Malaya Journal of Matematik, Vol. 6, No. 1, 80-84, 2018

https://doi.org/10.26637/MJM0601/0012

Numerical solution of the first order linear fuzzy differential equations using He′s variational iteration method

M. Ramachandran[1]* and M. Shobanapriya[2]

Abstract

In this research, first order linear fuzzy differential equations is considered. This paper compares the He's variational iteration method (HVIM) and Leapfrog method [17] for solving these equations. He's variational iteration method is an analytical procedure for finding the solutions of problems which is based on the constructing a variational iterations. The Leapfrog method, based upon Taylor series, transforms the fuzzy differential equation into a matrix equation. The results of applying these methods to the first order linear fuzzy differential equations show the simplicity and efficiency of these methods.

Keywords

Fuzzy differential equations, Fuzzy initial value problems, Leapfrog method, He's variational iteration method.

AMS Subject Classification

65L80, 65L05.

[1] *Department of Mathematics, Government Arts and Science College, Sathyamangalam-638 401, Tamil Nadu, India.*

[2] *BT Assistant (Mathematics), Municipal Girls Higher Secondary School, Gobichettipalayam-638 476, Tamil Nadu, India.*

*****Corresponding author**: [1] dr.ramachandran64@gmail.com; [2]mshobanapriya@gmail.com

Article History: Received **24** September **2017**; Accepted **09** December **2017**

Contents

1. Introduction

Knowledge about dynamical systems modelled by differential equations is often incomplete or vague. It concerns, for example, parameter values, functional relationships, or initial conditions. The well-known methods for solving analytically or numerically initial value problems can only be used for finding a selected system behaviour, e.g., by fixing the unknown parameters to some plausible values. However, in this case, it is not possible to describe the whole set of system behaviours compatible with our partial knowledge.

The topics of fuzzy differential equations, which attracted a growing interest for some time, in particular, in relation to the fuzzy control, have been rapidly developed recent years. The concept of a fuzzy derivative was first introduced by S. L. Chang, L. A. Zadeh [2]. It was followed up by D. Dubois, H. Prade [3], who defined and used the extension principle. Other methods have been discussed by M. L. Puri, D. A. Ralescu [16] and R. Goetschel and W. Voxman [5]. Fuzzy differential equations and initial value problems were regularly treated by O. Kaleva [12, 13], S. Seikkala [20]. A numerical method for solving first order linear fuzzy differential equations has been introduced by M. Ma, M. Friedman and A. Kandel [15] via the standard Euler method.

Recently, T. Jayakumar, D. Maheskumar and K. Kanagarajan [14] solved the first order linear fuzzy differential equations using Runge-Kutta method of order five. S. Sekar and S. Senthilkumar [18] solved the same first order linear fuzzy differential equations using single term Haar wavelet series method. The objective of this paper is to use the He's variational iteration method to solve the first order linear fuzzy differential equations (discussed by T. Jayakumar, D.

Maheskumar and K. Kanagarajan [14] and S. Sekar and S. Senthilkumar [18]).

In this article we developed numerical methods for first order linear fuzzy differential equations to get discrete solutions via He's variational iteration method which was studied by Sekar et al. [19]. The subject of this paper is to try to find numerical solutions of first order linear fuzzy differential equations using He's variational iteration method and compare the discrete results with the Leapfrog method which is presented previously by Sekar et al. [17]. Finally, we show the method to achieve the desired accuracy. Details of the structure of the present method are explained in sections. We apply He's variational iteration method and Leapfrog method for first order linear fuzzy differential equations. In Section 4, it's proved the efficiency of the He's variational iteration method. Finally, Section 5 contains some conclusions and directions for future expectations and researches.

2. He's Variational Iteration Method

In this section, we briefly review the main points of the powerful method, known as the He's variational iteration method [6]-[11]. This method is a modification of a general Lagrange multiplier method proposed by [6]-[11]. In the variational iteration method, the differential equation

$$L[u(t)]+N[u(t)]=g(t) \tag{2.1}$$

is considered, where L and N are linear and nonlinear operators, respectively and $g(t)$ is an inhomogeneous term. Using the method, the correction functional

$$u_{n+1}(t)=u_n(t)+\int \lambda[L[u_n(s)]+N[\tilde{u}_n(s)]-g(s)]ds \tag{2.2}$$

is considered, where λ is a general Lagrange multiplier, u_n is the nth approximate solution and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n=0$. In this method, first we determine the Lagrange multiplier λ that can be identified via variational theory, i.e. the multiplier should be chosen such that the correction functional is stationary, i.e. $\delta\tilde{u}_{n+1}(u_n(t),t)=0$. Then the successive approximation $u_n, n\geq 0$ of the solution u will be obtained by using any selective initial function u_0 and the calculated Lagrange multiplier λ. Consequently $u=lim_{n\to\infty}u_n$. It means that, by the correction functional (2.2) several approximations will be obtained and therefore, the exact solution emerges at the limit of the resulting successive approximations.

3. General format for Fuzzy initial value problems

Consider a first-order fuzzy initial value differential equation is given by

$$y'(t)=f(t,y(t)),t\in[t_0,T],y(t_0)=y_0$$

where y is a fuzzy function of $t, f(t,y)$ is a fuzzy function of the crisp variable t and the fuzzy variable y, y' is the fuzzy derivative of y and $y(t_0=y_0$ is a parallelogram or a parallelogram shaped fuzzy number. We denote the fuzzy function y by $y=[\underline{y},\bar{y}$. It means that the r-level set of $y(t)$ for $t\in[t_0,T]$ is

$$[y(t)]_r=[\underline{y}(t;r),\bar{y}(t;r)],[y(t_0)]_r=[\underline{y}(t_0;r),\bar{y}(t_0;r)],r\in(0,1]$$

we write $f(t;y)=[\underline{f}(t;y),\overline{f}(t;y)]$ and $\underline{f}(t;y)=F[t,\underline{y},\bar{y}],\overline{f}(t;y)=G[t,\underline{y},\bar{y}]$. Because of $y'(t)=f(t,y)$ we have

$$\underline{f}(t;y(t;r))=F[t;\underline{y}(t;r),\bar{y}(t;r)]$$

$$\overline{f}(t;y(t;r))=F[t;\underline{y}(t;r),\bar{y}(t;r)]$$

By using the extension principle, we have the membership function

$$f(t;y(t))(s)=Sup\{y(t)(\tau)/s=f(t,\tau)\},s\in R$$

so fuzzy number $f(t;y(t))$. From this it follows that

$$[f(t;y(t))]_r=[\underline{f}(t,y(t;r)),\overline{f}(t,y(t;r))],r\in[0;1]$$

where

$$\underline{f}(t,y(t;r))=min\{f(t,u)/u\in[y(t)]_r\}$$

$$\overline{f}(t,y(t;r))=max\{f(t,u)/u\in[y(t)]_r\}$$

Definition 3.1. *A function $f:R\to R_F$ is said to be fuzzy continuous function, if for an arbitrary fixed $t_0\in R$ and $\varepsilon>0,\delta>0$ such that $|t-t_0<\delta\Rightarrow D[f(t),f(t_0)]|<\varepsilon$ exists.*

In this paper we also consider fuzzy functions which are continuous in metric D. Then the continuity of $f(t,y(t);r)$ guarantees the existence of the definition of $f(t,y(t);r)$ for $t\in[t_0,T]$ and $r\in[0,1]$ [M. Ma, M. Friedman and A. Kandel [15]]. Therefore, the functions G and F can be definite too.

4. Numerical Experiments

In this section, the exact solutions and approximated solutions obtained by He's variational iteration method and Leapfrog method. To show the efficiency of the He's variational iteration method, we have considered the following problem taken from C. Duraisamy and B. Usha [4] and T.Jayakumar, D.Maheskumar and K.Kanagarajan [14], with step size $r=0.1$ along with the exact solutions.

The discrete solutions obtained by the two methods, He's variational iteration method and Leapfrog method. The absolute errors between them are tabulated and are presented in Tables 1 – 4. To distinguish the effect of the errors in accordance with the exact solutions, graphical representations are given for selected values of $'r'$ and are presented in Figures 1 – 6 for the following problem, using three dimensional effects.

4.1 Example

Consider the initial value problem [C. Duraisamy and B. Usha [4]]

$$y'(t) = tf(t), t \in [0,1]$$

with initial condition $y(0) = (1.01 + 0.1r\sqrt{e}, 1.5 + 0.1r\sqrt{e})$

The exact solution at $t = 0.1$ is given by

$Y(0.1,r) = [(1.01+0.1r\sqrt{e})e^{0.0005}, (1.5+0.1r\sqrt{e})e^{0.0005}]$, $0 \le r \le 1$

4.2 Example

Consider the fuzzy initial value problem [M. Ma, M. Friedman and A. Kandel [15]]

$$y'(t) = y(t), t \in I = [0,1]$$

with initial condition $y(0) = (0.75 + 0.25r, 1.125 - 0.125r)$, $0 < r \le 1$

The exact solution is given by

$Y_1(t,r) = y_1(0;r)e^t$, $Y_2(t,r) = y_2(0;r)e^t$ which at $t = 1$

4.3 Example

Consider the fuzzy initial value problem [James J. Buckley and Thomas Feurihg [1]]

$$y'(t) = c_1 y^2(t) + c_2$$

with initial condition $y(0) = 0$
where $c_i > 0$, for $i = 1,2$ are triangular fuzzy numbers.

The exact solution is given by

$Y_1(t;r) = l_1(r)tan(w_1(r)t)$,

$Y_2(t;r) = l_2(r)tan(w_2(r)t)$,

with

$l_1(r) = \sqrt{c_{2,1}(r)/c_{1,1}(r)}, l_2(r) = \sqrt{c_{2,2}(r)/c_{1,2}(r)}$

$w_1(r) = \sqrt{c_{1,1}(r)/c_{2,1}(r)}, w_2(r) = \sqrt{c_{1,2}(r)/c_{2,2}(r)}$

where $[c_1]_r = [c_{1,1}(r), c_{1,2}(r)]$ and $[c_2]_r = [c_{2,1}(r), c_{2,2}(r)]$

$c_{1,1}(r) = 0.5 + 0.5r, c_{1,2}(r) = 1.5 - 0.5r$

$c_{2,1}(r) = 0.75 + 0.25r, c_{2,2}(r) = 1.25 - 0.25r$

The r-level sets of $y'(t)$ are

$Y_1'(t;r) = c_{2,1}(r)sec^2(w_1(r)t)$,

$Y_2'(t;r) = c_{2,2}(r)sec^2(w_2(r)t)$,

which defines a fuzzy number. We have

$f_1(t,y,r) = min\{|c_1u^2 + c_2|u \in [y_1(t;r), y_2(t;r)]$, $c_1 \in [c_{1,1}(r), c_{1,2}(r)], c_2 \in [c_{2,1}(r), c_{2,2}(r)]\}$

$f_2(t,y,r) = max\{|c_1u^2 + c_2|u \in [y_1(t;r), y_2(t;r)]$, $c_1 \in [c_{1,1}(r), c_{1,2}(r)], c_2 \in [c_{2,1}(r), c_{2,2}(r)]\}$

5. Conclusion

He's variational iteration method is a powerful, accurate, and flexible tool for solving many types of fuzzy differential equations (problems) in scientific computation. The obtained approximate solutions of the first order linear fuzzy differential equations are compared with exact solutions and it reveals that the He's variational iteration method works well for finding the approximate solutions. From the Tables 1 -– 4, one can observe that for most of the time intervals, the absolute error

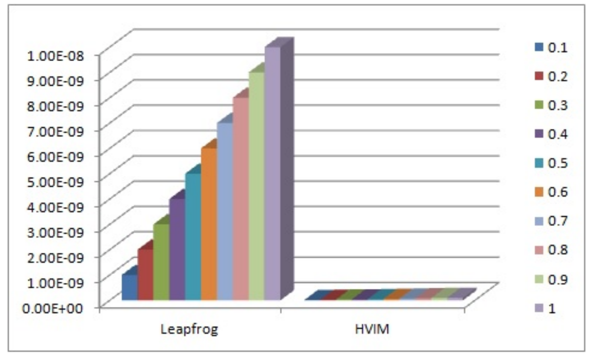

Figure 1. Error estimation of Example 4.1 at y_1

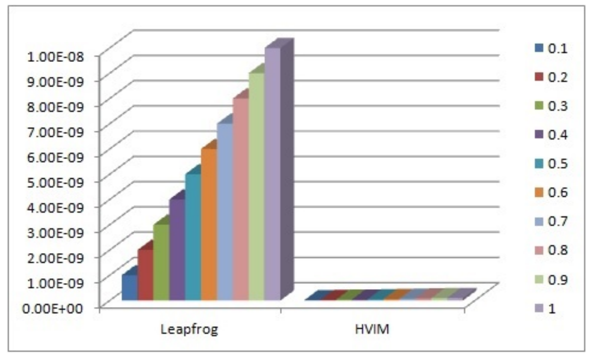

Figure 2. Error estimation of Example 4.1 at y_2

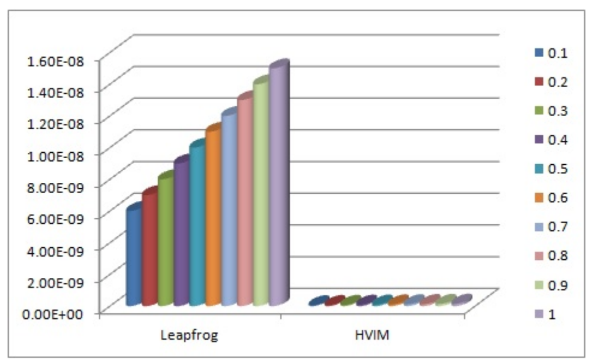

Figure 3. Error estimation of Example 4.2 at y_1

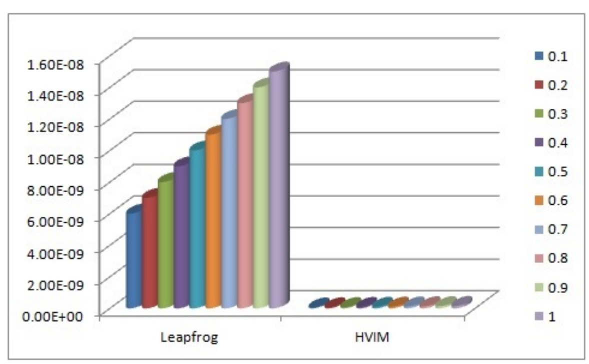

Figure 4. Error estimation of Example 4.2 at y_2

Table 1. He's variational iteration method – Error Calculations

	Example 4.1		Example 4.2	
r	y_1	y_2	y_1	y_2
0.1	1.00E-11	1.00E-11	6.00E-11	6.00E-11
0.2	2.00E-11	2.00E-11	7.00E-11	7.00E-11
0.3	3.00E-11	3.00E-11	8.00E-11	8.00E-11
0.4	4.00E-11	4.00E-11	9.00E-11	9.00E-11
0.5	5.00E-11	5.00E-11	1.00E-10	1.00E-10
0.6	6.00E-11	6.00E-11	1.10E-10	1.10E-10
0.7	7.00E-11	7.00E-11	1.20E-10	1.20E-10
0.8	8.00E-11	8.00E-11	1.30E-10	1.30E-10
0.9	9.00E-11	9.00E-11	1.40E-10	1.40E-10
1.0	1.00E-10	1.00E-10	1.50E-10	1.50E-10

Table 2. Leapfrog Method – Error Calculations

	Example 4.1		Example 4.2	
r	y_1	y_2	y_1	y_2
0.1	1.00E-09	1.00E-09	6.00E-09	6.00E-09
0.2	2.00E-09	2.00E-09	7.00E-09	7.00E-09
0.3	3.00E-09	3.00E-09	8.00E-09	8.00E-09
0.4	4.00E-09	4.00E-09	9.00E-09	9.00E-09
0.5	5.00E-09	5.00E-09	1.00E-08	1.00E-08
0.6	6.00E-09	6.00E-09	1.10E-08	1.10E-08
0.7	7.00E-09	7.00E-09	1.20E-08	1.20E-08
0.8	8.00E-09	8.00E-09	1.30E-08	1.30E-08
0.9	9.00E-09	9.00E-09	1.40E-08	1.40E-08
1.0	1.00E-08	1.00E-08	1.50E-08	1.50E-08

Table 3. He's variational iteration method – Error Calculations

	Example 4.3	
r	y_1	y_2
0.1	1.00E-11	1.00E-11
0.2	2.00E-11	2.00E-11
0.3	3.00E-11	3.00E-11
0.4	4.00E-11	4.00E-11
0.5	5.00E-11	5.00E-11
0.6	6.00E-11	6.00E-11
0.7	7.00E-11	7.00E-11
0.8	8.00E-11	8.00E-11
0.9	9.00E-11	9.00E-11
1.0	1.00E-10	9.90E-11

Table 4. Leapfrog Method – Error Calculations

	Example 4.3	
r	y_1	y_2
0.1	1.00E-09	1.00E-09
0.2	2.00E-09	2.00E-09
0.3	3.00E-09	3.00E-09
0.4	4.00E-09	4.00E-09
0.5	5.00E-09	5.00E-09
0.6	6.00E-09	6.00E-09
0.7	7.00E-09	7.00E-09
0.8	8.00E-09	8.00E-09
0.9	9.00E-09	9.00E-09
1.0	1.00E-08	9.90E-09

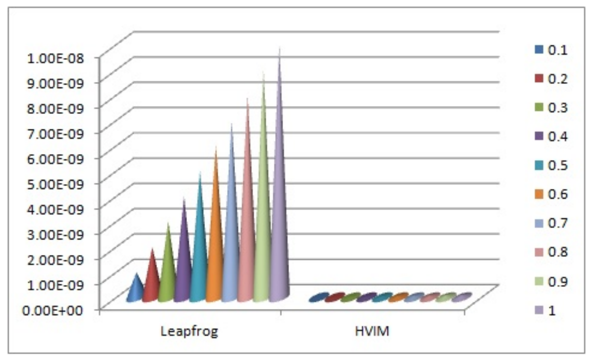

Figure 5. Error estimation of Example 4.3 at y_1

is less in He's variational iteration method when compared to the Leapfrog method [18], which yields a little error, along with the exact solutions. From the Figures 1 -– 6, it can be predicted that the error is very less in He's variational iteration method when compared to the Leapfrog method. Hence, He's variational iteration method is more suitable for studying first order linear fuzzy differential equations.

References

[1] J. J. Buckley and T. Feurihg, Introduction to Fuzzy Partial Differential Equations, *Fuzzy Sets and Systems*, 110(1) (2000), 43-54.

[2] S. L. Chang and L. A. Zadeh, On fuzzy mapping and control, *IEEE Transactions on Systems, Man, and Cybernetics*, 2 (1972), 30-34.

[3] D. Dubois and H. Prade, Towards fuzzy differential calculus.III. Differentiation, *Fuzzy Sets and Systems*, 8(3) (1982), 225-233.

[4] C. Duraisamy and B. Usha, Another Approach to Solution of Fuzzy Differential Equations, *Applied Mathematical Sciences*, 4(16) (2010), 777-790.

[5] R. Goetschel and W. Voxman, Elementary calculus, *Fuzzy Sets and Systems*, 18 (1986), 31-43.

[6] J. H. He, A new approach to nonlinear partial differential equations, *Commun Nonlinear Sci Numer Simulat*, 2(4) (1997), 203-205.

[7] J. H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, *Comput Methods Appl Mech Eng.*, 167(1998), 57-68.

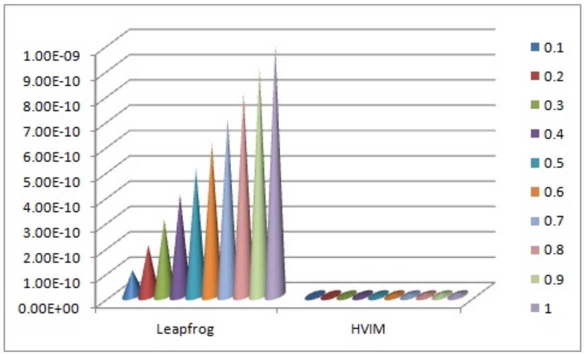

Figure 6. Error estimation of Example 4.3 at y_2

[8] J. H. He, Variational iteration method – a kind of nonlinear analytical technique: some examples, *Int. J. Nonlinear Mech.*, 34(1999), 699-708.

[9] J. H. He, Variational iteration method for autonomous ordinary differential systems, *Appl. Math. Comput.*, 114(2–3) (20000, 115-123.

[10] J. H. He, Variational iteration method – Some recent results and new interpretations, *J. Comput. Appl. Math.*, 207 (2007), 3–17.

[11] J. H. He, Variational principle for some nonlinear partial differential equations with variable coefficients, *Chaos Solitons Fractals*, 19 (2004), 847–851.

[12] O. Kaleva, Fuzzy differential equations, *Fuzzy Sets and Systems*, 24(3) (1987), 301-317.

[13] O. Kaleva, "The Cauchy problem for fuzzy differential equations," Fuzzy Sets and Systems, vol. 35, no. 3, pp. 389—396, 1990.

[14] T. Jayakumar, D. Maheskumar and K. Kanagarajan, Numerical Solution of Fuzzy Differential Equations by Runge Kutta Method of Order Five, *Applied Mathematical Sciences*, 6(60) (2012), 2989-3002.

[15] M. Ma, M. Friedman and A. kandel, Numerical Solutions of Fuzzy Differential Equations, *Fuzzy Sets and Systems*, 105 (1999), 133-138.

[16] M. L. Puri and D. A. Ralescu, Differentials of fuzzy functions, *Journal of Mathematical Analysis and Applications*, 91(2) (1983), 552-558.

[17] S. Sekar and K. Prabhavathi, Numerical solution of first order linear fuzzy differential equations using Leapfrog method, *IOSR Journal of Mathematics*, 10(5) (2014), 07-12.

[18] S. Sekar and S. Senthilkumar, Single Term Haar Wavelet Series for Fuzzy Differential Equations, *International Journal of Mathematics Trends and Technology*, 4(9) (2013), 181-188.

[19] S. Sekar and B. Venkatachalam alias Ravikumar, Numerical Solution of linear first order Singular Systems Using He's Variational Iteration Method, *IOSR Journal of Mathematics*, 11(5) (2015), 32-36.

[20] S. Seikkala, On the fuzzy initial value problem, *Fuzzy Sets and Systems*, 24(3) (1987), 319-330.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆