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On two boundary-value problems of functional integro-differential equations with nonlocal conditions

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Abstract

In this paper we establish the existence of solution for two boundary value problems of Fredholm functional integro-differential equations with nonlocal boundary conditions.

Keywords: Nonlocal boundary value problems, Fredholm functional integral equation, Fredholm functional integro-differential equation, compact in measure.

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1 Introduction

Mathematical modelling of real-life problems usually results in functional equations, of various types appear in many applications that arise in the fields of mathematical analysis, nonlinear functional analysis, mathematical physics, and engineering. An interesting feature of functional integral equations is their role in the study of many problems of functional integro-differential equations. Several different techniques were proposed to study the existence of solutions of the functional integral equations in appropriate function spaces. Although all of these techniques have the same goal, they differ in the function spaces and the fixed point theorems to be applied. Consider the following boundary value problems of Fredholm functional integro-differential equations.

$$x'(t) = f(t, \int_0^1 k(t,s)x'(s)ds), \qquad a.e. \qquad t \in (0,1) \tag{1.1}$$

with the nonlocal boundary condition

$$x(\tau) + \alpha\, x(\xi) = 0, \qquad \tau, \xi \in [0,1], \alpha \neq -1. \tag{1.2}$$

$$x''(t) = f(t, \int_0^1 k(t,s)x''(s)ds) \qquad a.e. \qquad t \in (0,1) \tag{1.3}$$

with the nonlocal boundary conditions

$$x(\tau) + \beta\, x(\xi) = 0, \qquad \beta \neq -1 \tag{1.4}$$

$$x'(\tau) + \alpha\, x'(\xi) = 0, \qquad \tau, \xi \in [0,1], \alpha \neq -1. \tag{1.5}$$

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Here we study the existence of at least one solution of each of the boundary value problems (1.1)-(1.2) and (1.3)-(1.5).
The existence of exactly one solution of them will be deduced.

2 Functional integral equation

Here we study the existence of at least one (and exactly one) integrable solution of the Fredholm functional integral equation.

$$y(t) = f(t, \int_0^1 k(t,s)y(s)ds) \tag{2.6}$$

under the following assumptions

(1) $f : I = [0,1] \times R \to R$ is measurable in $t \in [0,1]$ for all $x \in R$ and continuous in $x \in R$ for all $t \in [0,1]$ and there exists integrable function $a \in L^1[0,1]$ and positive constant $b > 0$ such that

$$| f(t,x) | \leq a(t) + b|x| \qquad a.e. \quad t \in I.$$

(2)

$$||a|| = \int_0^1 |a(t)|dt, \qquad t \in [0,1]$$

(3) $k : I = [0,1] \times [0,1] \to R$ is continuous $t \in [0,1]$ for every $s \in [0,1]$ and measurable in $s \in [0,1]$ for all $t \in [0,1]$, such that

$$\sup_t \int_0^1 k(t,s)dt \leq M$$

Now for the existence of at least one integrable solution of the functional integral equation (2.6) we have the following theorem.

Theorem 2.1. *Let the assumptions (1)-(3) be satisfied. If* $L = Mb < 1$*, then the functional integral equation (2.6) has at least one solution* $y \in L^1[0,1]$.

Proof. Let $L^1 = L^1[0,1]$ and define the set B_r by

$$B_r = \{y \in L^1 : ||y|| \leq r\} \subset L^1[0,1],$$

where

$$r = \frac{a}{1 - bM}.$$

Define the operator T associated with the Fredholm functional integral equation (2.6) by

$$Ty(t) = f(t, \int_0^1 k(t,s)y(s)ds).$$

To show that $T : B_r \to B_r$, let $y \in B_r$, then

$$\begin{aligned}
||Ty(t)||_{L^1} &= \int_0^1 |Ty(t)|dt \\
&= \int_0^1 |f(t, \int_0^1 k(t,s)y(s)ds)|dt \\
&\leq \int_0^1 [|a(t)| + b|\int_0^1 k(t,s)y(s)ds|]dt \\
&\leq \int_0^1 |a(t)|dt + b\int_0^1 \int_0^1 |k(t,s)y(s)|dsdt \\
&\leq \int_0^1 |a(t)|dt + bM\int_0^1 |y(s)|\,ds \\
&\leq ||a|| + bM\,||y|| \\
&\leq ||a|| + bMr = r. \\
&\leq a + bMr = r.
\end{aligned}$$

From this we observe that $T(B_r) \subset B_r$. Then $T : B_r \rightarrow B_r$, Moreover from our assumptions $(1)-(3)$ follows that the operator T is continuous.

To prove that T is a contraction with respect to measure of weak non compactness β on the set B_r.
Let $X \subset B_r$ and let $y \in X$. Futher $\epsilon > 0$ and take a measurable subset $D \subset [0,1]$ such that $\mu(D) \leq \epsilon$, then we get

$$
\begin{aligned}
||Ty(t)||_{L^1(D)} &= \int_D |Ty(t)\ |dt \\
&= \int_D |f(t, \int_0^1 k(t,s)y(s)ds)\ |dt \\
&\leq \int_D [|\ a(t)\ | + b| \int_D k(t,s)y(s)ds|]dt \\
&\leq \int_D |\ a(t)\ |dt + b\int_D \int_0^1 |k(t,s)y(s)|dsdt \\
&\leq \int_D |\ a(t)\ |dt\ +\ bM \int_D\ |y(s)|\ ds \\
&\leq\ ||\ a\ ||_{L^1(D)}\ +\ bM\ \int_D |y(s)|ds
\end{aligned}
$$

for this subset X, the measure of weak non compactness $\beta(X)$ is given by the formula

$$\beta(X) = \lim_{\epsilon \to 0} \{\sup_{y \in X} \{\sup_{D} \{\int_D |y(t)|dt\ :\ D \subset [0,1]\ \ \mu(D) \leq \epsilon\}\}$$

To value $\beta(TX)$ we notice that

$$\beta(X) = \lim_{\epsilon \to 0} \{\sup_{y \in X} \{\sup_{D} \{\int_D |a(t)|dt + bM \int_D |y(t)|dt\ :\ D \subset [0,1]\ \ \mu(D) \leq \epsilon\}\} = 0$$

Indeed,we have $\beta(TX) \leq bM\beta(X)$.
Since all conditions of Schauder fixed point theorem (see[15]), are satisfied, then the operator T has at least one fixed point $y \in L^1[0,1]$, which completes the proof. ■

Now for the uniqueness of the solution of the Fredholm functional integral equation (2.6) Consider following assumptions

(1*) $f : I = [0,1] \times R \rightarrow R$ is measurable in $t \in [0,1]$ for all $x \in R$ and satisfies
the lipschitz such that

$$|f(t,x) - f(t,y)| \leq b|x-y|, \qquad b > 0 \tag{2.7}$$

(2*) $f(t,0) \in L^1[0,1]$ $\quad \int_0^1 |f(t,0)|dt \leq a.$

Theorem 2.2. *Let the assumptions (1*),(2*) and (3) be satisfied. If $Mb < 1$, then the functional integral equation (2.6) has a unique solution $y \in L^1[0,1]$.*

Proof. From (2.7) we can obtain

$$|f(t,x)| \leq |f(t,0)| + b\,|x|.$$

This shows that the assumptions of Theorem (2.1) are satisfied
Now let y_1,y_2 be two solution of functional integral equation (2.6)

$$y_1(t) = f(t, \int_0^1 k(t,s)y_1(s)ds)$$

$$y_2(t) = f(t, \int_0^1 k(t,s)y_2(s)ds)$$

$$\begin{aligned}
||y_1(t) - y_2(t)||_{L^1} &= \int_0^t |f(t, \int_0^1 k(t,s)y_1\, ds) - f(t, \int_0^1 k(t,s)y_2\, ds)|dt \\
&\leq b\int_0^t | \int_0^1 k(t,s)y_1(s)\, ds - \int_0^1 k(t,s)y_2(s)\, ds|dt \\
&\leq b\int_0^t | \int_0^1 k(t,s)\ (y_1(s) - y_2(s))\, ds|dt \\
&\leq b\int_0^t \int_0^1 |k(t,s)|\ |y_1(s) - y_2(s)|dsdt \\
&\leq bM\int_0^t |y_1(s) - y_2(s)|ds \\
&\leq bM||y_1 - y_2||_{L^1},
\end{aligned}$$

then

$$||y_1 - y_2|| \leq\ K||y_1 - y_2||$$

where $L = bM <\ 1$, then

$$||y_1 - y_2||(1 - k) \leq\ 0$$

and

$$||y_1 - y_2|| =\ 0$$

which implies that $y_1\ =\ y_2$ then the Fredholm functional integral equation (2.6) has a unique integrable solution.

3 Nonlocal boundary value problems

Here we study the existence of at least one (and exactly one) solution of each of the boundary value problems (1.1)-(1.2) and (1.3)-(1.5).
Consider the functional integro differential equation

$$x'(t)\ =\ f(t, \int_0^1\ k(t,s)\ x'(s)\ ds) \qquad a.e. \qquad t \in (0,1).$$

with the nonlocal boundary value condition

$$x(\tau) + \alpha\ x(\xi) = 0. \qquad\qquad \tau, \xi\ \in [0,1], \alpha\ \neq\ -1$$

Theorem 3.3. *Let the assumptions of theorem (2.1) be satisfied, then the nonlocal boundary value problem (1.1)-(1.2) has at least one integrable solution* $x\ \in\ L^1[0,1]$.

Proof. Let $x'(t) = y(t)$. Integrating both sides we get

$$x(t) = x(0) + \int_0^t y(s)ds,$$

$$x(\tau) = x(0) + \int_0^\tau y(s)ds$$

and

$$x(\xi) = x(0) + \int_0^\xi y(s)ds$$

Using the nonlocal boundary condition (1.2) we obtain

$$x(0) + \int_0^\tau y(s)ds\ =\ -\alpha\ x(0) - \alpha\ \int_0^\xi y(s)ds,$$

and

$$x(0) = -\frac{1}{1+\alpha}\int_0^\tau y(s)ds - \frac{\alpha}{1+\alpha}\int_0^\xi y(s)ds,$$

then

$$x(t) = \int_0^t y(s)ds - \frac{1}{1+\alpha}\int_0^\tau y(s)ds - \frac{\alpha}{1+\alpha}\int_0^\xi y(s)ds \tag{3.8}$$

where y satisfies the functional integral equation

$$y(t) \;=\; f(t, \int_0^1 k(t,s)y(s)ds).$$

This complete the proof of equivalent between the nonlocal problem (1.1)-(1.2) and the functional integral equation (2.6). This implies that there exists at least one solution $x \in L^1[0,1]$ of the nonlocal problem (1.1)-(1.2).■

Corollary 3.1. *Let the assumptions* (1^*),(2^*) *and (3) be satisfied, then the solution of nonlocal boundary value problem (1.1)-(1.2) has a unique integrable solution* $x \in L^1[0,1]$.

Consider the Fredholm functional integro-differential equation

$$x''(t) \;=\; f(t, \int_0^1 k(t,s)x''(s)ds) \qquad a.e. \quad t \in (0,1)$$

with the nonlocal boundary conditions

$$x(\tau) + \beta\, x(\xi) = 0,$$
$$x'(\tau) \;+\; \alpha\, x'(\xi) = 0.$$

Theorem 3.4. *Let the assumptions of theorem (2.1) be satisfied then the boundary value problems (1.3)-(1.5) has at least one integrable solution* $x \in L^1[0,1]$.

Proof. Let $x''(t) \;=\; y(t)$ integrating both sides we obtain

$$x'(t) \;=\; x'(0) + \int_0^t y(s)\,ds$$

and

$$x(t) \;=\; x(0) + tx'(0) + \int_0^t (t-s)\,y(s)ds.$$

then

$$x'(\tau) \;=\; x'(0) + \int_0^\tau y(s)\,ds,$$

and

$$x'(\xi) \;=\; x'(0) + \int_0^\xi y(s)\,ds.$$

Using the nonlocal condition (1.5) we obtain

$$x'(0) \;=\; -\frac{1}{1+\alpha}\int_0^\tau y(s)ds - \frac{\alpha}{1+\alpha}\int_0^\xi y(s)\,ds$$

and

$$x(\tau) \;=\; x(0) + \tau\, x'(0) + \int_0^\tau (\tau - s)\,y(s)\,ds,$$
$$x(\xi) \;=\; x(0) + \xi\, x'(0) + \int_0^\xi (\xi - s)\,y(s)\,ds,$$

$$x'(0) \;=\; -\frac{1}{1+\alpha}\int_0^\tau y(s)ds - \frac{\alpha}{1+\alpha}\int_0^\xi y(s)\,ds.$$

Using Boundary condition (1.4) we obtain

$$x(0) = \frac{-\beta\xi - \tau}{1+\beta}x'(0) - \frac{1}{1+\alpha}\int_0^\tau (\tau - s)y(s)ds - \frac{1}{1+\beta}\int_0^\xi (\xi - s)y(s)ds,$$

$$
\begin{aligned}
x(t) &= \frac{-\beta\xi-\tau}{1+\beta}[-\frac{1}{1+\beta}\int_0^\tau y(s)ds - \frac{1}{1+\alpha}\int_0^\xi y(s)ds] \\
&- \frac{1}{1+\beta}\int_0^\tau (\tau-s)y(s)ds - \frac{\beta}{1+\beta}\int_0^\xi (\xi-s)ds \\
&+ t[-\frac{1}{1+\alpha}\int_0^\tau y(s)ds - \frac{\alpha}{1+\alpha}\int_0^\xi y(s)ds] + \int_0^t (t-s)y(s)ds, \qquad (3.9)\\
x'(t) &= -\frac{1}{1+\alpha}\int_0^\tau y(s)ds - \frac{\alpha}{1+\alpha}\int_0^\xi y(s)ds] + \int_0^t y(s)ds,
\end{aligned}
$$

and y satisfies the functional integral equation

$$y(t) = f(t, \int_0^1 k(t,s)y(s)ds).$$

This complete the proof of equivalent between the nonlocal problem (1.3)-(1.5) and the functional integral equation (2.6). This implies that there exists at least one solution $x \in L^1[0,1]$ of the nonlocal problem (1.3)-(1.5). ■

Corollary 3.2. *Let the assumptions (1^*),(2^*) and (3) be satisfied, then the solution of nonlocal boundary value problem (1.3)-(1.5) has a unique integrable solution $x \in L^1[0,1]$.*

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