Malaya Journal of Matematik, Vol. 6, No. 1, 61-68, 2018

https://doi.org/10.26637/MJM0601/0009

On the exact and the approximate solutions of second-order fuzzy initial value problems with constant coefficients

Hülya Gültekin Çitil[1]*

Abstract

In this paper investigates the approximate solutions by the Adomian decomposition method and by the undetermined fuzzy coefficients method and the exact solutions by using the Hukuhara differentiability of second-order fuzzy linear initial value problems with constant coefficients. Comparison results of the solutions is given.

Keywords

Fuzzy initial value problem, second-order fuzzy differential equation, Hukuhara differentiability, Adomian decomposition method, the undetermined fuzzy coefficients method.

AMS Subject Classification

03E72, 34A07, 65L05.

[1] *Department of Mathematics, Faculty of Arts and Sciences, Giresun University, Giresun-28100, Turkey.*

*Corresponding author: [1] hulya.citil@giresun.edu.tr

Article History: Received **12** October 2017; Accepted **30** November 2017

Contents

1. Introduction

Fuzzy differential equations are studied by many researhers. Because, fuzzy differential equations forms the mathematical model of real world problems in which uncertainty. Fuzzy differential equations can be solved with several approach. The first approach was Hukuhara differentiability. Gültekin and Altınışık [15] have investigated the existence and uniqueness of solutions of two-point fuzzy boundary value problems for second-order fuzzy differential equations under the approach of Hukuhara differentiability. The fuzzy Sturm-Liouville equation have been defined under the approach of Hukuhara differentiability by Gültekin Çitil and Altınışık [16]. Bede [7] has proved that a large class of boundary value problems have not a solution.The second approach was generalized differentiability. Khastan and Nieto [19] have found new solutions for some fuzzy boundary value problems using the generalized differentiability. The third approach generate the fuzzy solution from the crips solution [9],[10], [17], [13]. But, many fuzzy initial and boundary value problems can not be solved exactly. Thus, to find approximate solutions of these problems is important. Some numeric methods such as the fuzzy Euler method, predictor-corrector method, Taylor method, Nyström method are introduced [1], [2], [4], [12], [18] . Adomian decomposition method was introduced by Adomian [3]. Wang and Guo [22] presented the second-order linear fuzzy differential equation with fuzzy initial conditions by Adomian method using the generalized differentiability. Guo at al. [14] have found the approximate solution of a class of-second-order linear differential equation with fuzzy boundary value conditions by the undetermined fuzzy coefficients method.

In this paper we investigate the approximate solutions by the undetermined fuzzy coefficients and by the Adomian decomposition method and the exact solutions by using the Hukuhara differentiability of second-order fuzzy linear initial value problems with constant coefficients. Thus, we give comparisons of the approximate solutions.

2. Preliminaries

Definition 2.1. *[20] A fuzzy number is a function $u : \mathbb{R} \to [0,1]$ satisfying the following properties:*
u is normal, u is convex fuzzy set, u is upper semi-continuous on $\mathbb{R}$, $cl\{x \in \mathbb{R} \mid u(x) > 0\}$ is compact where cl denotes the closure of a subset.

Let $\mathbb{R}_F$ denote the space of fuzzy numbers.

Definition 2.2. *[19] Let $u \in \mathbb{R}_F$. The α-level set of u, denoted , $[u]^\alpha$, $0 < \alpha \leq 1$, is $[u]^\alpha = \{x \in \mathbb{R} \mid u(x) \geq 0\}$ .If $\alpha = 0$,the support of u is defined $[u]^0 = cl\{x \in \mathbb{R} \mid u(x) > 0\}$. The notation, denotes explicitly the α-level set of u. The notation, $[u]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$ denotes explicitly the α-level set of u.We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches of u, respectively.*

The following remark shows when $[\underline{u}_\alpha, \overline{u}_\alpha]$ is a valid α-level set.

Remark 2.3. *[11, 19] The sufficient and necessary conditions for $[\underline{u}_\alpha, \overline{u}_\alpha]$ to define the parametric form of a fuzzy number as follows:*

$\underline{u}_\alpha$ is bounded monotonic increasing (nondecreasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha = 0$,
$\overline{u}_\alpha$ is bounded monotonic decreasing (nonincreasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha = 0$,
$\underline{u}_\alpha \leq \overline{u}_\alpha$, $0 \leq \alpha \leq 1$.

Definition 2.4. *[20] If A is a symmetric triangular numbers with supports $[\underline{a}, \overline{a}]$, the $\alpha-$level sets of $[A]^\alpha$ is*

$$[A]^\alpha = \left[\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right].$$

Definition 2.5. *[14, 19, 21] Let $u, v \in \mathbb{R}_F$.If there exists $w \in \mathbb{R}_F$ such that $u = v + w$, then w is called the Hukuhara difference of fuzzy numbers u and v,and it is denoted by $w = u \ominus v$.*

Definition 2.6. *[8, 14, 19] Let $f : [a,b] \to \mathbb{R}_F$ and $t_0 \in [a,b]$ .We say that f is Hukuhara differentiable at t_0, if there exists an element $f'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists f(t_0+h) \ominus f(t_0)$, $f(t_0) \ominus f(t_0-h)$ and the limits*

$$\lim_{h\to 0} \frac{f(t_0+h) \ominus f(t_0)}{h} = \lim_{h\to 0} \frac{f(t_0) \ominus f(t_0-h)}{h} = f'(t_0).$$

Lemma 2.7. *[6] If $g : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$ such that g', g'' are nonnegative and monotonic increasing on $[a,b]$, then $\forall c \in \mathbb{R}_F, f(x) = cg(x)$ is differentiable on $[a,b]$ and*

$$f'(x) = cg'(x), f''(x) = cg''(x), \forall x \in [a,b].$$

Definition 2.8. *[14] The undetermined fuzzy coefficient method is to seek an approximate solution as*

$$\tilde{y}_N(t) = \sum_{k=0}^{N} \tilde{\theta}_k \phi_k(t),$$

where, $\phi_k(t)$, $k = 0,1,\ldots,N$ are positive basic functions whose all differentiations are positive.

Lemma 2.9. *[5, 14] Let basis functions $\phi_k(t)$, $k = 0,1,\ldots,N$ and all of their differentiations be positive, without loss of generality. Then $\left(\underline{y}_N\right)^{(i)}(t) = \underline{y_N^{(i)}}(t)$ and $\left(\overline{y}_N^{(i)}\right)(t) = \overline{y_N^{(i)}}(t)$, $i = 0,1,2$.*

3. Second-order fuzzy linear initial value problems

3.1 The case of positive constant coefficient

Consider the fuzzy boundary value problem

$$y''(t) = \lambda y(t), \quad y(0) = A, \quad y'(0) = B, \tag{3.1}$$

where $\lambda > 0$, $[A]^\alpha = \left[\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right]$, $[B]^\alpha = \left[\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha, \overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right]$ are symmetric triangular fuzzy numbers.

3.1.1 The exact solution by Hukuhara differentiability

From the fuzzy differential equation in (3.1), we have differential equations

$$\underline{Y}''_\alpha(t) = \lambda \underline{Y}_\alpha(t), \quad \overline{Y}''_\alpha(t) = \lambda \overline{Y}_\alpha(t)$$

by using the Hukuhara differentiability. Then, the lower solution and the upper solution of the fuzzy differential equation in (3.1) are obtained as

$$\underline{Y}_\alpha(t) = \underline{c}_1(\alpha) e^{\sqrt{\lambda} t} + \underline{c}_2(\alpha) e^{-\sqrt{\lambda} t},$$

$$\overline{Y}_\alpha(t) = \overline{c}_1(\alpha) e^{\sqrt{\lambda} t} + \overline{c}_2(\alpha) e^{-\sqrt{\lambda} t}.$$

Using the boundary conditions, coefficients $\underline{c}_1(\alpha)$, $\underline{c}_2(\alpha)$, $\overline{c}_1(\alpha)$, $\overline{c}_2(\alpha)$ are solved as

$$\underline{c}_1(\alpha) = \frac{\sqrt{\lambda}\left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)}{2\sqrt{\lambda}},$$

$$\underline{c}_2(\alpha) = \frac{\sqrt{\lambda}\left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) - \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)}{2\sqrt{\lambda}},$$

$$\overline{c}_1(\alpha) = \frac{\sqrt{\lambda}\left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)}{2\sqrt{\lambda}},$$

$$\overline{c}_2(\alpha) = \frac{\sqrt{\lambda}\left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) - \left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)}{2\sqrt{\lambda}}.$$

3.1.2 The approximate solution by the undetermined fuzzy coefficients method (UFCM)

An approximate solution with the undetermined fuzzy coefficients method is

$$\widetilde{y}_N(t) = \sum_{k=0}^{N} \widetilde{\theta}_k \phi_k(t), \tag{3.2}$$

where, $\phi_k(t), k = 0, 1, ..., N$ are positive basic functions whose all differentiations are positive and the lower solution and upper solution are

$$\underline{y}_\alpha(t) = \sum_{k=0}^{N} \underline{\theta}_k(\alpha)\phi_k(t), \qquad \overline{y}_\alpha(t) = \sum_{k=0}^{N} \overline{\theta}_k(\alpha)\phi_k(t),$$

respectively. Substituing the equation (3.3) in the fuzzy differential equation (3.1) yields

$$\sum_{k=0}^{N} \underline{\theta}_k(\alpha)\phi_k''(t) - \lambda \sum_{k=0}^{N} \underline{\theta}_k(\alpha)\phi_k(t) = 0,$$
$$\sum_{k=0}^{N} \overline{\theta}_k(\alpha)\phi_k''(t) - \lambda \sum_{k=0}^{N} \overline{\theta}_k(\alpha)\phi_k(t) = 0.$$

Using the boundary conditions in (3.1)

$$\sum_{k=0}^{N} \underline{\theta}_k(\alpha)\phi_k(0) = \underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,$$
$$\sum_{k=0}^{N} \underline{\theta}_k(\alpha)\phi_k'(0) = \underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha,$$
$$\sum_{k=0}^{N} \overline{\theta}_k(\alpha)\phi_k(0) = \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,$$
$$\sum_{k=0}^{N} \overline{\theta}_k(\alpha)\phi_k'(0) = \overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha,$$

are obtained. Taking

$$\phi_k''(t) + \lambda\phi_k(t) = \rho_k, \quad \phi_k(0) = \zeta_{0k}, \quad \phi_k'(0) = \xi_{0k},$$

we obtain

$$\sum_{k=0}^{N} \underline{\theta}_k(\alpha)\rho_k = 0, \tag{3.3}$$
$$\sum_{k=0}^{N} \underline{\theta}_k(\alpha)\zeta_{0k} = \underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \tag{3.4}$$
$$\sum_{k=0}^{N} \underline{\theta}_k(\alpha)\xi_{0k} = \underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha, \tag{3.5}$$
$$\sum_{k=0}^{N} \overline{\theta}_k(\alpha)\rho_k = 0, \tag{3.6}$$
$$\sum_{k=0}^{N} \overline{\theta}_k(\alpha)\zeta_{0k} = \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \tag{3.7}$$
$$\sum_{k=0}^{N} \overline{\theta}_k(\alpha)\xi_{0k} = \overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha. \tag{3.8}$$

Let write the (3.3)-(3.8) as $S(t)X(\alpha) = Y(\alpha)$, where

$$S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}, \; S_1 = \begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_N \\ \zeta_{00} & \zeta_{01} & \cdots & \zeta_{0N} \\ \xi_{00} & \xi_{01} & \cdots & \xi_{0N} \end{pmatrix},$$
$$S_2 = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{pmatrix},$$

$$X(\alpha) = \left(\underline{\theta}_0 \; \underline{\theta}_1 \; ....\underline{\theta}_N \; \overline{\theta}_0 \; \overline{\theta}_1 ....\overline{\theta}_N\right)^T, \;\; Y(\alpha) = \left(0 \; \underline{A}_\alpha \; \underline{B}_\alpha \; 0 \; \overline{A}_\alpha \; \overline{B}_\alpha\right)^T.$$

From this, $\underline{\theta}_0, \underline{\theta}_1, ...\underline{\theta}_N, \overline{\theta}_0, \overline{\theta}_1, ..., \overline{\theta}_N$ are solved and the approximate solution is obtained [14].

3.1.3 The approximate solution by the Adomian decomposition method (ADM)

The equation (3.1) is written as

$$\underline{y}_\alpha''(t) = \lambda \underline{y}_\alpha(t), \;\; \overline{y}_\alpha''(t) = \lambda \overline{y}_\alpha(t) \tag{3.9}$$

by using the Hukuhara differentiability.In the operator form, the first equation in (3.9) becomes $L\underline{y}_\alpha = \lambda \underline{y}_\alpha$,where the differential operator L is given by $L = \frac{d^2}{dx^2}$. Operating with L^{-1} on both sides of the above equation and using the initial conditions we obtain

$$\underline{y}_\alpha(t) = \underline{y}_\alpha(0) + t\underline{y}_\alpha'(0) + L^{-1}\left(\lambda \underline{y}_\alpha\right)$$

$$\underline{y}_\alpha(t) = \left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t + \lambda L^{-1}\left(\underline{y}_\alpha\right)$$

Let take

$$\underline{y}_\alpha(t) = \sum_{n=0}^{\infty} \underline{y}_{n\alpha}(t).$$

Then

$$\sum_{n=0}^{\infty} \underline{y}_{n\alpha}(t) = \left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t + \lambda L^{-1}\left(\sum_{n=0}^{\infty} \underline{y}_{n\alpha}(t)\right)$$

is obtained. From this,

$$\underline{y}_{0\alpha}(t) = \left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right) + \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t$$

$$\begin{aligned}
\underline{y}_{1\alpha}(t) &= \lambda L^{-1}\left(\underline{y}_{0\alpha}(t)\right) \Rightarrow \\
\underline{y}_{1\alpha}(t) &= \lambda\left(\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2}+\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right),
\end{aligned}$$

$$\begin{aligned}
\underline{y}_{2\alpha}(t) &= \lambda L^{-1}\left(\underline{y}_{1\alpha}(t)\right) \Rightarrow \\
\underline{y}_{2\alpha}(t) &= \lambda\left(\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^3}{6}+\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^4}{24}\right), \\
&\dots
\end{aligned}$$

are obtained. Then, the lower approximate solution with the Adomian decomposition method of the problem (3.1) becomes

$$\begin{aligned}
\underline{y}_{\alpha}(t) = & \left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t \\
&+\lambda\left(\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2}+\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right) \\
&+\lambda\left(\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^3}{6}+\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^4}{24}\right) \\
&+\dots
\end{aligned}$$

Similarly, upper approximate solution with the Adomian decomposition method of the problem (3.1) becomes

$$\begin{aligned}
\overline{y}_{\alpha}(t) = & \left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t \\
&+\lambda\left(\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2}+\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right) \\
&+\lambda\left(\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^3}{6}+\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^4}{24}\right) \\
&+\dots
\end{aligned}$$

Example 3.1. *Consider the fuzzy boundary value problem*

$$y''(t) = y(t), t > 0, \tag{3.10}$$

$$y(0) = [1+0.5\alpha, 2-0.5\alpha], \; y'(0) = [3+0.5\alpha, 4-0.5\alpha]. \tag{3.11}$$

Using the Hukuhara differentiability, the lower exact solution and the upper exact solution of the fuzzy differential equation (3.10)are obtained as

$$\underline{Y}_{\alpha}(t) = \underline{c}_1(\alpha)e^{t} + \underline{c}_2(\alpha)e^{-t}, \;\; \overline{Y}_{\alpha}(t) = \overline{c}_1(\alpha)e^{t} + \overline{c}_2(\alpha)e^{-t}. \tag{3.12}$$

Using the boundary conditions (3.11),

$$\underline{c}_1(\alpha) = 2+0.5\alpha, \quad \underline{c}_2(\alpha) = -1, \quad \overline{c}_1(\alpha) = 3-0.5\alpha, \quad \overline{c}_2(\alpha) = -1.$$

are obtained. Now, let find the approximate solution with the undetermined fuzzy coefficients method . Let be

$$\phi_k(t) = t^k, k = 0, 1, 2.$$

Then, the the lower approximate solution and the upper approximate solution are

$$\underline{y}_{\alpha}(t) = \underline{\theta}_0 + \underline{\theta}_1 t + \underline{\theta}_2 t^2, \;\; \overline{y}_{\alpha}(t) = \overline{\theta}_0 + \overline{\theta}_1 t + \overline{\theta}_2 t^2. \tag{3.13}$$

From the fuzzy differential equation (3.10) becomes

$$-\underline{\theta}_0 - \underline{\theta}_1 t + \underline{\theta}_2\left(2-t^2\right) = 0, \;\; -\overline{\theta}_0 - \overline{\theta}_1 t + \overline{\theta}_2\left(2-t^2\right) = 0. \tag{3.14}$$

Using (3.14), the boundary conditions (3.11) and taking $t = \frac{1}{2}$,

$$\begin{aligned}
\underline{\theta}_0 &= 1+0.5\alpha, \;\; \underline{\theta}_1 = 3+0.5\alpha, \\
\underline{\theta}_2 &= 1.42857142857+0.42857142857\alpha.
\end{aligned}$$

$$\begin{aligned}
\overline{\theta}_0 &= 2-0.5\alpha, \;\; \underline{\theta}_1 = 4-0.5\alpha, \\
\overline{\theta}_2 &= 2.28571428571+0.42857142857\alpha.
\end{aligned}$$

Substituing these values in (3.13), the lower approximate solution and the upper approximate solution with the undetermined fuzzy coefficients method are obtained as

$$\begin{aligned}
\underline{y}_{\alpha}(t) &= (1+0.5\alpha)+(3+0.5\alpha)t \\
&\quad +(1.42857142857+0.42857142857\alpha)t^2,
\end{aligned}$$

$$\begin{aligned}
\overline{y}_{\alpha}(t) &= (2-0.5\alpha)+(4-0.5\alpha)t \\
&\quad +(2.28571428571+0.42857142857\alpha)t^2.
\end{aligned}$$

The approximate lower and upper solutions with the undetermined fuzzy coefficients method for $t = 0.01$ are

$$\underline{y}_{\alpha}(t) = 1.03142857142+0.50500428571\alpha,$$
$$\overline{y}_{\alpha}(t) = 2.04228571428-0.50500428571\alpha.$$

From the equations (3.12) the exact lower and upper solution for $t = 0.01$ are

$$\underline{Y}_{\alpha}(t) = 1.03005050042+0.50502508354\alpha,$$
$$\overline{Y}_{\alpha}(t) = 2.0401006675-0.50502508354\alpha.$$

By the Adomian decomposition method, we obtain the solution of (3.10)-(3.11) as

$$\underline{y}_{\alpha}(t) = (1+0.5\alpha)+(3+0.5\alpha)t+(1+0.5\alpha)\frac{t^2}{2}+(3+0.5\alpha)\frac{t^3}{6},$$

$$\overline{y}_{\alpha}(t) = (2-0.5\alpha)+(4-0.5\alpha)t+(2-0.5\alpha)\frac{t^2}{2}+(4-0.5\alpha)\frac{t^3}{6}.$$

The approximate lower and upper solutions by the Adomian decomposition method for $t = 0.01$ are

$$\underline{y}_{\alpha}(t) = 1.0300505+0.50502508333\alpha,$$

$$\overline{y}_{\alpha}(t) = 2.04100066667-0.50502508333\alpha$$

The lower exact solution and the lower approximate solutions by the UFCM by the ADM

α	$\underline{Y}_{\alpha}(t)$	$\underline{y}_{\alpha}(t)$ (**UFCM**)	$\underline{y}_{\alpha}(t)$(**ADM**)	$Error$(**UFCM**)	$Error$(**ADM**)
0	1.03005050042	1.03142857142	1.0300505	0.001378071	$4.2000003 \times 10^{-10}$
0.1	1.08055300877	1.08192899999	1.08055300833	0.00137599122	$4.4000004 \times 10^{-10}$
0.2	1.13105551712	1.13242942856	1.13105551666	0.00137391144	$4.6000004 \times 10^{-10}$
0.3	1.18155802548	1.18292985713	1.18155802499	0.00137183165	$4.9000004 \times 10^{-10}$
0.4	1.23206053383	1.2334302857	1.23206053333	0.00136975187	$4.9999982 \times 10^{-10}$
0.5	1.28256304219	1.28393071427	1.28256304166	0.00136767208	$5.3000004 \times 10^{-10}$
0.6	1.33306555054	1.33443114284	1.33306554999	0.0013655923	$5.5000005 \times 10^{-10}$
0.7	1.38356805889	1.38493157141	1.38356805833	0.00136351252	$5.5999982 \times 10^{-10}$
0.8	1.43407056725	1.43543199998	1.43407056666	0.00136143273	$5.9000005 \times 10^{-10}$
0.9	1.4845730756	1.48593242855	1.48457307499	0.00135935295	$6.1000005 \times 10^{-10}$
1	1.53507558396	1.53643285713	1.53507558333	0.00135727317	$6.3000005 \times 10^{-10}$

The upper exact solution and the upper approximate solutions by the UFCM and by the ADM

α	$\overline{Y}_\alpha(t)$	$\bar{y}_\alpha(t)$(**UFCM**)	$\bar{y}_\alpha(t)$(**ADM**)	$Error$(**UFCM**)	$Error$(**ADM**)
0	2.0401006675	2.04228571428	2.04100066667	0.00218504678	0.00089999917
0.1	1.98959815915	1.99178528571	1.99049815834	0.00218712656	0.00089999919
0.2	1.9390956508	1.94128485714	1.93999565001	0.00218920634	0.00089999921
0.3	1.88859314244	1.89078442857	1.88949314167	0.00219128613	0.00089999923
0.4	1.83809063409	1.840284	1.83899063334	0.00219336591	0.00089999925
0.5	1.78758812573	1.78978357143	1.78848812501	0.0021954457	0.00089999928
0.6	1.73708561738	1.73928314286	1.73798561667	0.00219752548	0.00089999929
0.7	1.68658310903	1.68878271429	1.68748310834	0.00219960526	0.00089999931
0.8	1.63608060067	1.63828228572	1.63698060001	0.00220168505	0.00089999934
0.9	1.58557809232	1.58778185715	1.58647809167	0.00220376483	0.00089999935
1	1.53507558396	1.53728142857	1.53597558334	0.00220584461	0.00089999938

3.2 The case of negative constant coefficient

Consider the fuzzy boundary value problem

$$y''(t) = -\lambda y(t),\;\; y(0) = A,\;\; y^{'}(0) = B, \tag{3.15}$$

where $\lambda > 0$, $[A]^\alpha = \left[\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right]$,
$[B]^\alpha = \left[\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha, \overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right]$ are symmetric triangular fuzzy numbers.

3.2.1 The exact solution by Hukuhara differentiability

From the fuzzy differential equation in (3.15), we have differential equations

$$\underline{Y}^{''}_\alpha(t) = -\lambda\overline{Y}_\alpha(t),\;\; \overline{Y}^{''}_\alpha(t) = -\lambda\underline{Y}_\alpha(t)$$

by using the Hukuhara differentiability and $-\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = \left[-\overline{y}_\alpha, -\underline{y}_\alpha\right]$. Then, the lower solution and the upper solution of the fuzzy differential equation in (3.15) are

$$\begin{aligned}\underline{Y}_\alpha(t) &= -c_1(\alpha)e^{\sqrt{\lambda}t} - c_2(\alpha)e^{-\sqrt{\lambda}t} + c_3(\alpha)\,\sin\left(\sqrt{\lambda}t\right)\\ &\quad + c_4(\alpha)\cos\left(\sqrt{\lambda}t\right),\end{aligned}$$

$$\begin{aligned}\overline{Y}_\alpha(t) &= c_1(\alpha)e^{\sqrt{\lambda}t} + c_2(\alpha)e^{-\sqrt{\lambda}t} + c_3(\alpha)\sin\left(\sqrt{\lambda}t\right)\\ &\quad + c_4(\alpha)\cos\left(\sqrt{\lambda}t\right).\end{aligned}$$

Using the boundary conditions, the coefficient $c_1(\alpha)$, $c_2(\alpha)$, $c_3(\alpha)$, $c_4(\alpha)$ are obtained as

$$c_1(\alpha) = 0.5 + 0.5\alpha,\quad c_2(\alpha) = 0,\quad c_3(\alpha) = 3.5,\quad c_4(\alpha) = 1.5.$$

3.2.2 The approximate solution by the undetermined fuzzy coefficients method (UFCM)

The undetermined fuzzy coefficients method is to seek an approximate solution as

$$\widetilde{y}_N(t) = \sum_{k=0}^{N}\widetilde{\theta}_k\phi_k(t), \tag{3.16}$$

where, $\phi_k(t)$, $k = 0, 1, ..., N$ are positive basic functions whose all differentiations are positive and and the lower solution and upper solution are

$$\underline{y}_\alpha(t) = \sum_{k=0}^{N}\underline{\theta}_k(\alpha)\phi_k(t), \qquad \overline{y}_\alpha(t) = \sum_{k=0}^{N}\overline{\theta}_k(\alpha)\phi_k(t)$$

respectively. Substituting the equation (3.16) in the fuzzy differential equation (3.15) and using $-\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = \left[-\overline{y}_\alpha, -\underline{y}_\alpha\right]$ yields

$$\sum_{k=0}^{N}\underline{\theta}_k(\alpha)\phi_k^{''}(t) + \lambda\sum_{k=0}^{N}\overline{\theta}_k(\alpha)\phi_k(t) = 0,$$

$$\sum_{k=0}^{N}\overline{\theta}_k(\alpha)\phi_k^{''}(t) + \lambda\sum_{k=0}^{N}\underline{\theta}_k(\alpha)\phi_k(t) = 0.$$

Using the boundary conditions in (3.15)

$$\sum_{k=0}^{N}\underline{\theta}_k(\alpha)\phi_k(0) = \underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\;\; \sum_{k=0}^{N}\underline{\theta}_k(\alpha)\phi_k^{'}(0) = \underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha,$$

$$\sum_{k=0}^{N}\overline{\theta}_k(\alpha)\phi_k(0) = \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\;\; \sum_{k=0}^{N}\overline{\theta}_k(\alpha)\phi_k^{'}(0) = \overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha,$$

are obtained. Taking

$$\phi_k^{''}(t) = \rho_k,\;\; \lambda\phi_k(t) = \varepsilon_k,\;\; \phi_k(0) = \zeta_{0k},\quad \phi_k^{'}(0) = \xi_{0k}$$

we obtain

$$\sum_{k=0}^{N}\underline{\theta}_k(\alpha)\rho_k + \sum_{k=0}^{N}\overline{\theta}_k(\alpha)\varepsilon_k = 0, \tag{3.17}$$

$$\sum_{k=0}^{N}\underline{\theta}_k(\alpha)\zeta_{0k} = \underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \tag{3.18}$$

$$\sum_{k=0}^{N}\underline{\theta}_k(\alpha)\xi_{0k} = \underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha, \tag{3.19}$$

$$\sum_{k=0}^{N}\overline{\theta}_k(\alpha)\rho_k + \sum_{k=0}^{N}\underline{\theta}_k(\alpha)\varepsilon_k = 0, \tag{3.20}$$

$$\sum_{k=0}^{N}\overline{\theta}_k(\alpha)\zeta_{0k} = \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \tag{3.21}$$

$$\sum_{k=0}^{N}\overline{\theta}_k(\alpha)\xi_{0k} = \overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha. \tag{3.22}$$

Let write the (3.17)-(3.22) as $S(t)X(\alpha) = Y(\alpha)$, where

$$\begin{aligned} S &= \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix},\; S_1 = \begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_N \\ \zeta_{00} & \zeta_{01} & \cdots & \zeta_{0N} \\ \xi_{00} & \xi_{01} & \cdots & \xi_{0N} \end{pmatrix}, \\ S_2 &= \begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \ldots & \varepsilon_N \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{pmatrix}, \end{aligned}$$

$X(\alpha) = \left(\underline{\theta}_0\ \underline{\theta}_1\ ....\underline{\theta}_N, \overline{\theta}_0\ \overline{\theta}_1....\overline{\theta}_N\right)^T$, $Y(\alpha) = \left(0\ \underline{A}_\alpha\ \underline{B}_\alpha\ 0.\overline{A}_\alpha\ \overline{B}_\alpha\right)^T$.

From this, $\underline{\theta}_0$, $\underline{\theta}_1$, ...$\underline{\theta}_N$, $\overline{\theta}_0$, $\overline{\theta}_1$, ..., $\overline{\theta}_N$ are solved and the approximate solution is obtained.

3.2.3 The approximate solution by the Adomian decomposition method (ADM)

The equation in (3.15) is written as

$$\underline{y}^{''}_\alpha(t) = -\lambda\overline{y}_\alpha(t),\;\; \overline{y}^{''}_\alpha(t) = -\lambda\underline{y}_\alpha(t) \tag{3.23}$$

by using the Hukuhara differentiability and $-\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = \left[-\overline{y}_\alpha, -\underline{y}_\alpha\right]$. In the operator form, the first equation in (3.23) becomes $L\underline{y}_\alpha =$

$-\lambda\overline{y}_\alpha$, where the differential operator L is given by $L=\frac{d^2}{dx^2}$. Operating with L^{-1} on both sides of the above equations and using the initial conditions we obtain

$$\underline{y}_\alpha(t)=\underline{y}_\alpha(0)+\underline{y}'_\alpha(0)t+L^{-1}\left(-\lambda\overline{y}_\alpha\right),$$
$$\overline{y}_\alpha(t)=\overline{y}_\alpha(0)+\overline{y}'_\alpha(0)t+L^{-1}\left(-\lambda\underline{y}_\alpha\right)$$

$$\underline{y}_\alpha(t)=\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t-\lambda L^{-1}\left(\overline{y}_\alpha\right),$$
$$\overline{y}_\alpha(t)=\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t-\lambda L^{-1}\left(\underline{y}_\alpha\right)$$

Let take

$$\underline{y}_\alpha(t)=\sum_{n=0}^{\infty}\underline{y}_{n\alpha}(t),\ \overline{y}_\alpha(t)=\sum_{n=0}^{\infty}\overline{y}_{n\alpha}(t)$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty}\underline{y}_{n\alpha}(t) &= \left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t\\ &\quad -\lambda L^{-1}\left(\sum_{n=0}^{\infty}\overline{y}_{n\alpha}(t)\right)\end{aligned}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{y}_{n\alpha}(t) &= \left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t\\ &\quad -\lambda L^{-1}\left(\sum_{n=0}^{\infty}\underline{y}_{n\alpha}(t)\right)\end{aligned}$$

is obtained. From this,

$$\underline{y}_{0\alpha}(t)=\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t$$

$$\overline{y}_{0\alpha}(t)=\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t$$

$$\begin{aligned}\underline{y}_{1\alpha}(t) &= -\lambda L^{-1}\left(\overline{y}_{0\alpha}(t)\right)\Rightarrow\\ \underline{y}_{1\alpha}(t) &= -\lambda\left(\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2}+\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right),\\ &\ldots\end{aligned}$$

$$\begin{aligned}\overline{y}_{1\alpha}(t) &= -\lambda L^{-1}\left(\underline{y}_{0\alpha}(t)\right)\Rightarrow\\ \overline{y}_{1\alpha}(t) &= -\lambda\left(\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2}+\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right),\\ &\ldots\end{aligned}$$

are obtained. Then, the lower approximate solution by the Adomian decomposition method of the problem (3.1) becomes

$$\begin{aligned}\underline{y}_\alpha(t) &= \left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t\\ &\quad -\lambda\left(\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2}+\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right)\\ &\quad +\ldots\end{aligned}$$

Similarly, upper approximate solution by the Adomian decomposition method of the problem (3.1) becomes

$$\begin{aligned}\overline{y}_\alpha(t) &= \left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)t\\ &\quad -\lambda\left(\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\frac{t^2}{2}+\left(\overline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\frac{t^3}{6}\right)\\ &\quad +\ldots\end{aligned}$$

Example 3.2. *Consider the fuzzy boundary value problem*

$$y''(t)=-y(t),t>0, \tag{3.24}$$

$$y(0)=[1+0.5\alpha,2-0.5\alpha],\ \ y'(0)=[3+0.5\alpha,4-0.5\alpha]. \tag{3.25}$$

Using the Hukuhara differentiability and using

$$-\left[\underline{y}_\alpha,\overline{y}_\alpha\right]=\left[-\overline{y}_\alpha,-\underline{y}_\alpha\right]$$

in the fuzzy differential equation (3.24), we have

$$\underline{Y}''_\alpha(t)+\overline{Y}_\alpha(t)=0,\ \ \overline{Y}''_\alpha(t)+\underline{Y}''_\alpha(t)=0$$

Then, using the boundary conditions (3.25), the lower exact solution and the upper exact solution of the fuzzy differential equation (3.24) are obtained as

$$\underline{Y}_\alpha(t) = (-0.5+0.5\alpha)e^t+3.5\sin(t)+1.5\cos(t), \tag{3.26}$$
$$\overline{Y}_\alpha(t) = (0.5-0.5\alpha)e^t+3.5\sin(t)+1.5\cos(t). \tag{3.27}$$

Let be $\phi_k(t)=t^k, k=0,1,2$. Then, the the lower approximate solution and the upper approximate solution are

$$\underline{y}_\alpha(t)=\underline{\theta}_0+\underline{\theta}_1 t+\underline{\theta}_2 t^2,\ \ \overline{y}_\alpha(t)=\overline{\theta}_0+\overline{\theta}_1 t+\overline{\theta}_2 t^2. \tag{3.28}$$

Substituting these equations in (3.24) and using the equation

$$-\left[\underline{y}_\alpha,\overline{y}_\alpha\right]=\left[-\overline{y}_\alpha,-\underline{y}_\alpha\right]$$

$$\overline{\theta}_0+\overline{\theta}_1 t+\overline{\theta}_2 t^2+2\underline{\theta}_2=0,\ \ \underline{\theta}_0+\underline{\theta}_1 t+\underline{\theta}_2 t^2+2\overline{\theta}_2=0 \tag{3.29}$$

are obtained. Using (3.29), , the boundary conditions (3.25) and taking $t=\frac{1}{2}$, yields

$$\begin{aligned}\underline{\theta}_0 &= 1+0.5\alpha,\ \ \underline{\theta}_1=3+0.5\alpha,\\ \underline{\theta}_2 &= -1.11111111111-0.33333333333\alpha,\\[6pt] \overline{\theta}_0 &= 2-0.5\alpha,\ \ \overline{\theta}_1=4-0.5\alpha,\\ \overline{\theta}_2 &= -1.77777777777+0.33333333333\alpha\end{aligned}$$

Substituting these values in the equations (3.28), the approximate lower and upper solutions with the undetermined fuzzy coefficients method of (3.24) are obtained as

$$\begin{aligned}\underline{y}_\alpha(t) &= (1+0.5\alpha)+(3+0.5)t\\ &\quad +(-1.11111111111-0.33333333333\alpha)t^2,\\[6pt] \overline{y}_\alpha(t) &= (2-0.5\alpha)+(4-0.5\alpha)t\\ &\quad +(-1.77777777777+0.33333333333\alpha)t^2.\end{aligned}$$

The approximate lower and upper solutions with the undetermined fuzzy coefficients method for $t = 0.01$ are

$$\underline{y}_\alpha(t) = 1.02988888889 + 0.50496666667\alpha,$$
$$\overline{y}_\alpha(t) = 2.03982222223 - 0.50496666667\alpha.$$

From the equations (3.27),the exact lower and upper solution for $t = 0.01$ are

$$\underline{Y}_\alpha(t) = 0.99558575885 + 0.50502508354\alpha,$$
$$\overline{Y}_\alpha(t) = 2.00563592592 - 0.50502508354\alpha.$$

By the Adomian decomposition method, we obtain the approximate solution of (3.24), (3.25) as

$$\underline{y}_\alpha(t) = (1+0.5\alpha) + (3+0.5\alpha)t + (-2+0.5\alpha)\frac{t^2}{2} + (-4+0.5\alpha)\frac{t^3}{6}$$

$$\overline{y}_\alpha(t) = (2-0.5\alpha) + (4-0.5\alpha)t + (-1-0.5\alpha)\frac{t^2}{2} + (-3-0.5\alpha)\frac{t^3}{6}$$

The approximate lower and upper solutions with the Adomian decomposition methodfor $t = 0.01$ are

$$\underline{y}_\alpha(t) = 1.02990066667 + 0.50005008333\alpha,$$
$$\overline{y}_\alpha(t) = 2.03994983334 - 0.50005008333\alpha$$

The lower exact solution and the lower approximate solutions by the UFCM and the ADM

α	$\underline{Y}_\alpha(t)$	$\underline{y}_\alpha(t)$(**UFCM**)	$\underline{y}_\alpha(t)$(**ADM**)	$Error$(**UFCM**)	$Error$(**ADM**)
0	0.99558575885	1.02988888889	1.02990066667	0.03430313004	0.03431490782
0.1	1.0460882672	1.08038555556	1.079905675	0.03429728836	0.0338174078
0.2	1.09659077556	1.13088222222	1.12991068334	0.03429144666	0.03331990778
0.3	1.14709328391	1.18137888889	1.17991569167	0.03428560498	0.03282240776
0.4	1.19759579227	1.23187555556	1.2299207	0.03427976329	0.03232490773
0.5	1.24809830062	1.28237222223	1.27992570834	0.03427392161	0.03182740772
0.6	1.29860080897	1.33286888889	1.32993071667	0.03426807992	0,0313299077
0.7	1.34910331733	1.38336555556	1.379935725	0.03426223823	0.03083240767
0.8	1.39960582568	1.43386222223	1.42994073333	0.03425639655	0.03033490765
0.9	1.45010833404	1.48435888889	1.47994574167	0.03425055485	0.02983740763
1	1.50061084239	1.53485555556	1.52995075	0.03424471317	0.02933990761

The upper exact solution and the upper approximate solutions by the UFCM and the ADM

α	$\overline{Y}_\alpha(t)$	$\overline{y}_\alpha(t)$(**UFCM**)	$\overline{y}_\alpha(t)$(**ADM**)	$Error$(**UFCM**)	$Error$(**ADM**)
0	2.00563592592	2.03982222223	2.03994983334	0.03418629631	0.03431390742
0.1	1.95513341757	1.98932555556	1.98994482501	0.03419213799	0.03481140744
0.2	1.90463090921	1.9388288889	1.93993981667	0.03419797969	0.03530890746
0.3	1.85412840086	1.88833222223	1.88993480834	0.03420382137	0.03580640748
0.4	1.8036258925	1.83783555556	1.83992980001	0.03420966306	0.03630390751
0.5	1.75312338415	1.7873388889	1.78992479168	0.03421550475	0.03680140753
0.6	1.7026208758	1.73684222223	1.73991978334	0.03422134643	0.03729890754
0.7	1.65211836744	1.68634555556	1.68991477501	0.03422718812	0.03779640757
0.8	1.60161585909	1.63584888889	1.63990976668	0.0342330298	0.03829390759
0.9	1.55111335073	1.58535222223	1.58990475834	0.0342388715	0.03879140761
1	1.50061084238	1.53485555556	1.53989975001	0.03424471318	0.03928890763

4. Conclusion

In this paper, the exact and the approximate solutions of second-order fuzzy linear initial value problems with positive and negative constant coeffients are investigated. The exact solutions are found by using the Hukuhara differentiability and the approximate solutions are found by using the undetermined fuzzy coefficient method and Adomian decomposition method. Then, the values of the exact solutions and the approximate solutions for each $\alpha = 0, 0.1, 0.2, 0,3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$ are computed. Consequently, the errors of the lower and upper approximate solutions by the Adomian method are the less than the errors of the the lower and upper approximate solutions by the undetermined fuzzy coefficient method for the case of positive constant coefficient. But for the case of negative constant coefficient, while the error of the lower approximate solution by the Adomian method are the less than the error of lower approximate solution by the undetermined fuzzy coefficient method, the error of the upper approximate solution by the undetermined fuzzy coefficient method are the less than the error of the upper approximate solution by Adomian method.

References

[1] S. Abbasbandy and T. Allahviranloo, Numerical solutions of fuzzy differential equations by Taylor method, *Comput. Methods Appl. Math*.,2 (2002), 113-124.

[2] S. Abbasbandy, T. Allahviranloo, O. Lopez-Pouso and J. J. Nieto, Numerical methods for fuzzy differential inclusions, *Journal of Computers Mathematics with Applications*, 48 (2004), 1633-1641.

[3] G. Adomian, A review of the decomposition method and some results for nonlinear equations, *Math. Compute Model*, 7 (1990), 17-43.

[4] T. Allahviranloo, N. Ahmady and E. Ahmady, Numeric solutions of fuzzy differential equations by predictor-corrector method, *Inf. Sci*., 177 (2007), 1633-1647.

[5] T. Allahviranloo, Nth-order fuzzy linear differential equations, *İnf. Sci*., 178 (2008), 1309-1324.

[6] B. Bede and S. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equation, *Fuzzy Sets Syst*., 151 (2005), 581-599.

[7] B. Bede, A Note on "Two-point boundary value problems associated with non-linear fuzzy differential equations", *Fuzzy Sets and Systems*, 157 (2006), 986-989.

[8] B. Bede, Note on "Numerical solutions of fuzzy differential equations by predictor method", *Inf. Sci*., 178 (2008), 1917-1922.

[9] J.J. Buckley and T. Feuring, Fuzzy differential equations, *Fuzzy Sets and Systems*, 110 (2000), 43-54.

[10] J.J. Buckley, T. Feuring, Fuzzy initial value problem for Nth-order linear differential equation, *Fuzzy Sets and Systems*, 121 (2001), 247-255.

[11] D. Dubois and H.Prade, Operations on fuzzy numbers, *Int. J. Syst. Sci*., 9 (1978), 613-626.

[12] M. Friedman, M. Ma and A. Kandel, Numerical solution of fuzzy differential and integral equations, *Fuzzy Sets and Systems*, 106 (1999), 35-48.

[13] N.A. Gasilov, S.E. Amrahov and A.G. Fatullayev, A geometric approach to solve fuzzy linear systems of differential equations, Appl. Math. Inf. Sci., 5 (2011), 484-495.

[14] X. Guo, D. Shang and X. Lu, Fuzzy approximate solutions of second-order fuzzy linear boundary value problems, *Boundary Value Problems*, 2013, doi:10.1186/1687-2770-2013-212.

[15] H. Gültekin and N. Altınışık, On solution of two-point fuzzy boundary value problems, *Bulletin of Society for Mathematical Services & Standarts*, 3(2) (2014), 43-53.

[16] H. Gültekin Çitil and N. Altınışık, On the eigenvalues and the eigenfunctions of the Sturm-Liouville fuzzy boundary value

problem, *Journal of Mathematical and Computational Science,* 7(4) (2017), 786-805.

[17] E. Hüllermeir, An approach to modeling and simulation of uncertain dynamical systems, *Internat. J. Uncertanity, Fuzziness, Knowledge-Based Systems*, 5 (1997), 117-137.

[18] A.Khastan and K. Ivaz, Numerical solution of fuzzy differential equations by Nyström method, *Chaos Solitons Fractals*, 41 (2009), 859-868.

[19] A. Khastan and JJ. Nieto, A boundary value problem for second order fuzzy differential equations, *Nonlinear Analysis*, 72 (2010), 3583-3593.

[20] H-K. Liu, Comparations results of two-point fuzzy boundary value problems, *International Journal of Computational and Mathematical Sciences*, 5:1 (2011).

[21] M. L. Puri and D. A. Ralescu, Differentials for fuzzy functions, *J. Math. Anal.Appl.*, 91 (1983), 552-558.

[22] L. Wang and S. Guo, Adomian method for second-order fuzzy differential equation, *World Academy of Science, Engineering and Technology*, 76 (2011), 979-982.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆