(h_a f(x) - Jh_a(x), r\right) \geq N'\left(\vartheta\left(\frac{x}{6}\right), \frac{r}{2}\right), x \in X\right\} \quad or \\
&\inf \left\{L^0 \in (0,\infty) : N\left(h_a f(x) - Jh_a(x), r\right) \geq N'\left(\beta(x), r\right), x \in X\right\} \quad or \\
&\inf \left\{L^{1-1} \in (0,\infty) : N\left(h_a f(x) - Jh_a(x), r\right) \geq N'\left(\beta(x), r\right), x \in X\right\}
\end{aligned} \tag{4.65}$$

Then from (4.63) and (4.65), we can conclude,

$$\inf \left\{L^{1-i} \in (0,\infty) : N\left(h_a f(x) - Jh_a(x), r\right) \geq N'\left(\beta(x), r\right), x \in \mathcal{X}\right\}$$

Hence property (FP1) holds. The rest of the proof is similar lines to the of Theorem 3.5. This completes the proof of the theorem. □

From Theorem 4.4, we obtain the following corollary concerning the stability for the functional equation (1.5).

**Corollary 4.4.** *Suppose that a odd function $h_a : X \to Y$ satisfies the inequality*

$$N\left(D\,h_a(x,y,z,t),r\right) \geq \begin{cases} N'\left(\epsilon,r\right), \\ N'\left(\epsilon\left(||x||^s+||y||^s+||z||^s\right),r\right), \\ N'\left(\epsilon\,||x||^s\,||y||^s\,||z||^s,r\right), \\ N'\left(\epsilon\left\{||x||^s\,||y||^s\,||z||^s+\left(||x||^{3s}+||y||^{3s}+||z||^{3s}\right)\right\},r\right), \end{cases} \tag{4.66}$$

*for all $x,y,z,t \in X$ and all $r > 0$, where $\epsilon, s$ are constants with $\epsilon > 0$. Then there exists a unique aditive mapping $A : X \to Y$ such that*

$$N\left(f_a(x)-A(x),r\right) \geq \begin{cases} N'\left(\epsilon,\frac{r|6-d|}{4}\right), \\ N'\left((8+2^s)\epsilon||x||^s,r|6-d|\right), & s<1 \;\; or \;\; s>1; \\ N'\left(\epsilon||x||^{3s},r|6-d|\right), & s<\frac{1}{3} \;\; or \;\; s>\frac{1}{3}; \\ N'\left((10+2^s)\epsilon||x||^{3s},r|6-d|\right), & s<\frac{1}{3} \;\; or \;\; s>\frac{1}{3}; \end{cases} \tag{4.67}$$

*for all $x \in X$ and all $r > 0$.*

The following theorem provide the stability result of (1.5) for $h$ is even function using fixed point method. The proof of the following Theorem and Corollary is similar to that of Theorem 4.4 and Corollary 4.4. Hence the details of the proof is omitted.

**Theorem 4.5.** *Let $h_q : X \to Y$ be a even mapping for which there exist a function $\vartheta : X^3 \to Z$ with the condition*

$$\lim_{n\to\infty} N'\left(\vartheta\left(\mu_i^n x,\mu_i^n y,\mu_i^n z\right),\mu_i^{2n} r\right) = 1 \;\; \forall\, x,y,z \in X, r>0 \tag{4.68}$$

*where*

$$\mu_i = \begin{cases} 6 & if \quad i=0 \\ \frac{1}{6} & if \quad i=1 \end{cases} \tag{4.69}$$

*and satisfying the functional inequality*

$$N\left(D\,h_q(x,y,z),r\right) \geq N'\left(\vartheta(x,y,z),r\right) \;\; \forall\, x,y,z \in X, r>0. \tag{4.70}$$

*If there exists $L = L(i)$ such that the function*

$$x \to \beta(x) = \vartheta\left(x,\frac{x}{2},0,0\right),$$

*has the property*

$$N'\left(L\frac{1}{\mu_i^2}\beta(\mu_i x),r\right) = N'\left(\beta(x),r\right) \;\; \forall\, x \in X, r>0. \tag{4.71}$$

*Then there exists a unique quadratic function $Q : X \to Y$ satisfying the functional equation (1.5) and*

$$N\left(h_q(x)-Q(x),r\right) \geq N'\left(\beta(x),\frac{L^{1-i}}{1-L}r\right) \;\; \forall\, x \in X, r>0. \tag{4.72}$$

**Corollary 4.5.** *Suppose that a even function $h_q : X \to Y$ satisfies the inequality*

$$N\left(D\,h_q(x,y,z),r\right) \geq \begin{cases} N'\left(\epsilon,r\right), \\ N'\left(\epsilon\left(||x||^s+||y||^s+||z||^s\right),r\right), \\ N'\left(\epsilon\,||x||^s\,||y||^s\,||z||^s,r\right), \\ N'\left(\epsilon\left\{||x||^s\,||y||^s\,||z||^s+\left(||x||^{3s}+||y||^{3s}+||z||^{3s}\right)\right\},r\right), \end{cases} \tag{4.73}$$

*for all $r > 0$ and all $x,y,z \in X$, where $\epsilon, s$ are constants with $\epsilon > 0$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$N\left(h_q(x)-A(x),r\right) \geq \begin{cases} N'\left(7\epsilon,r|36-d|\right), \\ N'\left((15+2^{s+1})\epsilon||x||^s,2r|36-d|\right), & s<2 \;\; or \;\; s>2; \\ N'\left(\epsilon||x||^{3s},\frac{r|36-d|}{2}\right), & s<\frac{2}{3} \;\; or \;\; s>\frac{2}{3}; \\ N'\left((19+2^{s+1})\epsilon||x||^{3s},2r|36-d|\right), & s<\frac{2}{3} \;\; or \;\; s>\frac{2}{3}; \end{cases} \tag{4.74}$$

*for all $x \in X$ and all $r > 0$.*

The following theorem provide the stability result of (1.5) for mixed case in fixed point method.

**Theorem 4.6.** *Let $f : X \to Y$ be a mapping for which there exist a function $\vartheta : X^3 \to Z$ with the condition (4.56) and (4.68) satisfying the functional inequality*

$$N\left(D\, f(x,y,z),r\right) \geq N'\left(\vartheta(x,y,z),r\right),\ \forall\, x,y,z,t \in X, r > 0. \tag{4.75}$$

*If there exists $L = L(i)$ such that the function*

$$x \to \beta(x) = \vartheta\left(x, \frac{x}{2}, 0, 0\right),$$

*has the properties (4.59) and (4.71) for all $x \in X$. Then there exists a unique additive function $A : X \to Y$ and a unique quadratic function $Q : X \to Y$ satisfying the functional equation (1.5) and*

$$N\left(f(x) - A(x) - Q(x),r\right) \geq N_3\left(\beta(x),r\right),\ \forall\, x \in X, r > 0. \tag{4.76}$$

The following corollary is the immediate consequence of Corollaries 4.4, 4.5 and Theorem 4.6 concerning the stability for the functional equation (1.5) in fixed point method.

**Corollary 4.6.** *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(D\, f(x,y,z,t),r\right) \geq \begin{cases} N'\left(\epsilon,r\right), \\ N'\left(\epsilon\left(||x||^s + ||y||^s + ||z||^s\right),r\right), \\ N'\left(\epsilon\, ||x||^s\, ||y||^s\, ||z||^s, r\right), \\ N'\left(\epsilon\left\{||x||^s\, ||y||^s\, ||z||^s + \left(||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right)\right\},r\right), \end{cases} \tag{4.77}$$

*for all $r > 0$ and all $x, y, z, t \in X$, where $\epsilon, s$ are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - A(x) - Q(x),r\right) \geq \begin{cases} N'\left(8\epsilon, r\left[\frac{|6-d|}{4} + |36-d|\right]\right), & \\ N'\left((23 + 2^s + 2^{s+1})\epsilon||x||^s, r\left[|6-d| + 2|36-d|\right]\right), & s \neq 1,2; \\ N'\left(2\epsilon||x||^{3s}, r\left[|6-d| + \frac{|36-d|}{2}\right]\right), & s \neq \frac{1}{3}, \frac{2}{3}; \\ N'\left((29 + 2^s + 2^{s+1})\epsilon||x||^{4s}, r\left[|6-d| + 2|36-d|\right]\right), & s \neq \frac{1}{3}, \frac{2}{3}; \end{cases} \tag{4.78}$$

*for all $x \in X$ and all $r > 0$.*

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