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Triparametric self information function and entropy

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Abstract

In this paper we start with a triparametric self information function and triparametric entropy. Some familiar entropies are derived as particular cases. A measure called information deviation and some generalization of Kullback's information are obtained under some boundary conditions.

Keywords: Shannon entropy, Kullback's information, joint entropy, generalized inaccuracy, information deviation.

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1 Introduction

Shannon[10] first introduced the idea of self-information function in the form

$$f(x) = -\log_2 x, \quad 0 < x \leq 1. \tag{1.1}$$

In this paper we use the method of averaging self-informations introduced by Shannon. Like Shannon we introduce a triparametric self-information function defined by

$$f_3(x;\alpha,\beta,\gamma) = \frac{k(x^{\alpha/\gamma} - x^{\beta/\gamma})}{x}, \quad 0 < x \leq 1,\ \alpha \geq 0,\ \beta \geq 0,\ \ \gamma > 0,\ \alpha \neq \beta \neq \gamma \tag{1.2}$$

Where k is a constant, depending upon the real valued parameters $\alpha,\ \beta,\ \gamma$ and k is ascertained by a suitable pair $(x,\ f_3)$. We apply the following conditions on f_3:

(i) $f_3 \to\ 0$ as $x \to 0$.
(ii) $f_3 = 0$, when $x = 1$.
(iii) $f_3 = 1$, $x = \frac{1}{2}$, then $k = \left(2^{\frac{\gamma-\alpha}{\gamma}} - 2^{\frac{\gamma-\beta}{\gamma}}\right)^{-1}$.

The function shows the following particular behaviors:

(I) If $\alpha,\ \beta$ are fixed, then for $x < \frac{1}{2}$, $f_3 \to \infty$ as $\gamma \to \infty$ and for $x > \frac{1}{2}$, $f_3 \to 0$ as $\gamma \to \infty$.
(II) For any fixed γ, $f_3 \to -(2x)^{\frac{\alpha-\gamma}{\gamma}} \log_2 x$ as $\alpha \to \beta$.
(III) If $\beta = \gamma$ and $\alpha \to \gamma(\alpha < \gamma)$, then $f_3 \to -\log_2 x$.

Self-information function is different from information function. Different authors, namely Darcozy [4], Aczel [1], Arndt [2], Chaundy and Mcleod [3], Havrda and Charvat [5], Kannapan [6], Sharma and Taneja [11], Mittal

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[9] and some others have solved some typical functional equations and have used their solutions as entropy, inaccuracy, directed divergence etc., In the capacity of finite measures only in complete probability distributions. The method of averaging self-informations includes the case of generalized probability distributions. Moreover, we have discussed in this paper, information measures in the capacity of even an infinite range, because a parameter can have negative values also corresponding to phenomenal circumstances. Further since it is uncertain and difficult to choose an arbitrary functional equation and to find its suitable solutions to be used as information measures, it becomes easier if we choose any suitable parametric self-information function that can satisfy a number of effective boundary conditions. We have given a most simple and general choice in (1.2).

Section 2 describes a triparametric entropy from which other familiar entropies have been deduced as particular cases. We have given a number of this entropy in section 3 as joint entropy, triparmetric information functions, generalized information function, generalized inaccuracy, a new information called information deviation and lastly generalizations of Kullback's information.

2 Triparametric entropy

Let $P = (p_1, p_2, ..., p_n)$ be a finite discrete probability distribution, where $0 < p_i \leq 1$, $\sum_{i=1}^{n} p_i \leq 1$. Then, averaging the function $f_3(p_i; \alpha, \beta, \gamma)$ with respect to P, we define the triparametric entropy as

$$H(P;\alpha,\beta,\gamma) = \left(2^{\frac{\gamma-\alpha}{\gamma}} - 2^{\frac{\gamma-\beta}{\gamma}}\right)^{-1} \left[\sum_{i=1}^{n}(p_i^{\alpha/\gamma} - p_i^{\beta/\gamma})\right] \Big/ \sum_{i=1}^{n} p_i, \tag{2.1}$$

where $\alpha, \beta, \gamma > 0$, $\alpha \neq \beta \neq \gamma$.
When P is complete, we have

$$H(P;\alpha,\beta,\gamma) = \left(2^{\frac{\gamma-\alpha}{\gamma}} - 2^{\frac{\gamma-\beta}{\gamma}}\right)^{-1} \left[\sum_{i=1}^{n}(p_i^{\alpha/\gamma} - p_i^{\beta/\gamma})\right], \tag{2.2}$$

where $\alpha \geq 0$, $\beta \geq 0$, $\gamma > 0$, $\alpha \neq \beta \neq \gamma$.

2.1 Some familiar entropies

From (2.2), we get the following entropies as particular cases:
(i) $\gamma = 1$ gives Sharma and Taneja's entropy [11] of type (α,β) in the form

$$H(P;\alpha,\beta) = \left(2^{1-\alpha} - 2^{1-\beta}\right)^{-1} \left[\sum_{i=1}^{n}(p_i^{\alpha} - p_i^{\beta})\right], \quad \alpha \neq \beta \tag{2.3}$$

and

$$\underset{\alpha\to\beta}{Lt} H(P;\alpha,\beta) = \left(\sum_{i=1}^{n} p_i^{\beta} \log_2 \frac{1}{p_i}\right) 2^{\beta-1}.$$

(ii) Putting $\alpha = \gamma = 1$, we get Darcozy's entropy [4] of type β as

$$H(P;\beta) = \left(2^{1-\beta} - 1\right)^{-1} \left[\sum_{i=1}^{n}(p_i^{\beta} - 1)\right], \quad \beta > 0,\ \beta \neq 1 \tag{2.4}$$

(iii) When $\beta = \gamma$ and $\alpha \to \gamma$, then (2.2) reduces to

$$H(P) = \sum_{i=1}^{n} p_i \log_2 \frac{1}{p_i}\ , \tag{2.5}$$

which is Shannon entropy.
(iv) When $n > 2$, then $H \to \infty$ as $\gamma \to \infty$; when $n = 1$, then $H = 0$, $p_1 = 1$ and when $n = 2$, then $H = 1$.

3 Application of the entropy (2.2)

3.1 Joint entropy

For joint probability distribution, a relation similar to (2.2) also holds in the form

$$H(PQ;\alpha,\beta,\gamma)=\left(2^{\frac{\gamma-\alpha}{\gamma}}-2^{\frac{\gamma-\beta}{\gamma}}\right)^{-1}\left[\sum_{k=1}^{n}\sum_{j=1}^{m}(p_{kj}^{\alpha/\gamma}-p_{kj}^{\beta/\gamma})\right], \tag{3.1}$$

$$0<p_{kj}\le 1,\ \sum_{k=1}^{n}\sum_{j=1}^{m}p_{kj}=1;\ \alpha\ge 0,\ \beta\ge 0,\gamma>0,\ \alpha\ne\beta.$$

Theorem 3.1. *If $P=(p_1,p_2,...,p_n)$ be the distribution of input symbols of a source, $Q=(q_1,q_2,...,q_m)$ be that of output symbols and $PQ=(p_{k1},p_{k2},...,p_{km};\ \ k=1,2,...,n)$ be the joint distribution of input and output symbols; also*

$$R_k=\left(\frac{p_{k1}}{p_k},\frac{p_{k2}}{p_k},...,\frac{p_{km}}{p_k}\right)$$

be the conditional distribution of output symbols and

$$R_j=\left(\frac{p_{1j}}{q_j},\frac{p_{2j}}{q_j},...,\frac{p_{nj}}{q_j}\right)$$

be the conditional distribution of input symbols, where

$$p_{kj}/p_k=p_{j|k},(j=1,2,...,m);\ \ p_{kj}/q_j=p_{k|j},\ (k=1,2,...,n);$$

$$\sum_{j=1}^{m}p_{kj}=p_k\ \textit{and}\ \sum_{k=1}^{n}p_{kj}=q_j,$$

then

$$H(PQ;\alpha,\beta,\gamma)=\sum_{k=1}^{n}p_k^{\frac{\beta}{\gamma}}H(R_k;\alpha,\beta,\gamma)+\left(2^{\frac{\gamma-\alpha}{\gamma}}-2^{\frac{\beta-\gamma}{\gamma}}\right)^{-1}\left[\sum_{k=1}^{n}\left(p_k^{\frac{\alpha}{\gamma}}-p_k^{\frac{\beta}{\gamma}}\right)\sum_{j=1}^{m}p_{j|k}^{\frac{\alpha}{\gamma}}\right]. \tag{3.2}$$

Putting $\alpha=\gamma=1$ and using $\sum_{j=1}^{m}p_{j|k}=1$ in (3.2), we have

$$H(PQ;\ \beta)=\sum_{k=1}^{n}p_k^{\beta}H_1(R_k;\beta)+H_1(P;\beta). \tag{3.3}$$

Theorem 3.2. *If $p_{kj}=p_kq_j$, then*

$$H(PQ;\ \alpha,\beta,\gamma)=\sum_{k=1}^{n}p_k^{\frac{\alpha}{\gamma}}H(R_k;\alpha,\beta,\gamma)+\sum_{j=1}^{m}q_j^{\frac{\beta}{\gamma}}H(R_j;\alpha,\beta,\gamma)$$

$$=\sum_{k=1}^{n}p_k^{\frac{\alpha}{\gamma}}H(Q;\alpha,\beta,\gamma)+\sum_{j=1}^{m}q_j^{\frac{\beta}{\gamma}}H(P;\alpha,\beta,\gamma). \tag{3.4}$$

3.2 Triparametric information function

With the help of equation (2.2), we define a triparametric information function in the form

$$F_3(x)=F_3(x;\alpha,\beta,\gamma)=\left(2^{\frac{\gamma-\alpha}{\gamma}}-2^{\frac{\gamma-\beta}{\gamma}}\right)^{-1}(x^{\alpha/\gamma}-x^{\beta/\gamma}) \tag{3.5}$$

$\alpha\ge 0,\ \beta\ge 0,\ \ \gamma>0,\ \alpha\ne\beta\ne\gamma$ and $0<x\le 1$.
Where $F_3(0)=0$, but $F_3(1)=0$ and $F_3\left(\frac{1}{2}\right)=\frac{1}{2}$ always.
Thus

$$H(P;\alpha,\beta,\gamma)=\sum_{k=1}^{n}F(p_k),\ \ 0<p_k\le 1,\quad \sum_{ki=1}^{n}p_k=1. \tag{3.6}$$

Putting $a=\alpha/\gamma,\ \ b=\beta/\gamma$ in (3.5), we have

$$F_3(x)=F(x;\alpha,\beta,\gamma)=\left(2^{1-a}-2^{1-b}\right)^{-1}(x^a-x^b),\quad -\infty<a,b<\infty,\quad a\ne b. \tag{3.7}$$

Now, from practical point of view, as far as an inaccuracy in a measure is concerned, a measure is associated with at least two probability distributions, corresponding to which at least two variables u and v are needed. This suggests the choice of at least four parameters a,b,c and d.

3.3 Generalized information function

Concerning an association of two variables u, v and four parameters a, b, c, d, an information measure similar to (3.7) is introduced by

$$F_4(u,v) = F(u,v;a,b,c,d) = G[u^a v^b - u^c v^d], \tag{3.8}$$

$$0 < u,v \leq 1, \quad -\infty < a,b,c,d < \infty, \quad a \neq b \neq c \neq d$$

as the generalized information function, which possesses the characteristic of becoming both bounded and unbounded.

3.3.1 Boundary conditions

(i) At $u = 1$, $v = \frac{1}{2}$, $F_4\left(1, \frac{1}{2}\right) = \frac{1}{2}$, so that $G = (2^{1-b} - 2^{1-d})^{-1}$, where $b \neq d$.

If $a + b = c + d$, where $a \neq c$, then $F_4\left(\frac{1}{2}, \frac{1}{2}\right) = 0$. Similarly at $u = \frac{1}{2}$, $v = 1$, $F_4\left(\frac{1}{2}, 1\right) = \frac{1}{2}$ so that $G = (2^{1-a} - 2^{1-c})^{-1}$, where $a \neq c$.

(ii) At $u = 1$, $v = \frac{1}{2}$, $F_4\left(1, \frac{1}{2}\right) = \frac{1}{2}$ so that $G = (2^{-b} - 2^{-d})^{-1}$, where $b \neq d$.
At $u = \frac{1}{2}$, $v = 1$, $F_4\left(\frac{1}{2}, 1\right) = 1$, so that $G = (2^{-a} - 2^{-c})^{-1}$, where $a \neq c$.

3.3.2 Generalize inaccuracy

Let $P = (p_1, p_2, ..., p_n)$ and $Q = (q_1, q_2, ..., q_n)$ be two discrete probability distributions concerned with (3.8), where $0 < p_i \leq 1$, $0 < q_i$, $\sum_{i=1}^n p_i = 1$, $\sum_{i=1}^n q_i = 1$, $(u,v) = (p_i, q_i)$ or (q_i, p_i); $i = 1, 2, ..., n$.
We may then define the generalized inaccuracies by

$$I_4(P \parallel Q) = \sum_{i=1}^n F_4(p_i, q_i) = (2^{1-b} - 2^{1-d})^{-1} \left[\sum_{i=1}^n p_i^a q_i^b - \sum_{i=1}^n p_i^c q_i^d\right], b \neq d, \tag{3.9}$$

$$I_4(Q \parallel P) = \sum_{i=1}^n F_4(q_i, p_i) = (2^{1-b} - 2^{1-d})^{-1} \left[\sum_{i=1}^n q_i^a p_i^b - \sum_{i=1}^n q_i^c p_i^d\right], b \neq d, \tag{3.10}$$

which follows from (3.8) and boundary condition 3.3.1(i).
Given P and Q, we see that
(i) $I_4(P \parallel Q) \to +\infty$ or $-\infty$, according as $a \to -\infty$ or $c \to -\infty$ for $b < d$; or as $c \to -\infty$ or $a \to -\infty$ for $b > d$.
(ii) If $d = 1$, $c = 0$, then $I_4(P \parallel Q) \to (1 - 2^{1-b})^{-1}$ as $a \to \infty$.
(iii) If $d = 1$, $c = 0$, then $I_4(P \parallel Q) \to 1$ as $b \to \infty$.
It is to be noted that when $d = 1$, $c = 0$, then

$$I_4(Q \parallel P) = (2^{1-b} - 1)^{-1} \left[\sum_{i=1}^n p_i^a q_i^b - 1\right]. \tag{3.11}$$

3.3.3 Information deviations

If $d = 1, c = 0, a + b = 1,$ then we introduce the quantities

$$D(Q \| P \| Q) = \underset{b \to 1}{Lt}\, I_4(P \| Q) = H(Q) - H(Q \| P) \tag{3.12}$$

and

$$D(P \| Q \| P) = \underset{b \to 1}{Lt}\, I_4(Q \| P) = H(P) - H(P \| Q) \tag{3.13}$$

as the information deviation of Q from P and of P from Q respectively, where

$$H(P) = \sum_{k=1}^n p_k \log_2 \frac{1}{p_k}, \quad H(Q) = \sum_{k=1}^n q_k \log_2 \frac{1}{q_k}$$

are Shannon's [10] entropies and

$$H(Q\| P) = \sum_{k=1}^{n} q_k \log_2 \frac{1}{p_k}, \quad H(P\| Q) = \sum_{k=1}^{n} p_k \log_2 \frac{1}{q_k}$$

are Kerridge's [7] inaccuracies. Thus

$$D(Q\| P \|Q) = \sum_{k=1}^{n} q_k \log_2 \frac{p_k}{q_k}, \quad D(P\| Q \|P) = \sum_{k=1}^{n} p_k \log_2 \frac{q_k}{p_k} \tag{3.14}$$

3.3.4 Kullback's information and its generalizations

If we take the boundary conditions 3.3.1(ii), then

$$I_4(P \parallel Q) = \frac{1}{2} I_4^*(P \| Q), \tag{3.15}$$

where

$$I_4^*(P \parallel Q) = (2^{-b} - 2^{-d})^{-1} \left[\sum_{i=1}^{n} p_i^a q_i^b - \sum_{i=1}^{n} p_i^c q_i^d \right], \quad b \neq d. \tag{3.16}$$

Now if $d = 0,\ c = 1,\ a + b = 1$, then

$$\underset{b\to 0}{Lt}\, I_4(P \| Q) = \tfrac{1}{2}\, I(P \| P \| Q), \quad \underset{b\to 0}{Lt}\, I_4(Q \| P) = \tfrac{1}{2}\, I(Q \| Q \| P), \tag{3.17}$$

where

$$D(P\| P \|Q) = \sum_{k=1}^{n} p_k \log_2 \frac{p_k}{q_k} = H(P \| Q) - H(P) \tag{3.18}$$

and

$$D(Q\| Q \|P) = \sum_{k=1}^{n} q_k \log_2 \frac{q_k}{p_k} = H(Q \| P) - H(Q) \tag{3.19}$$

represents Kullback's [8] informations.

Information deviations and Kullback's informations are equal and opposite measures. The fact follows from

$$D(Q \parallel P \| Q) + I(Q \| Q \| P) \ = 0, D(P \parallel Q \| P) + I(P \| P \| Q) \ = 0 \tag{3.20}$$

It may be noted that information deviations and Kullback's informations become zero, if $p_k = q_k$ for $k = 1, 2, ..., n$.

3.3.5 Generalized Boundary Conditions

We shall now show that so far as our generalized inaccuracies (3.9) and (3.10) are concerned, there exist certain boundary conditions for which certain limiting functions of (3.9) and (3.10) may be taken as the generalized forms of Kullback's informations. For this, we generalized the boundary conditions in the following ways and get the results:
(i) Let $u = 1$, $v = \frac{1}{2}$, $F_4\left(1, \frac{1}{2}\right) = \frac{1}{2^m}$,
where m is real number ≥ 0. Then, we have for $d = 0,\ c = 1,\ a + b = 1$,

$$I^{(1)}(P, Q, m) = \underset{b\to 0}{Lt}\, I_4(P \| Q) = 2^{-m} \sum_{k=1}^{n} p_k \log_2 \frac{p_k}{q_k}, \quad m \geq 0 \tag{3.21}$$

to be called the first generalized Kullback's information. For $m = 0$ in (3.21), we get Kullback's information. The information (3.21) decreases as m increases.

(ii) let $u = 1$, $v = \frac{1}{2}$, $F_4\left(1, \frac{1}{2}\right) = \frac{1}{2^m}$, where m is real number ≥ 0. Also let $d = 0,\ c = 1 + m,\ a + b = 1 + m$, then we have

$$I^{(2)}(P, Q, m) = \underset{b\to 0}{Lt}\, I_4(P \| Q) = 2^{-m} \sum_{k=1}^{n} p_k^{m+1} \log_2 \frac{p_k}{q_k}, \quad m \geq 0 \tag{3.22}$$

to be called the second generalized Kullback's information. It is observed that $I^2(P,Q,m) \le I^1(P,Q,m)$.

For $m = 0$ in (3.22), we get Kullback's information.

(iii) let $u = 1,\ v = \frac{1}{2},\ F_4\left(1, \frac{1}{2}\right) = 2^{-1/m}$, where m is any positive real number. Then the values $d = 0,\ c = 1 + 1/m,\ a + b = 1 + 1/m$, lead to the information

$$I^{(3)}(P,Q,m) = \underset{b\to 0}{Lt}\, I_4(P\,\|Q) \ = 2^{-1/m} \sum_{k=1}^{n} p_k^{1/m+1} \log_2 \frac{p_k}{q_k}\ , \tag{3.23}$$

which may be called the third generalized Kullback's information. In this case

$\underset{m\to 0}{Lt}\, I^{(3)}(P,Q,m) = 0$ and $\underset{m\to\infty}{Lt}\, I^{(3)}(P,Q,m) = I(P\,\|P\,\|Q)$.

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