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Maximum principles for fourth order semilinear elliptic boundary value problems

Gajanan C. Lomte[a,*] and R. M. Dhaigude[b]

[a] *Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad-431004, India.*

[b] *Government Vidarbha Institute of Science and Humanities, Amravati-444 604, Maharashtra, India.*

Dedicated to 61^{th} *Birthday of Dr. D. B. Dhaigude*

Abstract

The paper is devoted to prove maximum principles for the certain functionals defined on solution of the fourth order semilinear elliptic equation. The maximum principle so obtained is used to prove the non-existence of nontrivial solutions of the fourth order semilinear elliptic equation with some zero boundary conditions. Hopf's maximum principle is main ingredient.

Keywords: Maximum principles, fourth order elliptic equations, integral bounds.

2010 MSC: 35J40, 35J61.

1 Introduction

The 'P- function' technique for deducing maximum principle results for partial differential equations of order ≥ 2 is well known. For instance, [5] Miranda shows that the P- function

$$P = |\nabla u(x)|^2 - u\Delta u \tag{1.1}$$

is subharmonic, where u is a classical solution to the biharmonic equation $\Delta^2 u = 0$. Since, then many others have employed this technique on various classes of fourth order partial differential equations. In [7], for example, Schaefer utilizes auxiliary functions of type (1.1) to study semilinear equations of the form

$$\Delta^2 u + \rho(x, y)f(u) = 0,$$

in a plane domain. Still other types of functions have been employed in the pursuit of maximum principle results for fourth order differential equations [1, 2, 4]. Recently [3] Dhaigude and Gosavi extend a maximum principle for a class of fourth order semilinear elliptic equations due to Schaefer [7] to a more general fourth order semilinear elliptic equation of the form

$$\Delta^2 u + a(x, y)\Delta u + b(x, y)f(u) = 0.$$

In this paper, we study the existence problem for fourth order semilinear elliptic equation of the form

$$\Delta^2 u + a(x, y)\Delta u + b(x, y)f(u) = 0.$$

*Corresponding author.
E-mail addresses: gclomte@gmail.com (Gajanan C. Lomte) and rmdhaigude@gmail.com (R. M. Dhaigude)

For simplicity, we use the summation convention and denote partial derivatives $\dfrac{\partial u}{\partial x_i}$ by $u_{,i}$ and $\dfrac{\partial^2 u}{\partial x_i^2}$ by $u_{,ii}$.

This paper is organized as follows. In section 2 we develop a maximum principle for a class of fourth order semilinear elliptic equations. The maximum principle will be used to deduce the non-existence of nontrivial solutions of the boundary value problem under consideration in the last section of this paper.

2 Maximum principles

Suppose Ω is a plane domain bounded by a sufficiently smooth curve $\partial\Omega$. The following Lemma [8] is useful to prove our results.

Lemma 2.1. *For a sufficiently smooth function v the inequality*

$$Nv_{,ik}v_{,ik} \geq (\Delta v)^2$$

holds in N dimensions.

Now, we prove the following maximum principles for the function P denoted by $P = |\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds$, which will be the main result of this paper.

Theorem 2.1. *Let $u \in C^4$ be a sufficiently smooth solution of*

$$\Delta^2 u + a(x,y)\Delta u + b(x,y)f(u) = 0 \tag{2.1}$$

where $a \leq 0, b > 0$ in Ω and

$$b(x,y)u(x,y)f(u) + a(x,y)|\nabla u|^2 \geq 0 \quad in \quad \Omega. \tag{2.2}$$

If φ satisfy

$$\varphi(s) \geq 0,\ \varphi^{'}(s) \geq 0\, for\ s \geq 0,\ \int_0^u \varphi(s)\,ds \leq 0 \tag{2.3}$$

then the function

$$P = |\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds \tag{2.4}$$

assumes its maximum on $\partial\Omega$.

Proof. We have, the function

$$P = |\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds.$$

By straightforward computations

$$P_{,k} = 2u_{,i}u_{,ik} - u_{,k}\Delta u - u(\Delta u)_{,k} + \varphi(u)\,u_{,k} \tag{2.5}$$

$$P_{,kk} = 2u_{,ik}u_{,ik} - (\Delta u)^2 - u\Delta^2 u + \varphi^{'}(u)|\nabla u|^2 + \varphi(u)\Delta u. \tag{2.6}$$

Using (2.1) in (2.6), we get

$$\Delta P = 2u_{,ik}u_{,ik} - (\Delta u)^2 + au\Delta u + buf + \varphi^{'}(u)|\nabla u|^2 + \varphi(u)\Delta u. \tag{2.7}$$

Using (2.4), we have

$$\begin{aligned} \Delta P + aP =& 2u_{,ik}u_{,ik} - (\Delta u)^2 + a|\nabla u|^2 + buf + \varphi^{'}(u)|\nabla u|^2 \\ &+ \varphi(u)\Delta u + a\int_0^u \varphi(s)\,ds. \end{aligned} \tag{2.8}$$

By Lemma 2.1 and assumption (2.2) and (2.3), we see that the right hand side of (2.8) is non-negative. Thus

$$\Delta P + aP \geq 0 \quad in \quad \Omega.$$

□

By maximum principle, the result follows.

Theorem 2.2. *Let $u \in C^4$ be a sufficiently smooth solution of*

$$\Delta^2 u + a(x,y)\Delta u + b(x,y)f(u) = 0 \tag{2.9}$$

where $a \leq 0, b > 0$ in Ω and

$$b(x,y)u(x,y)f(u) + a(x,y)|\nabla u|^2 \geq 0 \quad in \quad \Omega. \tag{2.10}$$

If φ satisfy

$$\varphi(s) \geq 0,\ \varphi^{'}(s) \geq 0\ for\, s \geq 0,\ \int_0^u \varphi(s)\,ds \leq 0 \tag{2.11}$$

then the function

$$P = \frac{1}{b}\left[|\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds\right] \tag{2.12}$$

assumes its maximum on $\partial\Omega$ unless $P < 0$ in Ω.

Proof. We have, the function

$$P = \frac{1}{b}\left[|\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds\right]$$

By straightforward computations

$$P_{,k} = \frac{1}{b}\left[\nabla(|\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds)\right] - \frac{b_{,k}}{b^2}\left(|\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds\right) \tag{2.13}$$

$$\begin{aligned} P_{,kk} = \frac{1}{b}&\left[\Delta(|\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds)\right] - \frac{b_{,k}}{b^2}\Big[\nabla(|\nabla u(x)|^2 - u\Delta u + \\ &\int_0^u \varphi(s)\,ds)\Big] - \frac{b_{,k}}{b^2}\left[\nabla(|\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds)\right] - \frac{b_{,kk}}{b^2}\Big(|\nabla u(x)|^2 - \\ &u\Delta u + \int_0^u \varphi(s)\,ds\Big) + \frac{2b_{,k}b_{,k}}{b^3}\left(|\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds\right). \end{aligned} \tag{2.14}$$

Using (2.12) and after some rearrangements, we have

$$\begin{aligned} \Delta P - 2b\nabla\left(\frac{1}{b}\right)\nabla P + \frac{\Delta b}{b}P = \\ \frac{1}{b}\left[\Delta(|\nabla u(x)|^2 - u\Delta u + \int_0^u \varphi(s)\,ds)\right]. \end{aligned} \tag{2.15}$$

$$\begin{aligned} \Delta P - 2b\nabla\left(\frac{1}{b}\right)\nabla P + \left[\frac{\Delta b}{b} + a\right]P = \\ \frac{1}{b}\Big[2u_{,ik}u_{,ik} - (\Delta u)^2 + a|\nabla u|^2 + buf \\ + \varphi^{'}(u)|\nabla u|^2 + \varphi(u)\Delta u + a\int_0^u \varphi(s)\,ds\Big]. \end{aligned} \tag{2.16}$$

Then it follows from Lemma 2.1 and assumptions (2.10) and (2.11) that P satisfies

$$\Delta P - 2b\nabla\left(\frac{1}{b}\right)\nabla P + \left[\frac{\Delta b}{b} + a\right]P \geq 0 \quad in \quad \Omega.$$

By Hopf's maximum principle [6], the result follows. □

The next Lemma [7]is useful in proving the non-existence result in the last section of the paper.

Lemma 2.2. *If (2.2) is satisfied and if u is a C^4 solution of (2.1) which vanishes on $\partial\Omega$, then*

$$\int_\Omega |\nabla u(x)|^2 dxdy \leq \frac{1}{2}A|\nabla u(x)|_M^2$$

where A is the area of Ω.

3 Applications

In this section as an application of our maximum principle we prove non-existence of nontrivial solutions $u \in C^4$ of the following boundary value problem

$$\Delta^2 u + a(x,y)\Delta u + b(x,y)f(u) = 0, \quad \text{in} \quad \Omega \tag{3.1}$$

$$u(x,y) = 0, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \tag{3.2}$$

and

$$\Delta^2 u + a(x,y)\Delta u + b(x,y)f(u) = 0, \quad \text{in} \quad \Omega \tag{3.3}$$

$$u(x,y) = 0, \quad \Delta u = 0 \quad \text{on} \quad \partial\Omega. \tag{3.4}$$

Theorem 3.1. *If* (2.2) *is satisfied then no non-trivial solution of* (3.1)-(3.2) *exists.*

Proof. It is by contradiction. Assume on the contrary that a nontrivial solution u of the given BVP (3.1)-(3.2) exists. We have P as defined in (2.4). Now, Theorem 2.1 and boundary condition (3.2) gives

$$u_{,i}u_{,i} - u\Delta u + \int_0^u \varphi(s)\,ds \le 0. \tag{3.5}$$

Further integrating (3.5) over Ω, we have

$$\int_\Omega \Big[u_{,i}u_{,i} - u\Delta u\Big]dxdy + \int_\Omega \Big(\int_0^u \varphi(s)\,ds\Big) \le 0. \tag{3.6}$$

Using Green's first identity

$$\int_\Omega \Big[v\Delta u + \nabla v\cdot\nabla u\Big]dxdy = \int_{\partial\Omega} v\frac{\partial u}{\partial n}d\sigma, \quad \text{with} \quad v = u \tag{3.7}$$

and (3.2) in (3.7), we get

$$2\int_\Omega |\nabla u|^2 dxdy + \int_\Omega \Big(\int_0^u \varphi(s)\,ds\Big) \le 0. \tag{3.8}$$

Consequently $|\nabla u| = 0$ in Ω and by continuity $u \equiv 0$ in $\Omega \cup \partial\Omega$. This is a contradiction. Hence there is no nontrivial solution of (3.1)-(3.2). □

Theorem 3.2. *If* (2.2) *is satisfied in a convex domain* Ω *then no nontrivial solution of* (3.3)-(3.4) *exists.*

Proof. It is by contradiction. Assume on the contrary that a nontrivial solution u of the given BVP (3.3) - (3.4) exists. We have P as defined in (2.4). Then by Theorem 2.1, P takes its maximum on the boundary $\partial\Omega$ at a point, say Q. By Hopf's second maximum principle, either $\frac{\partial P}{\partial n}(Q) > 0$ or P is constant in $\Omega \cup \partial\Omega$.

Case I. Suppose $\frac{\partial P}{\partial n}(Q) > 0$ holds. Differentiate P partially in the normal direction and use boundary condition (3.4) to get

$$\frac{\partial P}{\partial n}(Q) = 2\frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial n^2}. \tag{3.9}$$

We know the following relation from differential geometry,

$$\frac{\partial^2 u}{\partial n^2} + k\frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial s^2} = u_{,ii} = \Delta u \quad (see[8], p.46) \tag{3.10}$$

where $\frac{\partial}{\partial n}$ and $\frac{\partial}{\partial s}$ are normal and tangential derivatives respectively. The tangential component $\frac{\partial^2 u}{\partial s^2}$ is zero. Equation (3.10) becomes

$$\frac{\partial^2 u}{\partial n^2} = \Delta u - k\frac{\partial u}{\partial n}. \tag{3.11}$$

Using (3.11) and (3.4) in (3.9), we get

$$\frac{\partial P}{\partial n}(Q) = -2k\Big(\frac{\partial u}{\partial n}\Big)^2. \tag{3.12}$$

Since Ω is convex, $k > 0$. So $\frac{\partial P}{\partial n}(Q) > 0$ is impossible. Therefore, in this case no nontrivial solution exists.

Case II. Suppose P is a constant say c in $\Omega \cup \partial\Omega$. Then we have

$$|\nabla u|^2 = \left(\frac{\partial u}{\partial n}\right)^2 = c \quad \text{on} \quad \partial\Omega. \tag{3.13}$$

Now as $P = c$ in $\Omega \cup \partial\Omega$, we have $\frac{\partial P}{\partial n} = 0$ on $\partial\Omega$. But from (3.12) we have

$$\frac{\partial P}{\partial n}(Q) = -2kc.$$

For a bounded convex domain with a continuously turning tangent on the boundary, $k \neq 0$. Moreover $c \neq 0$, for if $c = 0$ then $|\nabla u|_M = 0$ and by Lemma 2.2 and reasoning as in Theorem 2.1 we are led to the conclusion that $u \equiv 0$ in Ω. Thus $P = c$ is impossible. As neither case is possible, we conclude that no nontrivial solution exists. □

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