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Existence results for multi-term time-fractional impulsive differential equations with fractional order boundary conditions

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Abstract
In this paper, we discuss the existence and uniqueness of solutions for a class of multi-term time-fractional impulsive integro-differential equations with state dependent delay subject to some fractional order integral boundary conditions. In our consideration, we apply the Banach, and Sadovskii fixed point theorems to obtain our main results under some appropriate assumptions. An example is given at the end to illustrate the applications of the established results.

Keywords
Fractional order differential equations, multi-term time fractional derivative, fractional impulsive conditions, fractional order integral boundary conditions.

AMS Subject Classification
26A33, 30E25, 34A12, 34A34, 34A37, 37C25, 45J05.

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Contents

1. Introduction

Let $\mathbb{E}$ be a Banach space and let $\mathscr{D}=[0,T]$. We consider the space $\mathscr{PC}_t := \mathscr{PC}([-\tau,t],\mathbb{E})$, $0\leq t\leq T<\infty$, $\tau>0$ formed by the functions $g:[-\tau,t]\to\mathbb{E}$, which are continuous except the points $t\neq t_k, k=1,2,\ldots,m$ such that right limit $g(t_k^+)$ and left limit $g(t_k^-)$ exists at t_k, and $g(t_k^-)=g(t_k)$. The space $\mathscr{PC}_t$ equipped with the norm $\|g\|_{\mathscr{PC}_t}=\sup\limits_{-\tau\leq s\leq t}\{\|g(s)\|_{\mathbb{E}}: h\in\mathscr{PC}_t\}$ is a Banach space. Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural and real numbers.

In this paper, we study the existence and uniqueness of solutions for the following class of multi-term time-fractional impulsive differential system of the form

$$\begin{aligned} {}^cD^{1+\alpha}y(t)+\sum_{j=1}^{n}{}^cD^{\beta_j}f_j(t,y_{\rho(t,y_t)}) \\ = f_0(t,y_{\rho(t,y_t)},K(y)(t)),\ t\in\mathscr{D},\ t\neq t_k, \quad (1.1)\end{aligned}$$

$$\begin{aligned} y(t)+g(y_{\rho(t_1,y_{t_1})},y_{\rho(t_2,y_{t_2})},\ldots,y_{\rho(t_m,y_{t_m})})(t) \\ =\phi(t),,\ t\in[-\tau,0],\ m\in\mathbb{N}, \quad (1.2)\end{aligned}$$

$$\begin{aligned} \Delta y(t_k) &= I_k(y(t_k^-)), \\ \Delta({}^cD^{\gamma}y(t_k)) &= J_k(y(t_k^-)),\ \gamma\in(0,1), \quad (1.3)\end{aligned}$$

$k=1,2,3,\ldots,m\in\mathbb{N}$. Subject to the fractional order boundary conditions

$$c({}^cD^{\delta}y(0))+d({}^cD^{\delta}y(T))=\int_0^T h(s,y_1(s),\ldots,y_m(s))ds, \quad (1.4)$$

where $\delta\in(0,1)$, $d\neq0$, and when $\delta=1$, i.e.

$$c(y'(0))+d(y'(T))=\int_0^T h(s,y_1(s),\ldots,y_m(s))ds, \quad (1.5)$$

where $c+d\neq0, d\neq0$, ${}^cD^{\eta}$ represents the Caputo derivative of order $\eta>0$, and $\beta_j>0$ for all $j=1,2,\ldots,n$ such that

$0 < \alpha \leq \beta_n \leq \beta_{n-1} \leq \ldots \leq \beta_1 < 1$. The functions $f_j : \mathscr{D} \times \mathscr{PC}_0 \to \mathbb{E}$ such that $f_j(0, y(0)) = 0$ for all $j = 1,2,3,...,n$, $f_0 : \mathscr{D} \times \mathscr{PC}_0 \times \mathbb{E} \to \mathbb{E}$, $g : \mathscr{PC}_0^m \to \mathbb{E}$ and $h : [0,T] \times \mathbb{E}^m \to \mathbb{E}$ are given functions which satisfy some appropriate conditions. For any $y \in \mathscr{PC}_T$ we denote by y_t, the element of $\mathscr{PC}_0$ define by $y_t = y(t+\theta), \theta \in [-\tau, 0]$. The function $\rho : \mathscr{D} \times \mathscr{PC}_0 \to [-\tau, T]$ is continuous and $\phi(0) \in \mathscr{PC}_0$. Let the function $t \to y_t$ be well defined and continuous from the set $\{\rho(s,\chi) : (s,\chi) \in [0,T] \times \mathscr{PC}_0\}$ into $\mathscr{PC}_0$. The equation (1.3) characterize the impulsive conditions with $0 = t_0 < t_1 < t_2 <, \cdots, < t_m < t_{m+1} = T$, $I_k, J_k \in \mathscr{C}(\mathbb{E}, \mathbb{E})$(the set of continuous functions from $\mathbb{E}$ to $\mathbb{E}$), $k = 1,2,\ldots,m$ are continuous functions. We have $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$ and $\Delta({}^cD^\gamma y(t_k)) = ({}^cD^\gamma y(t_k^+)) - ({}^cD^\gamma y(t_k^-))$, such that $y(t_k) = y(t_k^-)$ and $({}^cD^\gamma y(t_k)) = ({}^cD^\gamma y(t_k^-))$. The operator K is described by $K(y)(t) = \int_0^t \mathfrak{K}(t,s) y(s) ds$, where $\mathfrak{K} \in \mathscr{C}(\Omega, \mathbb{R}^+)$ is a positive and continuous operator on $\Omega := \{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$) and $K^0 = \sup_{t\in[0,T]} \int_0^t \mathfrak{K}(t,s) ds$.

During the last few decades, the fractional differential equations (in brief, FDEs) including Riemann-Liouville and Caputo derivative have been magnetized the interest of many researchers towards itself, inspired by demonstrated applications in widespread areas of science and engineering such as in models about medicine (modeling of human tissue under mechanical loads), electrical engineering(transmission of ultrasound waves), biochemistry (modeling of proteins and polymers) mathematically modeling, thermal systems, acoustics, modeling of materials or rheology and mechanical systems, etc. For fundamental certainties regarding to fractional systems, one can make reference to the papers [2, 5, 6, 13, 17, 22, 27, 32], the monographs [24, 30, 33] and references therein.

On the other hand, the theory of fractional impulsive differential equations (in brief, FIDEs) has been generated a great interest among many researchers due to the fact that many real world processes and phenomena which are effected by short term external influences may be efficiently and effectively described by FIDEs. The time of external influences is negligible in the comparison of total duration of entire processes and phenomena. Therefore, the theory of FIDEs arising from real world problems to describe the dynamics of processes has been analytically investigated by many authors with interesting papers [4, 6, 17, 18, 34, 37, 39].

The system (1.1) − (1.3) with boundary conditions (1.4) and (1.5) is a strong motivation of the applications of physical models with papers [19, 25, 27, 28]. Kosmatov [25], Vidushi and Dabas [19] considered the following impulsive model

$$\Delta y(t_k) = I_k(y(t_k^-)),\ \Delta({}^cD^\gamma y(t_k)) = J_k(y(t_k^-)),$$

where $\gamma \in (0,1)$, $\quad k = 1,2,3,\ldots,m \in \mathbb{N}$. In [27] Liu et al. established the existence results for fractional differential equations with fractional non separated boundary conditions. Recently Losada et al. [28] show the attractivity of solutions for the following fractional differential equation

$$ {}^cD^\beta y(t) = \sum_{j=1}^m {}^cD^{\beta_j} f_j(t, y_t) + f_0(t, y_t), \quad t > t_0,$$
$$ y(t) = \varphi(t), \quad t_0 - \sigma \leq t \leq t_0,$$

where $0 < \beta_i < \beta$, $\varphi \in \mathscr{C}([t_0 - \sigma, t_0], \mathbb{R})$, and $f_j : [t_0, \infty) \times \mathscr{C}([t_0 - \sigma, t_0], \mathbb{R}) \to \mathbb{R}$ are some appropriate functions. The use of nonlocal conditions was initiated by Byszewski [11]. Further Byszewski and Lakshimikantham [12] tried to show the feasibility of nonlocal conditions over the classical conditions, which have great applications in applied sciences like as chemical engineering, blood flow problems, underground water flow, and population dynamics [9], cellular systems [1] etc.

The fractional order diffusion wave equations (i.e. order $(1+\alpha) \in (1,2)$) have many applications in various fields of science and engineering. These equations represent propagation of mechanical waves through viscoelastic media, charge transport in amorphous semiconductors [16, 20, 21, 29], and may be used in thermodynamics, shear in fluids, the flow of fluid through fissured rocks [7]. In particular, the fractional delay diffusion wave equations describe the driver reaction time, time taken for a signal traveling to the controlled object and time consume by body to produce red blood cells and cell division time in the dynamics of viral persistence or exhaustion.

Our work is arranged as follows: In preliminaries section, we recall some basic definitions and fundamental results of fractional calculus. In main results section, the existence and uniqueness results are obtained for the system (1.1) − (1.3) with boundary conditions (1.4) and (1.5). At last, an example is provided to show the feasibility of the theory discussed in this paper.

2. Preliminaries

In this section, we recall some definitions and basic results of fractional calculus which will be used throughout the paper.

Definition 2.1. *The fractional integral of a function g with the lower limit zero of order $\eta > 0$ is defined by*

$$J^\eta g(t) = \frac{1}{\Gamma(\eta)} \int_0^t (t-\xi)^{\eta-1} g(\xi) d\xi, \quad t > 0,$$

and $J^0 g(t) := g(t)$, where $\Gamma(\cdot)$ is the Euler Gamma function. This fractional integral satisfies the properties $J^\eta \circ J^b = J^{\eta+b}$ for $b > 0$.

Definition 2.2. *The Riemann-Liouville fractional derivative of a function g with the lower limit zero of order $\eta > 0, n-1 < \eta < n, n \in \mathbb{N}$ is given by*

$$D^\eta g(t) = \frac{1}{\Gamma(n-\eta)} \frac{d^n}{dt^n} \int_0^t (t-\xi)^{n-\eta-1} g(\xi) d\xi,$$

where the function $g(t)$ has absolutely continuous derivative up to order $(n-1)$. Moreover $D^0 g(t) = g(t)$ and $D^\eta J^\eta g(t) = g(t)$ for $t > 0$.

Definition 2.3. *The Caputo fractional derivative of a function $g \in \mathscr{C}^n([0,\infty))$ with the lower limit zero of order $\eta > 0$ is given by*

$$^cD^\eta g(t) = \frac{1}{\Gamma(n-\eta)} \int_0^t (t-\xi)^{n-\eta-1} \frac{d^n}{d\xi^n} g(\xi) d\xi,$$

where $n-1 < \eta < n, n \in \mathbb{N}$.

We denote $\mathbb{D}^n g(t) = \frac{d^n}{dt^n} g(t),$ for every $n \in \mathbb{N}$. Moreover $(\mathbb{D}^n \circ J^n) g(t) = g(t)$ for all $n \in \mathbb{N}$, and

$$(J^n \circ \mathbb{D}^n) g(t) = g(t) - \sum_{d=0}^{n-1} \frac{g^{(d)}(0)}{d!} t^d, \quad t > 0,\ n \in \mathbb{N}.$$

Similarly, we have $(J^\eta \circ {}^cD^\eta) g(t) = g(t) - \sum_{k=0}^{n-1} \frac{g^{(k)}(0)}{k!} t^k$, for $n-1 < \eta \leq n$. If $g^{(d)}(0) = 0$, for $d = 1,2,3,...n-1$, then $(J^\eta \circ {}^cD^\eta) g(t) = g(t)$.

Definition 2.4. *A function $y \in \mathscr{PC}_T$ such that $(1+\alpha)^{th}$ derivatives of y exist on $\mathscr{D}$ is said to be the solution of the system (1.1) – (1.2) with (1.4) (or (1.5)), if y satisfies the equation (1.1) on $\mathscr{D}$, the impulsive conditions*

$$\Delta y(t_k) = I_k(y(t_k^-)), \quad \Delta({}^cD^\gamma y(t_k^-)) = J_k(y(t_k^-)),$$

where $k = 1,2,3,\ldots,m$, and boundary conditions given by (1.4) (or (1.5)).

To analyze the system (1.1) – (1.5), first we characterize the solution, adopting the methodology of Fackan at el. [18].

Lemma 2.5. *A piecewise continuous differential functions $y : [-\tau, T] \to \mathbb{E}$ is a solution of the following integral equation* $y(t) =$

$$\begin{cases} \phi(t) - g(y_{\rho(t_1,y_{t_1})}, y_{\rho(t_2,y_{t_2})}, ..., y_{\rho(t_m,y_{t_m})})(t), t \in [-\tau, 0]; \\ \phi(0) - g(y_{\rho(t_1,y_{t_1})}, y_{\rho(t_2,y_{t_2})}, ..., y_{\rho(t_m,y_{t_m})})(0) \\ +b_0 t - \sum_{j=1}^n \int_0^t \frac{(t-s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)} f_j(s, y_{\rho(s,y_s)}) ds \\ + \int_0^t \frac{(t-s)^\alpha}{\Gamma(1+\alpha)} f_0(s, y_{\rho(s,y_s)}, K(y)(s)) ds, \quad t \in [0, t_1]; \\ \vdots \\ \phi(0) - g(y_{\rho(t_1,y_{t_1})}, y_{\rho(t_2,y_{t_2})}, ..., y_{\rho(t_m,y_{t_m})})(0) \\ +b_0 t + \sum_{k=1}^m I_k(y(t_k^-)) + \sum_{k=1}^m \frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}} J_k(y(t_k^-))(t-t_k) \\ -\sum_{j=1}^n \int_0^t \frac{(t-s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)} f_j(s, y_{\rho(s,y_s)}) ds \\ + \int_0^t \frac{(t-s)^\alpha}{\Gamma(1+\alpha)} f_0(s, y_{\rho(s,y_s)}, K(y)(s)) ds, \quad t \in (t_m, T], \end{cases} \tag{2.1}$$

where

$$\begin{aligned} b_0 = &\frac{\Gamma(2-\delta)}{T^{1-\delta}} \left[\frac{1}{d} \int_0^T h(s, y_1(s), \ldots, y_m(s)) ds \right. \\ &+ \sum_{j=1}^n \int_0^T \frac{(T-s)^{\alpha-\beta_j-\delta}}{\Gamma(1+\alpha-\beta_j-\delta)} f_j(s, y_{\rho(s,y_s)}) ds \\ &- \int_0^T \frac{(T-s)^{\alpha-\delta}}{\Gamma(1+\alpha-\delta)} f_0(s, y_{\rho(s,y_s)}, K(y)(s)) ds \\ &\left. - \sum_{k=1}^m \frac{\Gamma(2-\gamma)}{\Gamma(2-\delta)} J_k(y(t_k^-)) t_k^{\gamma-1} \right] \end{aligned} \tag{2.2}$$

if and only if y is a solution of the impulsive fractional system (1.1) – (1.3) with the boundary condition (1.4).

Proof. Let us denote by $\widehat{f}_j(t) = f_j(t, y_{\rho(t,y_t)})$ and $\widehat{f}_0(t) = f_0(t, y_{\rho(t,y_t)}, K(y)(t))$. Then for $t \in [0, t_1]$, the system (1.1) – (1.3) transforms as

$$^cD^{1+\alpha} y(t) + \sum_{j=1}^n {}^cD^{\beta_j} \widehat{f}_j(t) = \widehat{f}_0(t), \tag{2.3}$$

$$y(t) + (g(y_{\rho(t_1,y_{t_1})}, y_{\rho(t_2,y_{t_2})}, ..., y_{\rho(t_m,y_{t_m})}))(t) = \phi(t), \tag{2.4}$$

for $t \in [-\tau, 0]$. Applying Rienamm Liouville fractional integral of order $(1+\alpha)$ of (2.3), we get

$$\begin{aligned} y(t) = &a_0 + b_0 t - \sum_{j=1}^n J^{1+\alpha-\beta_j} (J^{\beta_j} {}^cD^{\beta_j}) \widehat{f}_j(t) \\ &+ \int_0^t \frac{(t-s)^\alpha}{\Gamma(1+\alpha)} \widehat{f}_0(s) ds \\ = &a_0 + b_0 t - \sum_{j=1}^n \int_0^t \frac{(t-s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)} \widehat{f}_j(s) ds \\ &+ \int_0^t \frac{(t-s)^\alpha}{\Gamma(1+\alpha)} \widehat{f}_0(s) ds. \end{aligned} \tag{2.5}$$

By using initial conditions, we get

$$a_0 = \phi(0) - g(y_{\rho(t_1,y_{t_1})}, y_{\rho(t_2,y_{t_2})}, ..., y_{\rho(t_m,y_{t_m})})(0),$$

then

$$\begin{aligned} y(t) = &\phi(0) - g(y_{\rho(t_1,y_{t_1})}, y_{\rho(t_2,y_{t_2})}, ..., y_{\rho(t_m,y_{t_m})})(0) \\ &+ b_0 t - \sum_{j=1}^n \int_0^t \frac{(t-s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)} \widehat{f}_j(s) ds \\ &+ \int_0^t \frac{(t-s)^\alpha}{\Gamma(1+\alpha)} \widehat{f}_0(s) ds. \end{aligned} \tag{2.6}$$

Similarly, if $t \in (t_1, t_2]$, then

$$^cD^{1+\alpha} y(t) + \sum_{j=1}^n {}^cD^{\beta_j} \widehat{f}_j(t) = \widehat{f}_0(t), \tag{2.7}$$

$$y(t_1^+) = y(t_1^-) + I_1(y(t_1^-)), \tag{2.8}$$

$$^cD^\gamma y(t_1^+) = {}^cD^\gamma y(t_1^-) + J_1(y(t_1^-)). \tag{2.9}$$

Again taking Rienamm Liouville fractional integral of order $(1+\alpha)$ of (2.7), we obtain

$$y(t) = a_1 + b_1 t - \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)} \widehat{f}_j(s) + \int_0^t \frac{(t-s)^{\alpha}}{\Gamma(1+\alpha)} \widehat{f}_0(s)ds, \tag{2.10}$$

at $t = t_1$, rewrite (2.10), as

$$y(t_1^+) = a_1 + b_1 t_1 - \sum_{j=1}^{n} \int_0^{t_1} \frac{(t_1-s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)} \widehat{f}_j(s)ds + \int_0^{t_1} \frac{(t_1-s)^{\alpha}}{\Gamma(1+\alpha)} \widehat{f}_0(s)ds. \tag{2.11}$$

By the impulsive condition (2.9) and the fact $y(t_1) = y(t_1^-)$, we may write

$$y(t_1) + I_1(y(t_1^-)) = a_1 + b_1 t_1 + + \int_0^{t_1} \frac{(t_1-s)^{\alpha}}{\Gamma(1+\alpha)} \widehat{f}_0(s)ds - \sum_{j=1}^{n} \int_0^{t_1} \frac{(t_1-s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)} \widehat{f}_j(s)ds. \tag{2.12}$$

Now, from (2.12) and (2.6) at $t = t_1$, we get $a_1 = \phi(0) - (g(y_{\rho(t_1,y_{t_1})}, y_{\rho(t_2,y_{t_2})}, \ldots, y_{\rho(t_m,y_{t_m})}))(0) + b_0 t_1 - b_1 t_1 + I_1(y(t_1^-))$, hence (2.10) becomes

$$\begin{aligned} y(t) = &\phi(0) - g(y_{\rho(t_1,y_{t_1})}, y_{\rho(t_2,y_{t_2})}, \ldots, y_{\rho(t_m,y_{t_m})})(0) \\ &+ b_0 t_1 + b_1(t-t_1) + I_1(y(t_1^-)) \\ &- \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)} \widehat{f}_j(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha}}{\Gamma(1+\alpha)} \widehat{f}_0(s)ds. \end{aligned} \tag{2.13}$$

Since ${}^cD^\gamma C = 0$(C is a constant)and ${}^cD^\gamma J^{1+\alpha} y(t) = J^{1+\alpha-\gamma} y(t)$, then applying Caputo derivative of order γ on (2.6) and (2.13) with respect to t at $t = t_1$, we achieve

$$\begin{aligned} {}^cD^\gamma y(t_1^-) = &b_0 \frac{t_1^{1-\gamma}}{\Gamma(2-\gamma)} + \int_0^{t_1} \frac{(t_1-s)^{\alpha-\gamma}}{\Gamma(1+\alpha-\gamma)} \widehat{f}_0(s)ds \\ &- \sum_{j=1}^{n} \int_0^{t_1} \frac{(t_1-s)^{\alpha-\beta_j-\gamma}}{\Gamma(1+\alpha-\beta_j-\gamma)} \widehat{f}_j(s)ds, \\ {}^cD^\gamma y(t_1^+) = &b_1 \frac{t_1^{1-\gamma}}{\Gamma(2-\gamma)} + \int_0^{t_1} \frac{(t_1-s)^{\alpha-\gamma}}{\Gamma(1+\alpha-\gamma)} \widehat{f}_0(s)ds \\ &- \sum_{j=1}^{n} \int_0^{t_1} \frac{(t_1-s)^{\alpha-\beta_j-\gamma}}{\Gamma(1+\alpha-\beta_j-\gamma)} \widehat{f}_j(s)ds. \end{aligned}$$

Incorporate with fractional impulsive condition (2.9), we get $b_1 = b_0 + \frac{\Gamma(2-\gamma)}{t_1^{1-\gamma}} J_1(y(t_1^-))$, then (2.13) becomes

$$\begin{aligned} y(t) = &\phi(0) - g(y_{\rho(t_1,y_{t_1})}, y_{\rho(t_2,y_{t_2})}, \ldots, y_{\rho(t_m,y_{t_m})})(0) \\ &+ b_0 t + I_1(y(t_1^-)) + \frac{\Gamma(2-\gamma)}{t_1^{1-\gamma}} J_1(y(t_1^-))(t-t_1) \\ &- \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)} \widehat{f}_j(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha}}{\Gamma(1+\alpha)} \widehat{f}_0(s)ds. \end{aligned} \tag{2.14}$$

Similarly, for $t \in (t_2, t_3]$, we can write the solution of the form

$$\begin{aligned} y(t) = &\phi(0) - g(y_{\rho(t_1,y_{t_1})}, y_{\rho(t_2,y_{t_2})}, \ldots, y_{\rho(t_m,y_{t_m})})(0) \\ &+ b_0 t + I_1(y(t_1^-)) + I_2(y(t_2^-)) \\ &+ \frac{\Gamma(2-\gamma)}{t_1^{1-\gamma}} J_1(y(t_1^-))(t-t_1) \\ &+ \frac{\Gamma(2-\gamma)}{t_2^{1-\gamma}} J_2(y(t_2^-))(t-t_2) \\ &- \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)} \widehat{f}_j(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha}}{\Gamma(1+\alpha)} \widehat{f}_0(s)ds. \end{aligned} \tag{2.15}$$

In general, for $t \in (t_l, t_{l+1}], 1 \le l \le m$, we have the following result

$$\begin{aligned} y(t) = &\phi(0) - g(y_{\rho(t_1,y_{t_1})}, y_{\rho(t_2,y_{t_2})}, \ldots, y_{\rho(t_m,y_{t_m})})(0) + b_0 t \\ &+ \sum_{k=1}^{l} I_k(y(t_k^-)) + \sum_{k=1}^{l} \left(\frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}} J_k(y(t_k^-)) \right)(t-t_k) \\ &- \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)} \widehat{f}_j(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha}}{\Gamma(1+\alpha)} \widehat{f}_0(s)ds. \end{aligned} \tag{2.16}$$

Finally, using the fractional order boundary condition (1.4), where $({}^cD^\gamma y(0))$ is calculated from (2.6) and $({}^cD^\gamma y(T))$ from (2.16) when $l = m$, we obtain the value of b_0 given by (2.2). On summarizing, we obtain the desired integral equation (2.5).

Conversely, let y satisfies (2.5). Since $1 < 1+\alpha < 2$, by virtue of ${}^cD^{1+\alpha} C = 0$(C is a constant), ${}^cD^{1+\alpha} t = 0$ and ${}^cD^{1+\alpha} J^{1+\alpha} y(t) = y(t)$, for $t \in \mathscr{D} = [0,T] - \{t_1, t_2, \ldots, t_k\}$ we have

$$\begin{aligned} {}^cD^{1+\alpha} y(t) = &- \sum_{j=1}^{n} {}^cD^{1+\alpha} \int_0^t \frac{(t-s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)} \widehat{f}_j(s)ds \\ &+ {}^cD^{1+\alpha} \int_0^t \frac{(t-s)^{\alpha}}{\Gamma(1+\alpha)} \widehat{f}_0(s)ds \end{aligned}$$

$$= -\sum_{j=1}^{n} {}^{c}D^{1+\alpha} J^{1+\alpha-\beta_j} \widehat{f}_j(s)ds + {}^{c}D^{1+\alpha} J^{1+\alpha} \widehat{f}_0(s)ds = -\sum_{j=1}^{n} {}^{c}D^{\beta_j} \widehat{f}_j(s)ds + \widehat{f}_0(s)ds.$$

Next, to validate the impulsive conditions for $t \in (t_l, t_{l+1}], 1 \leq l \leq m$, we get

$$\begin{aligned} y(t_l^+) - y(t_l^-) &= \sum_{k=1}^{l} I_k(y(t_k^-)) - \sum_{k=1}^{l-1} I_k(y(t_k^-)) \\ &+ \sum_{k=1}^{l} \left(\frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}} J_k(y(t_k^-)) \right)(t_l - t_k) \\ &- \sum_{k=1}^{l-1} \left(\frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}} J_k(y(t_k^-)) \right)(t_l - t_k) \\ &= I_l(y(t_l^-)). \end{aligned}$$

Similarly, for fractional impulsive condition

$$\begin{aligned} &({}^{c}D^{\gamma} y(t_l^+)) - ({}^{c}D^{\gamma} y(t_l^-)) \\ &= \sum_{k=1}^{l} \left(\frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}} J_k(y(t_k^-)) \right) \frac{t_l^{1-\gamma}}{\Gamma(2-\gamma)} \\ &\quad - \sum_{k=1}^{l-1} \left(\frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}} J_k(y(t_k^-)) \right) \frac{t_l^{1-\gamma}}{\Gamma(2-\gamma)} \\ &= \left(\frac{\Gamma(2-\gamma)}{t_l^{1-\gamma}} J_l(y(t_l^-)) \right) \frac{t_l^{1-\gamma}}{\Gamma(2-\gamma)} = J_l(y(t_l^-)). \end{aligned}$$

Further, it is easy to show that boundary condition (1.4) holds. Thus y given by (2.5) satisfies the system (1.1) – (1.3) with the boundary conditions (1.4). This completes the proof of the lemma. □

Lemma 2.6. *A piecewise continuous differential functions $y : [-\tau, T] \to \mathbb{E}$ be a solution of the following integral equation* $y(t) =$

$$\begin{cases} \phi(t) - g(y_{\rho(t_1,y_{t_1})}, y_{\rho(t_2,y_{t_2})}, \dots, y_{\rho(t_m,y_{t_m})})(t), t \in [-\tau, 0]; \\ \phi(0) - g(y_{\rho(t_1,y_{t_1})}, y_{\rho(t_2,y_{t_2})}, \dots, y_{\rho(t_m,y_{t_m})})(0) + b_0 t \\ -\sum_{j=1}^{n} \int_0^t \frac{(t-s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)} f_j(s, y_{\rho(s,y_s)}) ds \\ +\int_0^t \frac{(t-s)^{\alpha}}{\Gamma(1+\alpha)} f_0(s, y_{\rho(s,y_s)}, K(y)(s)) ds, \quad t \in [0, t_1]; \\ \vdots \\ \phi(0) - g(y_{\rho(t_1,y_{t_1})}, y_{\rho(t_2,y_{t_2})}, \dots, y_{\rho(t_m,y_{t_m})})(0) + c_0 t \\ +\sum_{k=1}^{m} I_k(y(t_k^-)) + \sum_{k=1}^{m} \frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}} J_k(y(t_k^-))(t - t_k) \\ -\sum_{j=1}^{n} \int_0^t \frac{(t-s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)} f_j(s, y_{\rho(s,y_s)}) ds \\ +\int_0^t \frac{(t-s)^{\alpha}}{\Gamma(1+\alpha)} f_0(s, y_{\rho(s,y_s)}, K(y)(s)) ds, \quad t \in (t_m, T]. \end{cases} \tag{2.17}$$

where

$$\begin{aligned} c_0 = &\frac{d}{c+d} \left[\frac{1}{d} \int_0^T h(s, y_1(s), \dots, y_m(s)) ds \right. \\ &+ \sum_{j=1}^{n} \int_0^T \frac{(T-s)^{\alpha-\beta_j-1}}{\Gamma(\alpha-\beta_j)} f_j(s, y_{\rho(s,y_s)}) ds \\ &- \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f_0(s, y_{\rho(s,y_s)}, K(y)(s)) ds \\ &\left. - \sum_{k=1}^{m} \frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}} J_k(y(t_k^-)) \right] \end{aligned} \tag{2.18}$$

if and only if y is a solution of the impulsive fractional system (1.1) – (1.3) *with the boundary condition* (1.5).

Proof. The proof is similar to the proof of previous Lemma 2.5. Assume that notations be given as in the proof of Lemma 2.5. Now, applying the boundary condition (1.5), where $y'(0)$ is calculated from (2.6) and $y'(T)$ from (2.16) when $l = m$, we get the value of c_0 given by (2.18). The remaining part of the proof is the similar as in the proof of Lemma 2.5. □

Remark 2.7. *We note that the solution expression of the system* (1.1) – (1.3) *with the boundary condition* (1.4) *is independent of parameter c appearing in the boundary condition. Thus, by Lemma* 2.5 *we come to the conclusion that the parameter c may be of arbitrary nature in the boundary condition* (1.4).

3. Main Results

In this section, we deal with the existence and uniqueness of solution for the system (1.1) – (1.3) in light of the boundary conditions (1.4) and (1.5).

Now, we consider the following assumptions to establish the existence results:

(A_1) There exist positive constants $\mu_{f_0}, \mu_{f_0}^0$ and μ_{f_j} such that

$$\begin{aligned} &\|f_0(t, \psi, x) - f_0(t, \chi, y)\|_{\mathbb{E}} \\ &\qquad \leq \mu_{f_0} \|\psi - \chi\|_{\mathscr{PC}_0} + \mu_{f_0}^0 \|x - y\|_{\mathbb{E}}, \\ &\|f_j(t, \psi) - f_j(t, \chi)\|_{\mathbb{E}} \leq \mu_{f_j} \|\psi - \chi\|_{\mathscr{PC}_0}, \end{aligned}$$

for all $j = 1, 2, 3, \dots, n$, and $x, y \in \mathbb{E}$, $\psi, \chi \in \mathscr{PC}_0$.

(A_2) There exist positive constants μ_g, μ_h, μ_I and μ_J such that

$$\begin{aligned} \|g(\psi_{t_1}, \psi_{t_2}, \dots, \psi_{t_m}) - g(\chi_{t_1}, \chi_{t_2}, \dots, \chi_{t_m})\|_{\mathscr{PC}_0} &\leq \mu_g \|\psi - \chi\|_{\mathscr{PC}_0}, \\ \|h(t, x_1, \dots, x_m) - h(t, y_1, \dots, y_m)\|_{\mathbb{E}} &\leq \mu_h \|x - y\|_{\mathbb{E}}, \\ \|I_k(x) - I_k(y)\|_{\mathbb{E}} &\leq \mu_I \|x - y\|_{\mathbb{E}}, \\ \|J_k(x) - J_k(y)\|_{\mathbb{E}} &\leq \mu_J \|x - y\|_{\mathbb{E}}, \end{aligned}$$

for all $k = 1, 2, 3, \dots, m$, and $x, y \in \mathbb{E}$, $\psi, \chi \in \mathscr{PC}_0$.

The following result is based on Banach fixed point theorem.

Theorem 3.1. *Assume that the assumptions $(A_1)-(A_2)$ are fulfilled, then the system (1.1) $-$ (1.3) with the boundary condition (1.4) has a unique solution on $[-\tau,T]$ if $\Theta_m<1$, where*

$$\Theta_m := \Bigg[\mu_g+\frac{Tb}{d}\mu_h+\sum_{j=1}^{n}\Big(a_{\alpha-\beta_j-\delta}b+a_{\alpha-\beta_j}\Big)\mu_{f_j} + \Big(a_{\alpha-\delta}b+a_\alpha\Big)[\mu_{f_0}+\mu^0_{f_0}K^0]+m\mu_I + \sum_{k=1}^{m}\frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}}(T+T^\delta)\mu_J\Bigg],$$

where $b=T^\delta\Gamma(2-\delta)$ and $a_z:=\frac{T^{1+z}}{\Gamma(2+z)}$ for all $1+z>0$.

Proof. First, we convert our problem in a fixed point problem by introducing an operator $Q:\mathscr{PC}_T\to\mathscr{PC}_T$ defined by $(Qy)(t)=$

$$\begin{cases} \phi(t)-g(y_{\rho(t_1,y_{t_1})},y_{\rho(t_2,y_{t_2})},\dots,y_{\rho(t_m,y_{t_m})})(t), t\in[-\tau,0];\\ \phi(0)-g(y_{\rho(t_1,y_{t_1})},y_{\rho(t_2,y_{t_2})},\dots,y_{\rho(t_m,y_{t_m})})(0)+b_0t\\ -\sum_{j=1}^{n}\int_0^t\frac{(t-s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)}f_j(s,y_{\rho(s,y_s)})ds\\ +\int_0^t\frac{(t-s)^{\alpha}}{\Gamma(1+\alpha)}f_0(s,y_{\rho(s,y_s)},K(y)(s))ds, \quad t\in[0,t_1];\\ \vdots\\ \phi(0)-g(y_{\rho(t_1,y_{t_1})},y_{\rho(t_2,y_{t_2})},\dots,y_{\rho(t_m,y_{t_m})})(0)+b_0t\\ +\sum_{k=1}^{m}I_k(y(t_k^-))+\sum_{k=1}^{m}\frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}}J_k(y(t_k^-))(t-t_k)\\ -\sum_{j=1}^{n}\int_0^t\frac{(t-s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)}f_j(s,y_{\rho(s,y_s)})ds\\ +\int_0^t\frac{(t-s)^{\alpha}}{\Gamma(1+\alpha)}f_0(s,y_{\rho(s,y_s)},K(y)(s))ds, \quad t\in(t_m,T]. \end{cases} \tag{3.1}$$

where b_0 is defined by (2.2). Next, we show that Q has a unique fixed point.

Let $x,y\in\mathscr{PC}_T$. It is clear that $\|(Qx)-(Qy)\|_{\mathscr{PC}_T}\le\mu_g\|x-y\|_{\mathscr{PC}_T}$ holds for $t\in[-\tau,0]$.

For $t\in[0,t_1]$, we have

$$\begin{aligned} &\|(Qx)(t)-(Qy)(t)\|_{\mathbb{E}}\\ &\le\|g(x_{\rho(t_1,x_{t_1})},x_{\rho(t_2,x_{t_2})},\dots,x_{\rho(t_m,x_{t_m})})(0)\\ &\quad -g(y_{\rho(t_1,y_{t_1})},y_{\rho(t_2,y_{t_2})},\dots,y_{\rho(t_m,y_{t_m})})(0)\|_{\mathbb{E}}\\ &\quad +\frac{T\Gamma(2-\delta)}{T^{1-\delta}}\Bigg[\frac{1}{d}\int_0^T\|h(s,x_1(s),\dots,x_m(s))\\ &\quad -h(s,x_1(s),\dots,x_m(s))\|_{\mathbb{E}}ds\\ &\quad +\sum_{j=1}^{n}\int_0^T\frac{(T-s)^{\alpha-\beta_j-\delta}}{\Gamma(1+\alpha-\beta_j-\delta)}\|f_j(s,x_{\rho(s,x_s)})\\ &\quad -f_j(s,y_{\rho(s,y_s)})\|_{\mathbb{E}}ds\\ &\quad +\int_0^T\frac{(T-s)^{\alpha-\delta}}{\Gamma(1+\alpha-\delta)}\|f_0(s,x_{\rho(s,x_s)},K(x)(s))\\ &\quad -f_0(s,y_{\rho(s,y_s)},K(y)(s))\|_{\mathbb{E}}ds\\ &\quad +\sum_{k=1}^{m}\frac{\Gamma(2-\gamma)}{\Gamma(2-\delta)}\|J_k(x(t_k^-))-J_k(y(t_k^-))\|_{\mathbb{E}}t_k^{\gamma-1}\Bigg]\\ &\quad +\sum_{j=1}^{n}\int_0^t\frac{(t-s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)}\|f_j(s,x_{\rho(s,x_s)})\\ &\quad -f_j(s,y_{\rho(s,y_s)})\|_{\mathbb{E}}ds\\ &\quad +\int_0^t\frac{(t-s)^{\alpha}}{\Gamma(1+\alpha)}\|f_0(s,x_{\rho(s,x_s)},K(x)(s))\\ &\quad -f_0(s,y_{\rho(s,y_s)},K(y)(s))\|_{\mathbb{E}}ds\\ &\le\Bigg[\mu_g+\frac{Tb}{d}\mu_h+\sum_{j=1}^{n}\Big(a_{\alpha-\beta_j-\delta}b+a_{\alpha-\beta_j}\Big)\mu_{f_j}\\ &\quad +\Big(a_{\alpha-\delta}b+a_\alpha\Big)[\mu_{f_0}+\mu^0_{f_0}K^0]\\ &\quad +\sum_{k=1}^{m}\Gamma(2-\gamma)\mu_J t_k^{\gamma-1}T^\delta\Bigg]\|x-y\|_{\mathscr{PC}_T}. \end{aligned}$$

For $t\in(t_m,t_{m+1}]$, we sustain

$$\begin{aligned} &\|(Qx)(t)-(Qy)(t)\|_{\mathbb{E}}\\ &\le\|g(x_{\rho(t_1,x_{t_1})},x_{\rho(t_2,x_{t_2})},\dots,x_{\rho(t_m,x_{t_m})})(0)\\ &\quad -g(y_{\rho(t_1,y_{t_1})},y_{\rho(t_2,y_{t_2})},\dots,y_{\rho(t_m,y_{t_m})})(0)\|_{\mathbb{E}}\\ &\quad \frac{T\Gamma(2-\delta)}{T^{1-\delta}}\Bigg[\frac{1}{d}\int_0^T\|h(s,x_1(s),\dots,x_m(s))\\ &\quad -h(s,x_1(s),\dots,x_m(s))\|_{\mathbb{E}}ds\\ &\quad +\sum_{j=1}^{n}\int_0^T\frac{(T-s)^{\alpha-\beta_j-\delta}}{\Gamma(1+\alpha-\beta_j-\delta)}\|f_j(s,x_{\rho(s,x_s)})\\ &\quad -f_j(s,y_{\rho(s,y_s)})\|_{\mathbb{E}}ds\\ &\quad +\int_0^T\frac{(T-s)^{\alpha-\delta}}{\Gamma(1+\alpha-\delta)}\|f_0(s,x_{\rho(s,x_s)},K(x)(s))\\ &\quad -f_0(s,y_{\rho(s,y_s)},K(y)(s))\|_{\mathbb{E}}ds\\ &\quad +\sum_{k=1}^{m}\frac{\Gamma(2-\gamma)}{\Gamma(2-\delta)}\|J_k(x(t_k^-))-J_k(y(t_k^-))\|_{\mathbb{E}}t_k^{\gamma-1}\Bigg]\\ &\quad +\sum_{k=1}^{m}\|I_k(x(t_k^-))-I_k(y(t_k^-))\|_{\mathbb{E}}\\ &\quad +\sum_{k=1}^{m}|t-t_k|\frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}}\|J_k(y(t_k^-))-J_k(y(t_k^-))\|_{\mathbb{E}}\\ &\quad +\sum_{j=1}^{n}\int_0^t\frac{(t-s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)}\|f_j(s,x_{\rho(s,x_s)})\\ &\quad -f_j(s,y_{\rho(s,y_s)})\|_{\mathbb{E}}ds\\ &\quad +\int_0^t\frac{(t-s)^{\alpha}}{\Gamma(1+\alpha)}\|f_0(s,x_{\rho(s,x_s)},K(x)(s))\\ &\quad -f_0(s,y_{\rho(s,y_s)},K(y)(s))\|_{\mathbb{E}}ds\\ &\le\Bigg[\mu_g+\frac{Tb}{d}\mu_h+\sum_{j=1}^{n}\Big(a_{\alpha-\beta_j-\delta}b+a_{\alpha-\beta_j}\Big)\mu_{f_j}\\ &\quad +\Big(a_{\alpha-\delta}b+a_\alpha\Big)[\mu_{f_0}+\mu^0_{f_0}K^0] \end{aligned}$$

$$+m\mu_I+\sum_{k=1}^{m}\frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}}(T+T^{\delta})\mu_J\Big]\|x-y\|_{\mathscr{PC}_T}.$$

Thus

$$\|(Qx)-(Qy)\|_{\mathscr{PC}_T}\leq\Theta_m\|x-y\|_{\mathscr{PC}_T}.$$

Since $\Theta_m<1$, so by Banach contraction principal we conclude that the system (1.1) − (1.3) with the boundary condition (1.4) has a unique solution $y\in\mathscr{PC}_T$. This completes the proof of the theorem. □

Definition 3.2. *[10] Let $\mathscr{B}_r$ be a nonempty, bounded, convex and closed subset of a Banach space $\mathbb{E}$. Consider two operator Q_1 and Q_2 such that*

(i) *Q_1 is compact operator.*

(ii) *Q_2 is contraction mapping,*

then $Q:=Q_1+Q_2$ is called condensing map on $\mathscr{B}_r$.

Theorem 3.3. *[10, Sadovskii] Let $\mathscr{B}_r$ be a nonempty, bounded, convex and closed subset of a Banach space $\mathbb{E}$. Then a condensing map $Q:\mathscr{B}_r\to\mathscr{B}_r$ has a fixed point in $\mathscr{B}_r$.*

Next result is based on Sadovskii fixed point theorem. Therefore, we assume the following additional assumptions:

(A_3) There exists functions $\nu_{f_0},\nu_{f_j}\in L^1([0,T],\mathbb{R}^+)$ such that

$$\|f_0(t,\psi,x)\|_{\mathbb{E}}\leq\nu_{f_0}(t),\quad \|f_j(t,\psi)\|_{\mathbb{E}}\leq\nu_{f_j}(t),$$

for all $j=1,2,3,\ldots,n$, $x\in\mathbb{E}$, $\psi\in\mathscr{PC}_0$.

Theorem 3.4. *Assume that the assumptions $(A_2)-(A_3)$ are fulfilled, then the system (1.1) − (1.3) with the boundary condition (1.4) has at least one solution on $[-\tau,T]$ if $\Delta_m<1$, where*

$$\Delta_m=\left[\mu_g+\frac{Tb}{d}\mu_h+m\mu_I+\sum_{k=1}^{m}\frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}}(T^{\delta}+T)\mu_J\right]. \tag{3.2}$$

Proof. Let $h^0=\sup_{s\in[0,T]}\|h(s,0,\ldots,0)\|_{\mathbb{E}}$ and $g^0=\|h(0,0,\ldots,0)\|_{\mathbb{E}}$. Now we claim that there exists a positive number r such that $Q\mathscr{B}_r\subseteq\mathscr{B}_r$, where $\mathscr{B}_r:=\{y\in\mathscr{PC}_T:\|y\|_{\mathscr{PC}_T}\leq r\}$. If this is not true, then for each $r>0$, there would exists $y\in\mathscr{B}_r$ and $t\in[-\tau,T]$ such that $\|Qy\|_{\mathscr{PC}_T}>r$. For $y\in\mathscr{B}_r$ and $t\in[-\tau,T]$, we have

$$\begin{aligned}
r<&\|(Qy)\|_{\mathbb{E}}\\
=&\|\phi(0)\|+\|g(y_{\rho(t_1,y_{t_1})},y_{\rho(t_2,y_{t_2})},\ldots,y_{\rho(t_m,y_{t_m})})(0)\\
&-g(0,0,...,0)(0)+g(0,0,...,0)(0)\|_{\mathbb{E}}\\
&+T^{\delta}\Gamma(2-\delta)\Big[\frac{1}{d}\int_0^T\|h(s,x_1(s),\ldots,x_m(s))\\
&-h(s,0,\ldots,0)+h(s,0,0)\|_{\mathbb{E}}ds\\
&+\sum_{j=1}^{n}\int_0^T\frac{(T-s)^{\alpha-\beta_j-\delta}}{\Gamma(1+\alpha-\beta_j-\delta)}\|f_j(s,y_{\rho(s,y_s)})\|_{\mathbb{E}}ds\\
&+\int_0^T\frac{(T-s)^{\alpha-\delta}}{\Gamma(1+\alpha-\delta)}\|f_0(s,y_{\rho(s,y_s)},K(y)(s))\|_{\mathbb{E}}ds\\
&+\sum_{k=1}^{m}\frac{\Gamma(2-\gamma)}{\Gamma(2-\delta)}\|J_k(y(t_k^-))-J_k(0)+J_k(0)\|_{\mathbb{E}}t_k^{\gamma-1}\Big]\\
&+\sum_{k=1}^{m}\|I_k(y(t_k^-))-I_k(0)+I_k(0)\|_{\mathbb{E}}\\
&+\sum_{k=1}^{m}|t-t_k|\frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}}\|J_k(y(t_k^-))-J_k(0)+J_k(0)\|_{\mathbb{E}}\\
&+\sum_{j=1}^{n}\int_0^t\frac{(t-s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)}\|f_j(s,y_{\rho(s,y_s)})\|_{\mathbb{E}}ds\\
&+\int_0^t\frac{(t-s)^{\alpha}}{\Gamma(1+\alpha)}\|f_0(s,y_{\rho(s,y_s)},K(y)(s))\|_{\mathbb{E}}ds\\
\leq&\|\phi(0)\|_{\mathbb{E}}+(\mu_g r+g^0)+\frac{Tb}{d}(\mu_h r+h^0)\\
&+\sum_{j=1}^{n}\left(a_{\alpha-\beta_j-\delta}b+a_{\alpha-\beta_j}b\right)\|\nu_{f_j}\|_{L^1}\\
&+\sum_{k=1}^{m}\frac{\Gamma(2-\gamma)}{\Gamma(2-\delta)}(\mu_J r+\|J(0)\|_{\mathbb{E}})bt_k^{\gamma-1}\\
&+m(\mu_I r+\|I(0)\|_{\mathbb{E}})+\left(a_{\alpha-\delta}b+a_{\alpha}b\right)\|\nu_{f_0}\|_{L^1}\\
&+\sum_{k=1}^{m}T\frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}}(\mu_J r+\|J(0)\|_{\mathbb{E}}).
\end{aligned}$$

Now, dividing the last inequality by r and for sufficiently large value of r, we achive

$$\begin{aligned}
1<\Big[&\mu_g+\frac{Tb}{d}\mu_h+\sum_{k=1}^{m}\frac{\Gamma(2-\gamma)}{\Gamma(2-\delta)}\mu_J bt_k^{\gamma-1}\\
&+m\mu_I+\sum_{k=1}^{m}T\frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}}\mu_J\Big].
\end{aligned}$$

which contradict (3.2), hence $Q(\mathscr{B}_r)\in\mathscr{B}_r$ for some positive number r.

Now, decompose the operator $Q=Q_1+Q_2$ on $\mathscr{B}_r$, where

$$\begin{aligned}
Q_1y(t)=&\int_0^t\frac{(t-s)^{\alpha}}{\Gamma(1+\alpha)}f_0(s,y_{\rho(s,y_s)},K(y)(s))ds\\
&-\sum_{j=1}^{n}\int_0^t\frac{(t-s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)}f_j(s,y_{\rho(s,y_s)})ds\\
&+\frac{t\Gamma(2-\delta)}{T^{1-\delta}}\Big[\sum_{j=1}^{n}\int_0^T\frac{(T-s)^{\alpha-\beta_j-\delta}}{\Gamma(1+\alpha-\beta_j-\delta)}\\
&\times f_j(s,y_{\rho(s,y_s)})\\
&-\int_0^T\frac{(T-s)^{\alpha-\delta}}{\Gamma(1+\alpha-\delta)}f_0(s,y_{\rho(s,y_s)},K(y)(s))ds\Big].
\end{aligned} \tag{3.3}$$

for $t \in [0,T]$, and

$$Q_2y(t) = \begin{cases} \phi(t) \\ -g(y_{\rho(t_1,y_{t_1})}, y_{\rho(t_2,y_{t_2})}, \ldots, y_{\rho(t_m,y_{t_m})})(t), t \in [-\tau, 0]; \\ \phi(0) - g(y_{\rho(t_1,y_{t_1})}, y_{\rho(t_2,y_{t_2})}, \ldots, y_{\rho(t_m,y_{t_m})})(0) \\ +\frac{t\Gamma(2-\delta)}{T^{1-\delta}}\left[\frac{1}{d}\int_0^T h(s, y_1(s), \ldots, y_m(s))ds \right. \\ \left. -\sum_{k=1}^m \frac{\Gamma(2-\gamma)}{\Gamma(2-\delta)} J_k(y(t_k^-))t_k^{\gamma-1}\right] \\ +\sum_{k=1}^m I_k(y(t_k^-)) \\ +\sum_{k=1}^m \frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}} J_k(y(t_k^-))(t-t_k), t \in (t_m, T]. \end{cases} \tag{3.4}$$

In the further steps, we will show that Q_2 is contraction mapping and Q_1 is compact operator.

For $x, y \in \mathscr{B}_r$ and $t \in (t_m, t_{m+1}]$ we have

$$\begin{aligned} &\|Q_2x(t) - Q_2y(t)\|_{\mathbb{E}} \\ &\le \|g(x_{\rho(t_1,x_{t_1})}, x_{\rho(t_2,x_{t_2})}, \ldots, x_{\rho(t_m,x_{t_m})})(0) \\ &\quad - g(y_{\rho(t_1,y_{t_1})}, y_{\rho(t_2,y_{t_2})}, \ldots, y_{\rho(t_m,y_{t_m})})(0)\|_{\mathbb{E}} \\ &\quad \frac{T\Gamma(2-\delta)}{T^{1-\delta}}\left[\frac{1}{d}\int_0^T \|h(s, x_1(s), \ldots, x_m(s)) \right. \\ &\quad - h(s, x_1(s), \ldots, x_m(s))\|_{\mathbb{E}} ds \\ &\quad \left. + \sum_{k=1}^m \frac{\Gamma(2-\gamma)}{\Gamma(2-\delta)} \|J_k(x(t_k^-)) - J_k(y(t_k^-))\|_{\mathbb{E}} t_k^{\gamma-1}\right] \\ &\quad + \sum_{k=1}^m \|I_k(x(t_k^-)) - I_k(y(t_k^-))\|_{\mathbb{E}} \\ &\quad + \sum_{k=1}^m |t-t_k| \frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}} \|J_k(y(t_k^-)) - J_k(y(t_k^-))\|_{\mathbb{E}}. \end{aligned}$$

Thus

$$\begin{aligned} &\|Q_2x - Q_2y\|_{\mathscr{PC}_T} \\ &\le \left[\mu_g + \frac{Tb}{d}\mu_h + m\mu_I \right. \\ &\quad \left. + \sum_{k=1}^m \frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}} (T^\delta + T)\mu_J\right] \|x-y\|_{\mathscr{PC}_T}. \end{aligned}$$

By the assumption Q_2 is a contraction mapping.

Let y^n be a sequence in $\mathscr{B}_r$. Then by the continuity of f_0 and f_j we have

$$\begin{aligned} &\|Q_1y^n(t) - Q_1y(t)\|_{\mathbb{E}} \\ &\le \frac{T^{1+\alpha}}{\Gamma(2+\alpha)} \|f_0(s, y^n_{\rho(s,y^n_s)}, K(y^n)(s)) \\ &\quad - f_0(s, y_{\rho(s,y_s)}, K(y)(s))\|_{\mathbb{E}} \\ &\quad - \sum_{j=1}^n \frac{T^{1+\alpha-\beta_j}}{\Gamma(2+\alpha-\beta_j)} \\ &\quad \times \|f_j(s, y^n_{\rho(s,y^n_s)}) - f_j(s, y_{\rho(s,y_s)})\|_{\mathbb{E}} ds \\ &\quad + T^\delta \Gamma(2-\delta)\left[\sum_{j=1}^n \frac{T^{1+\alpha-\beta_j-\delta}}{\Gamma(2+\alpha-\beta_j-\delta)} \right. \\ &\quad \times \|f_j(s, y^n_{\rho(s,y^n_s)}) - f_j(s, y_{\rho(s,y_s)})\|_{\mathbb{E}} \\ &\quad - \frac{T^{1+\alpha-\delta}}{\Gamma(2+\alpha-\delta)} \|f_0(s, y^n_{\rho(s,y^n_s)}, K(y^n)(s)) \\ &\quad \left. - f_0(s, y_{\rho(s,y_s)}, K(y)(s))\|_{\mathbb{E}} ds\right], \end{aligned}$$

which implies that Q_1 is continuous on $\mathscr{B}_r$.

Also, Q_1 is uniformly bounded on $\mathscr{B}_r$.

$$\begin{aligned} &\|Q_1y\|_{\mathscr{PC}_T} \\ &\le \frac{T^{1+\alpha}}{\Gamma(2+\alpha)} \|\nu_{f_0}\|_{L^1} \\ &\quad - \sum_{j=1}^n \frac{T^{1+\alpha-\beta_j}}{\Gamma(2+\alpha-\beta_j)} \|\nu_{f_j}\|_{L^1} \\ &\quad + T^\delta\Gamma(2-\delta)\left[\sum_{j=1}^n \frac{T^{1+\alpha-\beta_j-\delta}}{\Gamma(2+\alpha-\beta_j-\delta)} \|\nu_{f_j}\|_{L^1} \right. \\ &\quad \left. - \frac{T^{1+\alpha-\delta}}{\Gamma(2+\alpha-\delta)} \|\nu_{f_0}\|_{L^1}\right], \end{aligned}$$

Next, we show that Q_1 is equicontinuous. Let $\tau_1, \tau_2 \in (t_l, t_{l+1}], 1 \le l \le m$ such that $\tau_1 < \tau_2$ and $y \in \mathscr{B}_r$. Using the fact that f_0 and f_j are bounded on a compact set $f_0(t, \psi, x) \le M_{f_0}$ and $f_j(t, \psi) \le M_{f_j}$, we have

$$\begin{aligned} &\|Q_1y(\tau_2) - Q_1y(\tau_1)\|_{\mathbb{E}} \\ &\le \| \int_0^{\tau_2} \frac{(\tau_2 - s)^\alpha}{\Gamma(1+\alpha)} f_0(s, y_{\rho(s,y_s)}, K(y)(s))ds \\ &\quad - \int_0^{\tau_1} \frac{(\tau_1 - s)^\alpha}{\Gamma(1+\alpha)} f_0(s, y_{\rho(s,y_s)}, K(y)(s))ds\|_{\mathbb{E}} \\ &\quad + \sum_{j=1}^n \| \int_0^{\tau_2} \frac{(\tau_2 - s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)} f_j(s, y_{\rho(s,y_s)})ds \\ &\quad - \int_0^{\tau_1} \frac{(\tau_1 - s)^{\alpha-\beta_j}}{\Gamma(1+\alpha-\beta_j)} f_j(s, y_{\rho(s,y_s)})ds\|_{\mathbb{E}} \\ &\quad + \frac{(\tau_2 - \tau_1)\Gamma(2-\delta)}{T^{1-\delta}} \\ &\quad \times \left[\sum_{j=1}^n \int_0^T \frac{(T-s)^{\alpha-\beta_j-\delta}}{\Gamma(1+\alpha-\beta_j-\delta)} \|f_j(s, y_{\rho(s,y_s)})\| ds \right. \\ &\quad \left. + \int_0^T \frac{(T-s)^{\alpha-\delta}}{\Gamma(1+\alpha-\delta)} \|f_0(s, y_{\rho(s,y_s)}, K(y)(s))\|_{\mathbb{E}} ds\right] \end{aligned}$$

$$\leq \int_0^{\tau_1} \frac{|(\tau_2-s)^{\alpha}-(\tau_1-s)^{\alpha}|}{\Gamma(1+\alpha)} M_{f_0} ds + \frac{(\tau_2-\tau_1)^{1+\alpha}}{\Gamma(2+\alpha)} M_{f_0} ds + \sum_{j=1}^{n} \int_0^{\tau_1} \frac{|(\tau_2-s)^{\alpha-\beta_j}-(\tau_1-s)^{\alpha-\beta_j}|}{\Gamma(1+\alpha-\beta_j)} M_{f_j} ds + \frac{(\tau_2-\tau_1)^{1+\alpha-\beta_j}}{\Gamma(2+\alpha-\beta_j)} M_{f_j} + \frac{(\tau_2-\tau_1)\Gamma(2-\delta)}{T^{1-\delta}} \times \left[\sum_{j=1}^{n} a_{\alpha-\beta_j-\delta} M_{f_j} + a_{\alpha-\delta} M_{f_0}\right] \to 0,$$

as $\tau_2 \to \tau_1$. So Q_1 is equicontinuity for each $(t_l, t_{l+1}], 1 \leq l \leq m$; hence Q_1 is relative compact on $\mathscr{B}_r$. We conclude by utilizing PC-Arzela Ascoli theorem [37, Theorem 2.12] that Q_1 is compact operator. Hence, we obtain that the $Q = Q_1 + Q_2$ is condensing map on $\mathscr{B}_r$. Therefore, by using Sadovskii fixed point theorem the system (1.1) – (1.3) with the boundary condition (1.4) has a fixed point in $\mathscr{B}_r$. □

Next, we state the existence results for the system (1.1) – (1.3) with the boundary condition (1.5) without proof since the proof can be produced on similar lines as we obtained for the system (1.1) – (1.3) with the boundary condition (1.4).

Theorem 3.5. *Assume that the assumptions $(A_1) - (A_2)$ are fulfilled, then the system (1.1) – (1.3) with the boundary condition (1.5) has a unique solution on $[-\tau, T]$ if $\Theta_m < 1$, where*

$$\Theta_m := \left[\mu_g + \frac{Tp}{d}\mu_h + \sum_{j=1}^{n}\left(a_{\alpha-\beta_j-1}p + a_{\alpha-\beta_j}\right)\mu_{f_j} + \left(a_{\alpha-1}p + a_{\alpha}\right)[\mu_{f_0} + \mu_{f_0}^0 K^0] + m\mu_I + \sum_{k=1}^{m}\frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}}(p+T)\mu_J\right],$$

where $p = \frac{Td}{c+d}$.

Theorem 3.6. *Assume that the assumptions $(A_2) - (A_3)$ are fulfilled, then the system (1.1) – (1.3) with the boundary condition (1.5) has at least one solution on $[-\tau, T]$ if $\Delta_m < 1$, where*

$$\Delta_m = \left[\mu_g + \frac{Tp}{d}\mu_h + m\mu_I + \sum_{k=1}^{m}\frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}}(p+T)\mu_J\right]. \tag{3.5}$$

4. Example

Consider the following impulsive fractional differential equation

$$ {}^cD^{1+\frac{7}{8}}y(t) + {}^cD^{\frac{5}{8}}\left[\frac{\sin t y(t-\sigma(y(t)))e^{-t}}{(t+5)^2+y^2(t-\sigma(y(t))}\right] + {}^cD^{\frac{3}{8}}\left[\frac{te^{-2t}\sin\left(\frac{y(t-\sigma(y(t)))}{5}\right)}{(t+5)+y^2(t-\sigma(y(t))}\right] = \frac{e^t y(t-\sigma(y(t)))}{36+y^2(t-\sigma(y(t))} + \int_0^t \cos(t-s)\frac{y(s)e^s}{20+y^2(s)}ds, \tag{4.1}$$

for $t \in (0,1) - \frac{1}{2}$.

$$\Delta y(\frac{1}{2}^-) = \frac{\|y(\frac{1}{2}^-)\|}{15+\|y(\frac{1}{2}^-)\|}, \tag{4.2}$$

$$\Delta({}^cD^{\frac{4}{5}}y(\frac{1}{2}^-)) = \frac{\|y(\frac{1}{2}^-)\|}{17+\|y(\frac{1}{2}^-)\|}, \tag{4.3}$$

$$y(t) + \sum_{i=1}^{m}\int_0^1 H(t-\xi)y(t-\sigma(y(t)))(\xi)d\xi = \phi(t), \tag{4.4}$$

for $t \in [-\tau, 0]$. Subject to the fractional order $\delta = \frac{1}{2}$ boundary conditions

$$4({}^cD^{\frac{1}{2}}y(0)) + 5({}^cD^{\frac{1}{2}}y(T)) = \frac{1}{20}\int_0^T [\sin(s-1)y_1 + s^2y_2 + \sin(y_3+1)]ds, \tag{4.5}$$

and when $\delta = 1$ i.e.

$$4(y'(0)) + 5(y'(T)) = \frac{1}{20}\int_0^T [\sin(s-1)y_1 + s^2y_2 + \sin(y_3+1)]ds, \tag{4.6}$$

where $\phi \in \mathscr{C}([-\tau,0],\mathbb{E})$, $K(t-\xi) \in \mathscr{PC}([-\tau,0],\mathbb{E})$, and $\sigma \in \mathscr{C}(\mathbb{E},[0,\infty)), 0 < t_1 = \frac{1}{2} < 1$. Set $\omega > 0$, and chose $\mathscr{PC}^{\omega}$ such that

$$\mathscr{PC}^{\omega} = \{\phi \in \mathscr{PC}((0,\infty],\mathbb{E}) : \lim_{t\to -\tau} e^{\omega t}\phi(t)\, exist\},$$

with the norm $\|\phi\|_{\omega} = \sup_{t\in(0,\infty]} e^{\omega t}|\phi(t)|, \phi \in \mathscr{PC}^{\omega}$.

We set $\rho(t,\varphi) = t - \sigma(\varphi(0))$, then $f_1(t,\varphi) = \frac{(\varphi)e^{-t}\sin t}{(t+5)^2+(\varphi)^2}$, $f_2(t,\varphi) = \frac{te^{-2t}\sin(\frac{(\varphi)}{5})}{(t+5)+(\varphi)^2}$, $f_0(t,\varphi,K(y)(t)) = \frac{e^t(\varphi)}{36+(\varphi)^2} + \int_0^t \cos(t-s)\frac{y(s)e^s}{20+y^2(s)}ds$, $g(\varphi)(t) = \sum_{i=1}^{m}\int_0^1 K(t-\xi)\varphi(\xi)d\xi$ and $h(s,y_1(s),ldots,y_3(s)) = [\sin(s-1)y_1 + s^2y_2 + \sin(y_3+1)]$. For convenience, we assume $H^0 = \sup_{t\in[0,1]}\int_0^1 H(t-\xi)d\xi < \frac{1}{3}$.

Further, we have

$$\|f_0(t,\psi,x)-f_0(t,\chi,y)\|_{\mathbb{E}} \le \frac{e}{36}\|\psi-\chi\|+\frac{e}{20}\|x-y\|_{\mathbb{E}},$$
$$\|f_1(t,\psi)-f_1(t,\chi)\|_{\mathbb{E}} \le \frac{1}{25}\|\psi-\chi\|,$$
$$\|f_2(t,\psi)-f_2(t,\chi)\|_{\mathbb{E}} \le \frac{1}{25}\|\psi-\chi\|,$$
$$\|h(t,x_1,\dots,x_3)-h(t,y_1,\dots,y_3)\|_{\mathbb{E}} \le \frac{1}{20}\|x-y\|_{\mathbb{E}},$$
$$\|I_k(x)-I_k(y)\|_{\mathbb{E}} \le \frac{1}{15}\|x-y\|_{\mathbb{E}},$$
$$\|J_k(x)-J_k(y)\|_{\mathbb{E}} \le \frac{1}{17}\|x-y\|_{\mathbb{E}},$$
$$\|g(\varphi)-g(\chi)\|_{\mathbb{E}} \le H^0\|\varphi-\chi\|,$$

for all $x,y \in \mathbb{E}, \psi,\chi \in \mathscr{PC}^{\omega}$ and $t \in [0,1]$ and moreover $\|f_0(t,\psi,x)\|_{\mathbb{E}} \le e^t$, $\|f_1(t,\psi)\|_{\mathbb{E}} \le e^{-t}$, and $\|f_2(t,\psi)\|_{\mathbb{E}} \le e^{-2t}$ $b = 0.8862$, and $p = 0.5556$. Now, we have the following results

(i) $\Big[\mu_g+\frac{Tb}{d}\mu_h+\sum_{j=1}^{n}\Big(a_{\alpha-\beta_j-\delta}b+a_{\alpha-\beta_j}\Big)\mu_{f_j}+\Big(a_{\alpha-\delta}b+a_{\alpha}\Big)[\mu_{f_0}+\mu^0_{f_0}K^0]+m\mu_I+\sum_{k=1}^{m}\frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}}(T+T^{\delta})\mu_J\Big] = H^0+0.6054<1$. Hence by Theorem 3.1, the system (1.1) − (1.3) with (1.4) admit a unique solution.

(ii) $\Big[\mu_g+\frac{Tb}{d}\mu_h+m\mu_I+\sum_{k=1}^{m}\frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}}(T^{\delta}+T)\mu_J\Big] = H^0+1.996<1$. Hence by Theorem 3.4, the system (1.1) − (1.3) with (1.4) admit at least one solution.

(iii) $\Big[\mu_g+\frac{Tp}{d}\mu_h+\sum_{j=1}^{n}\Big(a_{\alpha-\beta_j-1}p+a_{\alpha-\beta_j}\Big)\mu_{f_j}+\Big(a_{\alpha-1}p+a_{\alpha}\Big)[\mu_{f_0}+\mu^0_{f_0}K^0]+m\mu_I+\sum_{k=1}^{m}\frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}}(p+T)\mu_J\Big] = H^0+0.4951<1$. Hence by Theorem 3.5, the system (1.1) − (1.3) with (1.5) admit a unique solution.

(iv) $\Big[\mu_g+\frac{Tp\mu_h}{d}+m\mu_I+\sum_{k=1}^{m}\frac{\Gamma(2-\gamma)}{t_k^{1-\gamma}}(p+T)\mu_J\Big] = 0.1688+H^0<1$. Hence by Theorem 3.6, the system (1.1) − (1.3) with (1.5) admit at least one solution.

5. Conclusion

In this paper, an approach has been developed for some results on existence and uniqueness of solutions for a class of systems governed by (1.1) − (1.5) along with finite delay. The existence and uniqueness results have been established under more general settings via fractional order impulsive conditions (1.3) with fractional order and integer order boundary conditions (1.4) and (1.5), respectively. The differential system (1.1) − (1.5) describes diffusion wave character of a phenomena [32, 36]. Moreover, instantaneous forces present in the phenomena at certain points may be characterized more precisely by fractional order impulsive conditions (1.3) rather than integer one (see [19, 25]). The results are illustrated with a well-analyzed example in Section 4.

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