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Solution of two-dimensional non-linear Burgers' equations with nonlocal boundary condition

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Abstract
In this article, we tried to find the solution in Burgers' equations by iteration method. Also we have proposed a numerical method by using finite difference method.

Keywords
Two Dimensional Parabolic Equation, Periodic Boundary Condition, Finite Difference, Boundary Value.

AMS Subject Classification
65Bxx, 65Gxx, 65Mxx, 76Mxx, 65Nxx, 65Txx.

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Contents

1. Introduction

Burgers' equations are a special case of the Navier-Stokes equation. Burgers' equations are an important differential equation from fluid dynamics, and are used to describe various natural phenomena such as mathematically shown that the turbulence and modelling of gas dynamics, shock waves, etc. Numerical methods for Burges' problems: finite-element, finite-difference methods, spectral methods. Superpositions for nonlinear differential operators constructed in 1893 by Vessiot [10]. Key references can be found in [10, 11]. In this study, we use the superposition principle for nonlinear Burgers' equations.

Consider 2-dimensional nonlinear Burgers' problem taken from [2],

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{1.1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{1.2}$$

$$\begin{aligned} u(x,y,0) &= f_1(x,y), \\ v(x,y,0) &= g_1(x,y), \end{aligned} \tag{1.3}$$

$$\begin{aligned} u(x,y,t) &= f_2(x,y,t), \\ v(x,y,t) &= g_2(x,y,t). \end{aligned} \tag{1.4}$$

Here

$$\Omega=\{(x,y): a\le x\le b, a\le y\le b, 0<t<T\}$$

$\partial\Omega$ denotes the boundary of Ω, $u(x,y,t)$ and $v(x,y,t)$ are the velocity components to be determinant f_1, g_1 and f_2, g_2 are known functions and R is Reynolds number.

The analytical solution of equation (1.1) and (1.2) were given by Fletcher. Fletcher used to the transformation Hopf-Cole [3]. Fletcher made a comparison of the different numerical methods [4]. Wubs and Geode have made an expilicit-implicit method [5], Goyon used the many multilevel methods used by ADI [6]. Bahadır used the fully implicit finite-difference method[3]. Srivastava et al used The Crank- Nicolson scheme [7].

In this paper, an iterative method is presented to find numerical solutions problem. We use the superposition principle for nonlinear partial equations [10, 11].

Ω denotes the domain
$\Omega := \{(x,y,t): 0<x<1,\ 0<y<1, 0<t<T\}$

Partial differential equation (1.1) can be solved by splitting it into two one dimensional equation from [10] rather than discretising the complete two-dimensional Burgers' equation to give an aproximating equation based on two-dimensional computational molecule, as seen [8]. Let consider

$$\begin{aligned} u_x &\approx f(x,t,u) \\ u_y &\approx f(y,t,u). \end{aligned}$$

We can write equation (1.1) as equations (1.5) and (1.6)

$$\frac{1}{2}u_t - \frac{1}{2R}u_{xx} = uf(x,t,u), (x,t) \in \Omega \tag{1.5}$$

$$\frac{1}{2}u_t - \frac{1}{2R}u_{yy} = vf(y,t,u), (y,t) \in \Omega \tag{1.6}$$

Applying the same estimations for equation (1.2) we can write as equations (1.7) and (1.8)

$$\frac{1}{2}v_t - \frac{1}{2R}v_{xx} = ug(x,t,v), (x,t) \in \Omega \tag{1.7}$$

$$\frac{1}{2}v_t - \frac{1}{2R}v_{yy} = vg(y,t,v), (y,t) \in \Omega \tag{1.8}$$

with the periodic and initial boundary condition

$$\begin{aligned} u(x,0) &= \varphi(x), u(y,0) = \varphi(y)\ x,y \in [0,1] \\ u(0,t) &= u(1,t), u_x(1,t) = 0 \text{ for } x \\ u(0,t) &= u(1,t), u_y(1,t) = 0 \text{ for } y \\ v(x,0) &= \varphi(x), v(y,0) = \varphi(y)\ x,y \in [0,1], \\ v(0,t) &= v(1,t), v_x(1,t) = 0 \text{ for } x \\ v(0,t) &= v(1,t), v_y(1,t) = 0 \text{ for } y \end{aligned} \tag{1.9}$$

The functions $\varphi(x), \varphi(y)$ and $f(x,t,u)$, $f(y,t,u)$, $g(x,t,v)$ and $g(y,t,v)$ are defined functions on $[0,1]$ and $\partial\Omega \times (-\infty,\infty)$, respectively.

$\{u(x,t), v(x,t), u(y,t), v(y,t)\}$ are solutions. The periodic boundary conditions are encountered very often [1].

Definition 1.1. $\{u(x,t), v(x,t), u(y,t), v(y,t)\}$ *from the class* $(C^{2,1}(\Omega) \cap C^{1,0}(\partial\Omega))$ *is defined the classical solution of system (1.5)-(1.9).*

2. The Fourier Method for Burgers' Problem

Let these conditions are valid:

(a1) $\varphi(x), \varphi(y) \in C[0,1]$,
$\varphi(0) = \varphi(1), \varphi_x(1) = 0, \varphi(0) = \varphi(1), \varphi_y(1) = 0$,

(a2) Let $f(x,t,u)$,$g(x,t,v)$, $f(y,t,u)$ and $g(y,t,v)$ is continuous

(1)

$$\begin{aligned} |f(x,t,u) - f(x,t,\tilde{u})| &\leq L(x,t)\,|u-\tilde{u}| \\ |g(x,t,v) - g(x,t,\widetilde{v})| &\leq L(x,t)\,|v-\widetilde{v}|, \end{aligned} \tag{2.1}$$

where $L(x,t) \in L_2(\Omega),\ b(x,t) \geq 0$,

$$\begin{aligned} |f(y,t,u) - f(y,t,\tilde{u})| &\leq L(y,t)\,|u-\tilde{u}| \\ |g(y,t,v) - g(y,t,\widetilde{v})| &\leq L(y,t)\,|v-\widetilde{v}| \end{aligned} \tag{2.2}$$

where $L(y,t) \in L_2(\Omega),\ L(y,t) \geq 0$,
(2) $f(x,t,u), g(x,t,v) \in C[0,1],\ t\varepsilon[0,T]$,
(3) $f(x,t,u)|_{x=0} = f(x,t,u)|_{x=1},\ f_x(x,t,u)|_{x=1} = 0$,

$g(x,t,v)|_{x=0} = g(x,t,v)|_{x=1},\ g_x(x,t,v)|_{x=1} = 0.$
(4) $f(y,t,u), g(y,t,v) \in C[0,1],\ t\varepsilon[0,T]$,
(5) $f(y,t,u)|_{y=0} = f(y,t,u)|_{y=1},\ f_y(y,t,u)\big|_{y=1} = 0$,

$g(y,t,v)|_{y=0} = g(y,t,v)|_{y=1},\ g_y(y,t,v)\big|_{y=1} = 0.$

By Fourier method ,the solution of (1.5)-(1.6) and (1.7)-(1.8):

$$\begin{aligned} u(x,t) &= \frac{u_0(t)}{2} + \sum_{k=1}^{\infty} \left[u_{ck}(t)\cos 2kx + u_{sk}(t)\sin 2kx\right], \\ v(x,t) &= \frac{v_0(t)}{2} + \sum_{k=1}^{\infty} \left[v_{ck}(t)\cos 2kx + v_{sk}(t)\sin 2kx\right], \end{aligned} \tag{2.3}$$

$$\begin{aligned} u(y,t) &= \frac{u_0(t)}{2} + \sum_{k=1}^{\infty} \left[u_{ck}(t)\cos 2ky + u_{sk}(t)\sin 2ky\right], \\ v(y,t) &= \frac{v_0(t)}{2} + \sum_{k=1}^{\infty} \left[v_{ck}(t)\cos 2ky + v_{sk}(t)\sin 2ky\right], \end{aligned} \tag{2.4}$$

For equation (1.5) Fourier coefficient is:

$$\begin{aligned} u_0(t) &= \varphi_0 + \int_0^t\int_0^1 \alpha u(\alpha,\beta) f(\alpha,\beta,u(\alpha,\beta))\, d\xi d\tau, \\ u_{2k}(t) &= \varphi_{2k}\, e^{\frac{-(2\pi k)^2}{R}t} + \int_0^t\int_0^1 u(\alpha,\beta) f(\alpha,\beta,u(\alpha,\beta)) \sin 2k\alpha\, e^{\frac{-(2\pi k)^2}{R}(t-\tau)} d\alpha d\beta, \\ u_{2k-1}(t) &= \left(\varphi_{2k-1} - 4\pi kt\ \varphi_{2k}\right) e^{\frac{-(2\pi k)^2}{R}t} + \int_0^t\int_0^1 \alpha u(\alpha,\beta) f(\alpha,\beta,u(\alpha,\beta)) \cos 2k\alpha\, e^{\frac{-(2\pi k)^2}{R}(t-\tau)} d\alpha d\beta \\ &\quad -4\pi kt \int_0^t\int_0^1 u(\alpha,\beta) f(\alpha,\beta,u(\alpha,\beta)) \sin 2k\alpha\, e^{\frac{-(2\pi k)^2}{R}(t-\tau)} d\alpha d\beta \end{aligned} \tag{2.5}$$

For (1.6) equation Fourier coefficient is:

$$v_0(t) = \varphi_0 + \int_0^t\int_0^1 \lambda v(\lambda,\beta) f(\lambda,\beta,u(\lambda,\beta))\, d\lambda d\beta, \tag{2.6}$$

$$\begin{aligned} v_{2k}(t) &= \varphi_{2k} e^{\frac{-(2\pi k)^2}{R}t} + \int_0^t\int_0^1 v(\lambda,\beta) f(\lambda,\beta,u(\lambda,\beta)) \sin 2k\lambda\, e^{\frac{-(2\pi k)^2}{R}(t-\tau)} d\lambda d\beta, \\ v_{2k-1}(t) &= \left(\varphi_{2k-1} - 4\pi kt\ \varphi_{2k}\right) e^{\frac{-(2\pi k)^2}{R}t} + \int_0^t\int_0^1 \lambda v(\lambda,\beta) f(\lambda,\beta,u(\lambda,\beta)) \cos 2k\lambda\, e^{\frac{-(2\pi k)^2}{R}(t-\tau)} d\lambda d\beta \\ &\quad -4\pi kt \int_0^t\int_0^1 \lambda v(\lambda,\beta) f(\lambda,\beta,u(\lambda,\beta)) \sin 2k\lambda\, e^{\frac{-(2\pi k)^2}{R}(t-\tau)} d\lambda d\beta \end{aligned} \tag{2.7}$$

Same estimation for Fourier coefficient to (1.7)-(1.8) equation.

Definition 2.1. *Let* $\{u(t)\} = \left\{u_0(t), u_{2k}(t), u_{2k-1}(t), k = 1,...,n\right\}$, *is satisfied that*

$$2\max_{0\le t\le T}|u_0(t)| + 4\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|u_{2k}(t)| + \max_{0\le t\le T}\left|u_{2k-1}(t)\right|\right) < \infty, \text{ by } \mathbf{B}_1. \text{ Let}$$

$$\|u(t)\| = 2\max_{0\le t\le T}|u_0(t)| + 4\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|u_{2k}(t)| + \max_{0\le t\le T}\left|u_{2k-1}(t)\right|\right), \text{ be the norm in } \mathbf{B}_1.$$

$\mathbf{B}_1$ *is called the Banach spaces.*

Denote the set $\{v(t)\} = \left\{v_0(t), v_{2k}(t), v_{2k-1}(t), k = 1,...,n\right\}$ *is satisfied that* $2\max_{0\le t\le T}|v_0(t)| + 4\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|v_{2k}(t)| + \max_{0\le t\le T}\left|v_{2k-1}(t)\right|\right) < \infty$, *by* $\mathbf{B}_2$. *Let*

$$\|v(t)\| = 2\max_{0\le t\le T}|v_0(t)| + 4\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|v_{2k}(t)| + \max_{0\le t\le T}\left|v_{2k-1}(t)\right|\right), \text{ be the norm in } \mathbf{B}_2.$$

$\mathbf{B}_2$ *is called the Banach spaces.*

Theorem 2.2. *If (a1)-(a2) are satisfied then the equation (1.5)-(1.9) has a unique solution .*

Proof.

$$u_0^{(N+1)}(t) = u_0^{(0)}(t) + \int_0^t\int_0^1 \alpha u^{(N)}(\alpha,\beta) f\left(\alpha,\beta,u^{(N)}(\alpha,\beta)\right) d\alpha d\beta, \tag{2.8}$$

$$u_{2k}^{(N+1)}(t) = u_{2k}^{(0)}(t) + \int_0^t\int_0^1 u^{(N)}(\alpha,\beta) f\left(\alpha,\beta,u^{(N)}(\alpha,\beta)\right)\sin 2k\xi\, e^{\frac{-(2\pi k)^2}{R}(t-\tau)} d\alpha d\beta,$$

$$u_{2k-1}^{(N+1)}(t) = u_{2k-1}^{(0)}(t) + \int_0^t\int_0^1 \alpha u^{(N)}(\alpha,\beta) f\left(\alpha,\beta,u^{(N)}(\alpha,\beta)\right)\cos 2k\xi\, e^{\frac{-(2\pi k)^2}{R}(t-\tau)} d\alpha d\beta$$
$$-4\pi kt\int_0^t\int_0^1 u^{(N)}(\alpha,\beta) f\left(\alpha,\beta,u^{(N)}(\alpha,\beta)\right)\sin 2k\xi\, e^{\frac{-(2\pi k)^2}{R}(t-\tau)} d\alpha d\beta. \tag{2.9}$$

$$u_0^{(0)}(t) = \varphi_0, u_{2k}^{(0)}(t) = \varphi_{2k} e^{\frac{-(2\pi k)^2}{R}t}, u_{2k-1}^{(0)}(t) = \left(\varphi_{2k-1} - 4\pi kt\,\varphi_{2k}\right) e^{\frac{-(2\pi k)^2}{R}t}.$$

From the theorem $u^{(0)}(t)\varepsilon\mathbf{B}_1$, $t\varepsilon[0,T]$.

For N=0 in the above equation and using Cauchy inequality, Lipschitzs condition and taking the maximum, we have :

$$2\max_{0\le t\le T}\left|u_0^{(1)}(t)\right| \le 2|\varphi_0| + 2\sqrt{T}\|L(x,t)\|_{L_2(\Omega)}\left\|u^{(0)}(t)\right\|_{B_1}^2 + 2\sqrt{T}\left\|u^{(0)}(t)\right\|_{B_1}\|f(x,t,0)\|_{L_2(\Omega)}.$$

Applying Cauchy, Bessel, Hölder, Lipshitzs conditions and taking maximum:

$$4\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{2k}^{(1)}(t)\right| \le 4\sum_{k=1}^{\infty}|\varphi_{2k}| + \frac{\sqrt{3}R}{3}\|L(x,t)\|_{L_2(\Omega)}\left\|u^{(0)}(t)\right\|_{B_1}^2 + \frac{\sqrt{3}R}{3}\left\|u^{(0)}(t)\right\|_{B_1}\|f(x,t,0)\|_{L_2(\Omega)}.$$

According to the same estimations ,

$$\begin{aligned}
4\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{2k-1}^{(1)}(t)\right| \le\ & 4\sum_{k=1}^{\infty}\left|\varphi_{2k-1}\right| + \frac{2\sqrt{6T}}{3}\sum_{k=1}^{\infty}\left|\varphi_{2k}''\right| \\
& + \left(\frac{\sqrt{3}R}{3} + 4\sqrt{2T}\right)\|L(x,t)\|_{L_2(\Omega)}\left\|u^{(0)}(t)\right\|_{B_1}^2 \\
& + \left(\frac{\sqrt{3}R}{3} + 4\sqrt{2T}\right)\left\|u^{(0)}(t)\right\|_{B_1}\|f(x,t,0)\|_{L_2(\Omega)}.
\end{aligned}$$

$$\begin{aligned}
\left\|u^{(1)}(t)\right\|_{\mathbf{B}_1} =\ & \left\|u^{(1)}(t)\right\|_1 + \left\|u^{(1)}(t)\right\|_2 \\
\le\ & 4|\varphi_0| + 8\sum_{k=1}^{\infty}\left(|\varphi_{2k}| + \left|\varphi_{2k-1}\right|\right) + \frac{4\sqrt{6T}}{3}\sum_{k=1}^{\infty}\left|\varphi_{2k}''\right| \\
& + \left(2\sqrt{T} + \frac{2\sqrt{3}R}{3} + 4\sqrt{2T}\right)\|L(x,t)\|_{L_2(\Omega)}\left\|u^{(0)}(t)\right\|_{B_1}^2 \\
& + \left(2\sqrt{T} + \frac{2\sqrt{3}R}{3} + 4\sqrt{2T}\right)\left\|u^{(0)}(t)\right\|_{B_1}\|f(x,t,0)\|_{L_2(\Omega)} \\
& + \left(2\sqrt{T} + \frac{2\sqrt{3}R}{3} + 4\sqrt{2T}\right)\|L(y,t)\|_{L_2(\Omega)}\left\|u^{(0)}(t)\right\|_{B_1}\left\|v^{(0)}(t)\right\|_{B_2} \\
& + \left(2\sqrt{T} + \frac{2\sqrt{3}R}{3} + 4\sqrt{2T}\right)\left\|v^{(0)}(t)\right\|_{B_2}\|f(y,t,0)\|_{L_2(\Omega)}.
\end{aligned}$$

From the theorem $u^{(1)}(t) \in \mathbf{B}_1$.

Same estimation for N,

$$\begin{aligned}
& \left\|u^{(N+1)}(t)\right\|_{B_1} \\
\le\ & 4|\varphi_0| + 8\sum_{k=1}^{\infty}\left(|\varphi_{2k}| + \left|\varphi_{2k-1}\right|\right) + \frac{4\sqrt{6T}}{3}\sum_{k=1}^{\infty}\left|\varphi_{2k}''\right| \\
& + \left(2\sqrt{T} + \frac{2\sqrt{3}R}{3} + 4\sqrt{2T}\right)\|b(x,t)\|_{L_2(D)}\left\|u^{(N)}(t)\right\|_{B_1}^2 \\
& + \left(2\sqrt{T} + \frac{2\sqrt{3}R}{3} + 4\sqrt{2T}\right)\left\|u^{(N)}(t)\right\|_{B_1}\|f(x,t,0)\|_{L_2(D)} \\
& + \left(2\sqrt{T} + \frac{2\sqrt{3}R}{3} + 4\sqrt{2T}\right)\|b(y,t)\|_{L_2(D)}\left\|u^{(N)}(t)\right\|_{B_1}\left\|v^{(N)}(t)\right\|_{B_2} \\
& + \left(2\sqrt{T} + \frac{2\sqrt{3}R}{3} + 4\sqrt{2T}\right)\left\|v^{(N)}(t)\right\|_{B_2}\|f(y,t,0)\|_{L_2(D)}.
\end{aligned} \tag{2.10}$$

Since $u^{(N)}(t) \in \mathbf{B}_1$ and according to the assumptions of the theorem, we obtain $u^{(N+1)}(t) \in \mathbf{B}_1$,

$$\{u(t)\} = \left\{u_0(t), u_{2k}(t), u_{2k-1}(t),\ k = 1,2,...\right\} \in \mathbf{B}_1.$$

Same estimation for (1.7)-(1.8) equations

$$\{v(t)\} = \left\{v_0(t), v_{2k}(t), v_{2k-1}(t),\ k = 1,2,...\right\} \in \mathbf{B}_2.$$

Now we show that $u^{(N+1)}(t\ )$ and $v^{(N+1)}(t\ )$ converge in $\mathbf{B}_1$ and $\mathbf{B}_2(N\to\infty)$.

Applying Cauchy inequality, Bessel inequality , Hölder inequality,the Lipshitzs condition , $u^{(1)}(t) - u^{(0)}(t)$ for equation (1.5)-(1.6):

$$A_T = \left(2\sqrt{T} + \frac{2\sqrt{3}R}{3} + 4\sqrt{2T}\right)\left(\|L(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_{\mathbf{B}_1}^2 + \left\|u^{(0)}(t)\right\|_{\mathbf{B}_1}\|f(x,t,0)\|_{L_2(D)}\right).$$

$$\left\|u^{(1)}(t) - u^{(0)}(t)\right\|_1 \le A_T$$

$$B_T = \left(2\sqrt{T} + \frac{2\sqrt{3}R}{3} + 4\sqrt{2T}\right)\left(\|L(y,t)\|_{L_2(\Omega)}\left\|u^{(0)}(t)\right\|_{\mathbf{B}_1} + \|f(y,t,0)\|_{L_2(\Omega)}\right)\left\|v^{(0)}(t)\right\|_{\mathbf{B}_2}.$$

$$\left\|u^{(1)}(t) - u^{(0)}(t)\right\|_2 \le B_T$$

$$\begin{aligned}
\left\|u^{(1)}(t) - u^{(0)}(t)\right\|_{\mathbf{B}_1} \le\ & \left\|u^{(1)}(t) - u^{(0)}(t)\right\|_1 + \left\|u^{(1)}(t) - u^{(0)}(t)\right\|_2 \\
\le\ & A_T + B_T
\end{aligned}$$

For N :

$$\begin{aligned}
\left\|u^{(N+1)}(t) - u^{(N)}(t)\right\|_1 \le\ & \frac{A_T}{\sqrt{N!}}\left(2\sqrt{T} + \frac{2\sqrt{3}R}{3} + 4\sqrt{2T}\right)^N\|L(x,t)\|_{L_2(\Omega)}^N \\
\left\|u^{(N+1)}(t) - u^{(N)}(t)\right\|_2 \le\ & \frac{B_T}{\sqrt{N!}}\left(2\sqrt{T} + \frac{2\sqrt{3}R}{3} + 4\sqrt{2T}\right)^N\|L(y,t)\|_{L_2(\Omega)}^N
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
\left\|u^{(N+1)}(t) - u^{(N)}(t)\right\| \le\ & \left(2\sqrt{T} + \frac{2\sqrt{3}R}{3} + 4\sqrt{2T}\right)^N\|L(x,t)\|_{L_2(\Omega)}^N\frac{A_T}{\sqrt{N!}} \\
& + \left(2\sqrt{T} + \frac{2\sqrt{3}R}{3} + 4\sqrt{2T}\right)^N\|L(y,t)\|_{L_2(\Omega)}^N\frac{B_T}{\sqrt{N!}}
\end{aligned} \tag{2.12}$$

$u^{(N+1)} \to u^{(N)}$ for $N\to\infty$, for (1.5)-(1.6)

Same estimations for (1.7)-(1.8)

$$\begin{aligned}
\left\|v^{(N+1)}(t) - v^{(N)}(t)\right\|_{B_2} \le\ & \left(2\sqrt{T} + \frac{2\sqrt{3}R}{3} + 4\sqrt{2T}\right)^N\|L(x,t)\|_{L_2(\Omega)}^N\frac{A_T}{\sqrt{N!}} \\
& + \left(2\sqrt{T} + \frac{2\sqrt{3}R}{3} + 4\sqrt{2T}\right)^N\|L(y,t)\|_{L_2(\Omega)}^N\frac{B_T}{\sqrt{N!}}.
\end{aligned} \tag{2.13}$$

$$\lim_{N\to\infty}u^{(N+1)}(t) = u(t),\ \lim_{N\to\infty}v^{(N+1)}(t) = v(t).$$

Same estimations and using Gronwall's inequality

$$\begin{aligned}\left\|u(t)-u^{(N+1)}(t)\right\|_{B_1} \leq\ & \sqrt{2}\left(2\sqrt{T}+\frac{2\sqrt{3}R}{3}+4\sqrt{2}T\right)^{(N+1)}\|L(x,t)\|^{N+1}_{L_2(\Omega)}\frac{A_T^2}{\sqrt{N!}}\\ & x\exp 2\left(2\sqrt{T}+\frac{2\sqrt{3}R}{3}+4\sqrt{2}T\right)^{2}\|L(x,t)\|^{N+1}_{L_2(\Omega)}\\ & +\sqrt{2}\left(2\sqrt{T}+\frac{2\sqrt{3}R}{3}+4\sqrt{2}T\right)^{(N+1)}\|L(y,t)\|^{N+1}_{L_2(\Omega)}\frac{B_T^2}{\sqrt{N!}}\\ & x\exp 2\left(2\sqrt{T}+\frac{2\sqrt{3}R}{3}+4\sqrt{2}T\right)^{2}\|L(y,t)\|^{N+1}_{L_2(\Omega)}.\end{aligned}$$

$$\begin{aligned}\left\|v(t)-v^{(N+1)}(t)\right\|_{B_2} \leq\ & \sqrt{2}\left(2\sqrt{T}+\frac{2\sqrt{3}R}{3}+4\sqrt{2}T\right)^{(N+1)}\|L(x,t)\|^{N+1}_{L_2(\Omega)}\frac{A_T^2}{\sqrt{N!}}\\ & x\exp 2\left(2\sqrt{T}+\frac{2\sqrt{3}R}{3}+4\sqrt{2}T\right)^{2}\|L(x,t)\|^{N+1}_{L_2(\Omega)}\\ & +\sqrt{2}\left(2\sqrt{T}+\frac{2\sqrt{3}R}{3}+4\sqrt{2}T\right)^{(N+1)}\|L(y,t)\|^{N+1}_{L_2(\Omega)}\frac{B_T^2}{\sqrt{N!}}\\ & x\exp 2\left(2\sqrt{T}+\frac{2\sqrt{3}R}{3}+4\sqrt{2}T\right)^{2}\|L(y,t)\|^{N+1}_{L_2(\Omega)}.\end{aligned}$$

$u^{(N+1)}(t)$ and $v^{(N+1)}(t)$ $(N\to\infty)$ converge in B_1 and B_2, respectively.

For the uniqueness, (u,v) and $(\overline{u},\overline{v})$ are two solution of problem. If the same estimation is used, $|u(t)-\overline{u}(t)|$ and $|v(t)-\overline{v}(t)|$.

$$\|u(t)-\overline{u}(t)\|_{B_1}\leq\left(2\sqrt{T}+\frac{2\sqrt{3}R}{3}+4\sqrt{2}T\right)\left(\|u\|_{B_1}\|L(x,t)\|_{L_2(D)}+\|v\|_{B_2}\|L(y,t)\|_{L_2(D)}\right)\|u(t)-\overline{u}(t)\|_{B_1}$$

From Gronwall's inequality $u(t)=\overline{u}(t)$.

Same estimation for $\|v(t)-\overline{v}(t)\|_{B_2}$,we have $v(t)=\overline{v}(t)$ □

Theorem 2.3. *Solution (u,v) of the problem (1.5)-(1.9) depends continuously on the data φ under assumptions (a1)-(a2)* .

Proof. By using same operations, we obtain:

$$\begin{aligned}\|u(t)-\overline{u}(t)\|^2_{B_1} \leq\ & \|\varphi-\overline{\varphi}\|^2\\ & \times\exp 2\left(2\sqrt{T}+\frac{2\sqrt{3}R}{3}+4\sqrt{2}T\right)^{2}\left[\|L(x,t)\|_{L_2(\Omega)}+\|L(y,t)\|_{L_2(\Omega)}\right]^2.\end{aligned}$$

$$\begin{aligned}\|v(t)-\overline{v}(t)\|^2_{B_2} \leq\ & \|\varphi-\overline{\varphi}\|^2\\ & \times\exp 2\left(2\sqrt{T}+\frac{2\sqrt{3}R}{3}+4\sqrt{2}T\right)^{2}\left[\|L(x,t)\|_{L_2(\Omega)}+\|L(y,t)\|_{L_2(\Omega)}\right]^2.\end{aligned}$$

For $\varphi\to\overline{\varphi}$ then $u\to\overline{u}$ and $v\to\overline{v}$. □

3. Numerical Method

In this Section, we construct the numerical solutions of problems (1.5)-(1.9). In order to solve these problems numerically, we have to use linearization of the nonlinear terms:

$$\frac{1}{2}u_t^{(n)}-\frac{1}{2R}u_{xx}^{(n)}=u^{(n)}f(x,t,u^{(n-1)}),(x,t)\in\Omega \tag{3.1}$$

$$\frac{1}{2}u_t^{(n)}-\frac{1}{2R}u_{yy}^{(n)}=vf(y,t,u^{(n-1)}),(y,t)\in\Omega \tag{3.2}$$

$$\frac{1}{2}v_t^{(n)}-\frac{1}{2R}v_{xx}^{(n)}=ug(x,t,v^{(n-1)}),(x,t)\in\Omega \tag{3.3}$$

$$\frac{1}{2}v_t^{(n)}-\frac{1}{2R}v_{yy}^{(n)}=v^{(n)}g(y,t,v^{(n-1)}),(y,t)\in\Omega \tag{3.4}$$

Let $u^{(n)}(x,t)=u(x,t)$, $u^{(n)}(y,t)=u(y,t)$, $v^{(n)}(x,t)=v(x,t)$, $v^{(n)}(y,t)=v(y,t)$ and $f(x,t,u^{(n-1)})=\widetilde{f}(x,t)$, $f(y,t,u^{(n-1)})=\widetilde{f}(y,t)$, $g(x,t,v^{(n-1)})=.\widetilde{g}(x,t)$, $g(y,t,v^{(n-1)})=.\widetilde{g}(y,t)$. After this transformation, we obtain a linear problem:

$$\frac{1}{2}u_t-\frac{1}{2R}u_{xx}=u\widetilde{f}(x,t),(x,t)\in\Omega \tag{3.5}$$

$$\frac{1}{2}u_t-\frac{1}{2R}u_{yy}=v\widetilde{f}(y,t),(y,t)\in\Omega \tag{3.6}$$

$$\frac{1}{2}v_t-\frac{1}{2R}v_{xx}=u\widetilde{g}(x,t),(x,t)\in\Omega \tag{3.7}$$

$$\frac{1}{2}v_t-\frac{1}{2R}v_{yy}=v\widetilde{g}(y,t),(y,t)\in\Omega \tag{3.8}$$

with the conditions (1.9).

We divide the intervals $[0,1]$ and $[0,T]$ into subintervals N_x and N_t. We obtain equal lengths $h=\frac{1}{N_x}$ and $\tau=\frac{T}{N_t}$. We use implicit finite-difference approximation for discreazing equation (3.5):

$$\frac{1}{2\tau}\left(u_i^{k+1}-u_i^k\right)-\frac{1}{2h^2R}\left(u_{i-1}^{k+1}-2u_i^{k+1}+u_{i+1}^{k+1}\right)=u_i^kf_i^k, \tag{3.9}$$

where $1\leq i\leq N_x$ and $0\leq k\leq N_t$ are the indices, $u_i^k=u(x_i,t_k)$, $f_i^k=\widetilde{f}(x_i,t_k)$, $x_i=ih$, $t_k=k\tau$.

We obtain

$$-Pu_{i-1}^{k+1}+(1+2P)u_i^{k+1}-Pu_{i+1}^{k+1}=u_i^k(1+2\tau)f_i^k$$

where $P=\frac{\tau}{h^2R}$. By using Gauss Elimination Method, we can solve this equation. Equations (3.6), (3.7) and (3.8) can be solved in a similar way.

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