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On efficient abstract method for the study of Cauchy problem for a second order differential equation set on a singular cylindrical domain

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Abstract
In this work, we present an abstract approach for the study of a mixed initial value problem set in cylindrical domain with a cusp base. Maximal L^p regularity results for the strict solution are established.

Keywords
Fractional powers of linear operators; analytic semigroup, Abstract differential equation, Cuspidal point.

AMS Subject Classification
34G10, 34K10, 12H2O, 44A45.

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1. Introduction

The regularity analysis for boundary value problems set on singular domains has attracted the attention of many researchers. The solvability of such problems was discussed by means of different methods such as the variational methods or the well known potential theory. For more information, we can refer the reader to [13], [14], [15] and the references cited therein.
In this work, we consider the cusp domain $\Pi \subset \mathbb{R}^3$ defined by

$$\Pi = \left\{ (x,y,z) \in \mathbb{R}^3 : 0 < z < 1,\ \left(\frac{x}{z^\alpha}, \frac{y}{z^\alpha} \right) \in D \right\},\ \alpha > 1, \tag{1.1}$$

where

$$D = \left\{ (x,y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 1 \right\}.$$

We want to study the problem

$$\partial_t^2 u + \Delta u = h, \quad \text{on }]0,1[\times \Pi, \tag{1.2}$$

associated to the following mixed boundary conditions

$$\begin{cases} \partial_t^2 u\big|_{\{0\}\times D} + \Delta_{(x,y)} u\big|_{\{1\}\times D} = u_0, \\ \partial_t^2 u\big|_{\{1\}\times D} - \Delta_{(x,y)} u\big|_{\{0\}\times D} = u_1, \\ u\big|_{]0,1[\times \partial D} = 0. \end{cases} \tag{1.3}$$

Here, Δ stands for the standard Laplacian in $\mathbb{R}^3$ and u_0, u_1 are given functions.
We assume that the second member h is taken in the Lebesgue spaces $L^p(0,1;L^p(\Pi))$, $\frac{3}{2} < p < \infty$. Note here that the boundary conditions (1.3) are a considered as restriction of the following ones

$$\begin{cases} \partial_t^2 u\big|_{\{0\}\times D} + \Delta u\big|_{\{1\}\times D} = u_0, \\ \partial_t^2 u\big|_{\{1\}\times D} + \Delta u\big|_{\{0\}\times D} = u_1, \\ u\big|_{]0,1[\times \partial D} = 0. \end{cases} \tag{1.4}$$

which can be viewed in some sense as a particular case of the Ventcel boundary conditions often encountered in various concrete applications, see [19]. This situation will be the subject

of a forthcoming study. In our situation, the boundary of the domain Π given by (1.1) is at the origin of several difficulties which make the use of the classical tools a complicated task. Then, we have opted for the use of another approach based essentially on the theory of abstract differential equations. The effectiveness of this method was manifested in several works, see for example [2], [3], [5], [7], [8], [18] and the references cited therein. Our purpose is to establish existence, uniqueness and maximal regularity of the strict solution for (1.2)-(1.3). For this reason, some suitable compatibility conditions on u_0 and u_1 will be imposed.
We will prove that the study of our problem can be reduced to the study of an abstract differential equation associated to some nonlocal boundary conditions involving a linear unbounded operator. The techniques of investigation used here are based on the use of analytic semigroup's approach combined with the sum's operators theory developed in [6].
This work is organized as follows, in the next section, we show that our problem can be transformed by a natural changes of variables into an abstract second order differential problem. Section 3, is devoted to the study of the abstract version of the transformed problem. Finally, in section 4, we go back to our first problem in the non regular cylindrical domain and we justify our main results.

2. The abstract setting of the problem

2.1 Change of variables

First, we use the following change of variables

$$\begin{array}{ll} T: &]0,1[\times\Pi\rightarrow]0,1[\times\Omega, \\ & (t,x,y,z)\mapsto(t,\xi,\eta,\lambda),\ \lambda>\lambda_0 \end{array} \tag{2.1}$$

where

$$\begin{cases} (t,\xi,\eta,\lambda)=\left(t,\frac{x}{z^{\alpha}},\frac{y}{z^{\alpha}},\frac{\theta}{z^{\beta}}\right) \\ \Omega=D\times]\lambda_0,+\infty[, \\ \lambda_0=\frac{1}{\alpha-1}>0, \\ \beta=\alpha-1=\frac{1}{\theta}. \end{cases}$$

Putting

$$v(t,\xi,\eta,\lambda)=u(t,x,y,z) \text{ and } g(t,\xi,\eta,\lambda)=h(t,x,y,z).$$

Then, equation (1.2) is transformed into a new one given by

$$\partial_t^2 v-\left(\frac{\theta}{\lambda}\right)^{-2\alpha/\beta}\Delta v+\left(\frac{\theta}{\lambda}\right)^{-2\alpha/\beta}Pv=g, \tag{2.2}$$

with

$$\begin{aligned} &Pv(t,\xi,\eta,\lambda)\\ =&\left\{1+\alpha^2\theta^2\left(\frac{\xi}{\lambda}\right)^2\right\}\partial_\xi^2 v+\left\{1+\alpha^2\theta^2\left(\frac{\eta}{\lambda}\right)^2\right\}\partial_\eta^2 v\\ &+2\alpha^2\theta^2\xi\eta\left(\frac{1}{\lambda}\right)^2\partial_{\xi\eta}^2 v+2\alpha\theta\left(\frac{\xi}{\lambda}\right)\partial_{\xi\lambda}^2 v\\ &+2\alpha\theta\left(\frac{\xi_2}{\lambda}\right)\partial_{\eta\lambda}^2 v+\alpha(\alpha+1)\theta^2\xi\left(\frac{1}{\lambda}\right)^2\partial_\xi v\\ &+\alpha(\alpha+1)\theta^2\eta\left(\frac{1}{\lambda}\right)^2\partial_\eta v\\ &+\alpha\theta\left(\frac{1}{\lambda}\right)\partial_\lambda v. \end{aligned} \tag{2.3}$$

Now, let us introduce the following change of functions

$$\begin{cases} v=\left(\frac{\theta}{\lambda}\right)^s w, \\ \text{and} \\ f=((\alpha-1)\lambda)^{\frac{-3\alpha}{p(\alpha-1)}}g, \end{cases}$$

with

$$s=-\frac{\alpha}{\alpha-1}\left(\frac{3}{p}-2\right)=-\frac{\alpha}{\beta}\left(\frac{3}{p}-2\right).$$

Consequently, equation (2.2) becomes

$$k(\lambda)\partial_t^2 w-\Delta w+\frac{1}{\lambda}\mathscr{L}w=f, \tag{2.4}$$

with

$$k(\lambda)=\left(\frac{\lambda}{\theta}\right)^s,$$

and

$$\begin{aligned} &\mathscr{L}w(t,\xi,\eta,\lambda)\\ =&\frac{(\alpha\theta)^2}{\lambda}\left\{\xi^2\partial_\xi^2 w+\eta^2\partial_\eta^2 w+2\xi\eta\partial_{\xi\eta}^2 w\right\}\\ &+2\alpha\theta\left\{\xi\partial_{\xi\lambda}^2 w+\xi_2\partial_{\eta\lambda}^2 w\right\}\\ &+(\alpha\theta-2s)\partial_\lambda w\\ &+\frac{\alpha\theta}{\lambda}((\alpha+1)\theta-2s)\left\{\xi\partial_\xi w+\eta\partial_\eta w\right\}\\ &+\frac{s}{\lambda}(s+1-\alpha\theta)w. \end{aligned}$$

Concerning the boundary conditions, it is easy to see that (1.3) are transformed to

$$\begin{array}{ll} k(\lambda)\partial_t^2 w(1,\xi,\eta,\lambda)-\Delta_{(\xi,\eta)}w(0,\xi,\eta,\lambda)=k(\lambda)w_0, & \\ k(\lambda)\partial_t^2 w(0,\xi,\eta,\lambda)+\Delta_{(\xi,\eta)}w(0,\xi,\eta,\lambda)=k(\lambda)w_1, & \\ w(t,\xi,\eta,\lambda)=0, & (t,\xi,\eta,\lambda)\in]0,1[\times\partial\Omega, \end{array}$$

for all $(\xi,\eta,\lambda)\in\Omega$.

2.2 Statement of the abstract problem

Without loss of generality, we consider the following simplified problem

$$\partial_t^2 w - \Delta w = f, \tag{2.5}$$

under the corresponding conditions

$$\begin{array}{ll} \partial_t^2 w(1,\xi,\eta,\lambda) - \Delta_{(\xi,\eta)} w(0,\xi,\eta,\lambda) = w_0, & (\xi,\eta,\lambda) \in \Omega, \\ \partial_t^2 w(0,\xi,\eta,\lambda) + \Delta_{(\xi,\eta)} w(0,\xi,\eta,\lambda) = w_1, & (\xi,\eta,\lambda) \in \Omega, \\ w(t,\xi,\eta,\lambda) = 0, \quad (t,\xi,\eta,\lambda) \in]0,1[\times \partial\Omega. & \end{array} \tag{2.6}$$

Now, set $E = L^p(\Omega)$ and $X = L^p(0,1;E)$ with $\frac{3}{2} < p < \infty$ and consider the following vector-valued functions

$$\begin{array}{lll} w & : &]0,1[\to E \; ; \; t \longrightarrow w(t) \; ; \quad w(t)(\xi,\eta,\lambda) = w(t,\xi,\eta,\lambda), \\ f & : &]0,1[\to E \; ; \; t \longrightarrow f(t) \; ; \quad f(t)(\xi,\eta,\lambda) = f(t,\xi,\eta,\lambda). \end{array}$$

Set

$$\left\{ \begin{array}{l} D(M) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \\ M\psi = \Delta\psi, \; \psi \in D(M). \end{array} \right.$$

$$\left\{ \begin{array}{l} D(N) = W^{2,p}(D) \cap W_0^{1,p}(D), \\ N\psi = \Delta_{(\xi,\eta)}\psi, \; \psi \in D(N). \end{array} \right.$$

and define the following operators.

$$\left\{ \begin{array}{l} D(A) = \{\psi \in X \; : \psi(t) \in D(M), \text{ a.e } t \in]0,1[\}, \\ (A\psi)(t) = M(\psi(t)), \quad t \in]0,1[, \end{array} \right. \tag{2.7}$$

$$\left\{ \begin{array}{l} D(H) = \{\psi \in X \; : \psi(t) \in D(N), \text{ a.e } t \in]0,1[\}, \\ (H\psi)(t) = N(\psi(t)), \quad t \in]0,1[. \end{array} \right. \tag{2.8}$$

Consequently, the abstract version of (2.5)-(2.6) is given by

$$w''(t) + Aw(t) = f(t), \; t \in]0,1[, \tag{2.9}$$

$$w''(1) - Hw(0) = w_1, \; w''(0) + Hw(1) = w_0, \tag{2.10}$$

where w_0, w_1 are given elements of the complex Banach space E. Our purpose is to find a strict solution of Problem (2.9)-(2.10), i.e. a function w such that

$$\left\{ \begin{array}{l} w \in W^{2,p}(0,1;E) \cap L^p(0,1;D(A)), \\ w_0, w_1 \in D(H), \\ w \text{ satisfies (2.10)}. \end{array} \right.$$

In the sequel, we need the following results describing some spectral properties of the above cited operators.

Definition 2.1. *Let E be a complex Banach space and B be a closed linear operator in E, denoting by $\rho(B)$ its resolvent set. Then, B is said to be sectorial if there are constants $\mu \in \mathbb{R}$, $\delta \in (\pi/2, \pi)$, $M > 0$ such that*

$$\left\{ \begin{array}{l} \rho(B) \supset S_{\delta,\mu} = \{z \in \mathbb{C}, z \neq \mu : \arg(z-\mu) < \delta\}, \\ \left\|(B - zI)^{-1}\right\|_{L(E)} \leqslant \dfrac{C}{|z-\mu|}, \quad z \in S_{\delta,\mu}. \end{array} \right. \tag{2.11}$$

Lemma 2.2. *The densely closed linear operator $(A, D(A))$ defined by (2.7) is a sectorial operator with $\mu = 0$ and $\delta = \frac{\pi}{2}$.*

Proof. See section 3.1.1 in [16], in where a complete study of more general elliptic operator was discussed. The density of $D(A)$ follows from Proposition 2.1.1 in [11]. □

Similarly, one has

Lemma 2.3. *The densely closed linear operator $(H, D(H))$ defined by (2.8) is a sectorial operator with $\mu = 0$ and $\delta = \frac{\pi}{2}$.*

Remark 2.4. *In our situation, observe that*

1. *it is easy to see that*
$$AH = HA,$$
2. *the estimate (2.11) implies that*
$$Q = -\sqrt{-A},$$
is well defined. Furthermore, there exists a sector
$$\Pi_{\delta,r_0} = \{z \in \mathbb{C}^* : |\arg z| \leq \delta + \pi/2\} \cup \overline{B(0,r_0)},$$
(with some positive δ, r_0) and $C > 0$ such that
$$\left\{ \begin{array}{l} \rho(Q) \supset \Pi_{\delta,r_0} \\ \text{and} \\ \forall z \in \Pi_{\delta,r_0}, \; \left\|(Q - zI)^{-1}\right\| \leq \dfrac{C}{|z|}. \end{array} \right.$$
Thus, one has for all $t \in]0,1[$ and $\varphi \in E$,
$$e^{tQ}\varphi = \frac{1}{2i\pi} \int_{\gamma_1} e^{tz}(Q - zI)^{-1}\varphi dz,$$
where γ_1 is a suitable sectorial curve in the complex plane.

From Remark 1 in [4], it follows that

Lemma 2.5. *For any $\varphi \in E$, $k \in \mathbb{N}^*$ and $t \in]0,1[$, one has*

$$\left\{ \begin{array}{l} e^Q\varphi \in D\left(Q^k\right), \\ e^Q\varphi \in L^p(0,1;E), \end{array} \right.$$

and

$$\left\{ \begin{array}{l} Q^k e^{tQ} e^Q \varphi \in L^p(0,1;E), \\ Q^k e^{tQ} e^Q \varphi = e^{tQ} Q^k e^Q \varphi, \\ He^{tQ}\varphi = He^{tQ}\varphi. \end{array} \right.$$

3. Regularity results for the transformed problem

3.1 Representation of the solution

Using the Krein's method, we know that the unique solution w of (2.9)-(2.10) is given by

$$\begin{aligned} & w(t) \\ = \; & b_1 e^{tQ} + b_2 e^{(1-t)Q} + I(t) + J(t), \end{aligned}$$

where

$$\begin{aligned} I(t) &= \frac{Q^{-1}}{2}\int_0^t e^{Q(t-s)} f(s) ds, \\ J(t) &= \frac{Q^{-1}}{2}\int_t^1 e^{Q(s-t)} f(s) ds. \end{aligned}$$

The constant b_1 and b_2 are uniquely determined via the boundary conditions (2.10). For more informations about this technique, we refer the reader to [1] and the references therein. Using the commutativity of the two operators, namely $(A, D(A))$, $(H, D(H))$, we obtain the following abstract system

$$\begin{cases} \left(Q^2 e^Q - H\right) b_1 + \left(Q^2 - H e^Q\right) b_2 \\ = w_1 - Q^2 I(1) + HJ(0), \\ \\ \left(Q^2 + H e^Q\right) b_1 + \left(Q^2 e^Q + H\right) b_2 \\ = w_0 - Q^2 J(0) - HI(1). \end{cases} \tag{3.1}$$

The abstract determinant of (3.1) is given by

$$\Lambda = -\left(1 - e^{2Q}\right)\left(Q^4 + H^2\right),$$

and one has

Lemma 3.1. *The operator*

$$\begin{aligned} & \Lambda \\ = \; & -\left(1 - e^{2Q}\right)\left(Q^4 + H^2\right), \end{aligned}$$

is closed and boundedly invertible. Furthermore

$$\Lambda^{-1} \tag{3.2}$$
$$= -\left(Q^4 + H^2\right)^{-1}\left(1 - e^{2Q}\right)^{-1}, \tag{3.3}$$

Proof. Thanks to Proposition 2.3.6 in [16], we know that the operator $\left(1 - e^{2Q}\right)$ has a bounded inverse

$$\begin{aligned} & \left(1 - e^{2Q}\right)^{-1} \\ = \; & \frac{1}{2i\pi}\int_{\gamma_1} \frac{g(z) - 1}{g(z)} (Q - zI)^{-1} dz + I, \end{aligned}$$

where

$$g(z) = 1 - e^{2z},$$

and γ_1 is a suitable curve in the complex plane.
On the other hand, The Dore-Venni sum's theory allows us to confirm that

$$Q^4 + H^2,$$

is closed and $\left(Q^4 + H^2\right)^{-1} \in L(E)$. *Furthermore, one has*

$$\begin{aligned} & \left(Q^4 + H^2\right)^{-1} \\ = \; & -\frac{1}{2i\pi}\int_{\gamma_2} \frac{(AH)^{2z} H^{-2}}{\sin \pi z} dz, \end{aligned}$$

γ_2 is also a suitable curve in the complex plane, see Theorem 2.1 in [6].
Using the Dunford's operational calculus, we write

$$\begin{aligned} & \left(Q^4 + H^2\right)^{-1}\left(1 - e^{2Q}\right)^{-1} \\ = \; & \left(\frac{1}{2i\pi}\right)^2 \int_{\gamma_1 \times \gamma_2} \frac{(AH)^{2z} H^{-2}}{\sin \pi z} \frac{g(\mu) - 1}{g(\mu)} (Q - \mu I)^{-1} dz d\mu \\ & -\frac{1}{2i\pi}\int_{\Gamma_2} \frac{(AH)^{2z} H^{-2}}{\sin \pi z} dz \end{aligned}$$

At this level, Fubini's theorem allows us to write

$$\begin{aligned} & \left(Q^4 + H^2\right)^{-1}\left(1 - e^{2Q}\right)^{-1} = \\ & \left(-\frac{1}{2i\pi}\int_{\gamma_1} \frac{g(\mu) - 1}{g(\mu)} (Q - \mu I)^{-1} d\mu\right) \\ & (\times)\left(-\frac{1}{2i\pi}\int_{\gamma_2} \frac{(AH)^{2z} H^{-2}}{\sin \pi z} dz\right) \\ & -\frac{1}{2i\pi}\int_{\gamma_2} \frac{(AH)^{2z} H^{-2}}{\sin \pi z} dz \\ = \; & \left(1 - e^{2Q}\right)^{-1}\left(Q^4 + H^2\right)^{-1}. \end{aligned}$$

□

Summing up, we deduce that the formal solution of (2.9)-(2.10) is given by the formula

$$\begin{aligned} & w(t) \\ = \; & e^{tQ}\Lambda^{-1}\left[Hw_1 - Q^2 w_0\right] \\ & + e^{tQ} e^{Q}\Lambda^{-1}\left[Hw_0 + Q^2 w_1\right] \\ & + e^{(1-t)Q}\Lambda^{-1}\left[Hw_0 + Q^2 w_1\right] \\ & - e^{(1-t)Q} e^{Q}\Lambda^{-1}\left[Hw_1 - Q^2 w_0\right] \\ & + e^{tQ}\left(1 - e^{2Q}\right)^{-1} J(0) \\ & + e^{tQ} e^{Q}\left[2\Lambda^{-1} Q^4 I(1) + \left(1 - e^{2Q}\right)^{-1} I(1)\right] \\ & + e^{(1-t)Q}\left(1 - e^{2Q}\right)^{-1}\left[I(1) - J(0)\right] \\ & + I(t) + J(t). \end{aligned} \tag{3.4}$$

Putting

$$\begin{cases} d_0 = \Lambda^{-1}\left[Hw_1 - Q^2 w_0\right], \\ d_1 = \Lambda^{-1}\left[Hw_0 + Q^2 w_1\right], \\ \bar{d_0} = \left(1-e^{2Q}\right)^{-1} J(0), \\ \bar{d_1} = 2\Lambda^{-1}Q^4 I(1) + \; TI(1), \\ \bar{d_2} = \left(1-e^{2Q}\right)^{-1}\left[I(1) - J(0)\right]. \end{cases} \tag{3.5}$$

Then, w is given by the following compact expression

$$w(t) = G_1(t) + G_2(t) + G_3(t) + G_4(t),$$

with

$$\begin{cases} G_1(t) = e^{tQ}d_0 + e^{(1-t)Q}d_1 + e^{tQ}\bar{d_0}, \\ G_2(t) = e^{tQ}e^{Q}d_1 - e^{(1-t)Q}e^{Q}d_0, \\ G_3(t) = e^{tQ}\bar{d_0} + e^{tQ}e^{Q}\bar{d_1} + e^{(1-t)Q}\bar{d_2}, \\ G_4(t) = I(t) + J(t). \end{cases}$$

3.2 Study of regularity of the solution

From [6]), one has

Lemma 3.2. . *Let $f \in L^p(0,1;E)$, $\frac{3}{2} < p < \infty$. Then, the following applications*

$$t \to Q\int_0^t e^{(t-s)Q} f(s)ds,$$
$$t \to Q\int_t^1 e^{(s-t)Q} f(s)ds,$$

are well defined for a.e. $t \in]0,1[$ and belong to $L^p(0,1;E)$.

To establish more regularity results, we need to introduce for any closed operator B and $\delta \in]0,1[$, the following classical real interpolation spaces between $D(B)$ and E defined by

$$(E, D(B))_{\delta,p} = (D(B),E)_{1-\delta,p}$$

For more details about these spaces, see [9]. In our situation, since Q generates an analytic semigroup, it follows from [17] that beginlemma For all $\frac{3}{2} < p < \infty$ and $\varphi \in E$, one has

$$\begin{cases} Qe^{\cdot Q}\varphi \\ \in L^p(0,1;E) \Leftrightarrow \varphi \in (D(A),E)_{\frac{1}{2p}+\frac{1}{2},p}, \\ \text{and} \\ Ae^{\cdot Q}\varphi \\ \in L^p(0,1;E) \Leftrightarrow \varphi \in (D(A),E)_{\frac{1}{2p},p}. \end{cases}$$

Keeping in mind that $I(0)$ and $J(1)$ are bounded and due to the regularizing effect of e^Q, Λ^{-1} and T, the following regularity results are a direct consequence of Lemma 3.2.

Proposition 3.3. *Let d_0, d_1, $\bar{d_0}$, $\bar{d_1}$, $\bar{d_2}$ given by (3.5). Then, for $\frac{3}{2} < p < \infty$, one has*

1. *$t \to Q^2G_1(t) \in L^p(0,1;E)$ if and only if d_0, $d_1 \in \left(D(Q^2),E\right)_{\frac{1}{2p},p}$.*
2. *$t \to Q^2G_2(t) \in L^p(0,1;E)$.*
3. *$t \to Q^2G_3(t) \in L^p(0,1;E)$.*
4. *$t \to Q^2G_4(t) \in L^p(0,1;E)$.*

The above proposition can be equivalently formulated as follows

Theorem 3.4. *Let $f \in L^p(0,1;E)$, $\frac{3}{2} < p < \infty$. and w_0, $w_1 \in E$ Then, the following assertions are equivalents*

1. *$Hw_1, Aw_0, Hw_0, Aw_1 \in (D(A),E)_{\frac{1}{2p},p}$.*
2. *w given by (3.4) is the unique strict solution of (2.9)-(2.10) satisfying the maximal regularity property, that is*

$$w'' \text{ and } Aw \in L^p(0,1;E).$$

In order to state our main results, we recall the definition of the following Besov space for $\upsilon \in (0,1)$

$$\mathscr{B}^{\upsilon}_{p,p}(\Omega) := \left\{\psi \in L^p(\Omega) : \int_\Omega\int_\Omega \frac{|\psi(\xi_1)-\psi(\xi_2)|^p}{\|\xi_1-\xi_2\|^{1+p\upsilon}} d\xi_1\, d\xi_2 < \infty\right\}$$

where ξ_1 and ξ_2 are a two generic points of $\mathbb{R}^3$, see [9], p. 680, and one has

Lemma 3.5. *Let A the operator defined by (2.7) and $\frac{3}{2} < p < \infty$. Then*

$$\begin{aligned} &(D(A),E)_{\frac{1}{2p}+\frac{1}{2},p} \\ = &\left\{\psi \in L^p\left(0,1;\mathscr{B}^{\upsilon}_p(\Omega)\right) : \ \psi = 0 \text{ on } \partial\Omega\right\} \\ := &\, L^p\left(0,1;\mathring{\mathscr{B}}^{\upsilon}_p(\Omega)\right). \end{aligned}$$

Proof. One has

$$(D(A),E)_{\frac{1}{2p}+\frac{1}{2},p} = L^p\left(0,1;\left(W^{2,p}_0(\Omega)\cap L^p(\Omega)\right)_{\frac{1}{2p}+\frac{1}{2},p}\right)$$

First, it is well known that

$$\begin{aligned} L^p\left(0,1;W^{2,p}_0(\Omega)\right) &\subset L^p\left(0,1;\left(W^{2,p}_0(\Omega)\cap L^p(\Omega)\right)_{\frac{1}{2p}+\frac{1}{2},p}\right) \\ &\subset L^p\left(0,1;L^p(\Omega)\right). \end{aligned}$$

From [9], Proposition 3, P. 683, one has

$$\left(W^{2,p}(\Omega)\cap L^p(\Omega)\right)_{\frac{1}{2p}+\frac{1}{2},p} = \mathscr{B}^{1-\frac{1}{p}}_p(\Omega)$$

and as in p. 708 in the same work cited above, we get

$$\begin{aligned}&\left(W_0^{2,p}(\Omega)\cap L^p(\Omega)\right)_{\frac{1}{2p}+\frac{1}{2},p}\\ =\;&\left\{\psi\in\mathscr{B}_p^{1-\frac{1}{p}}(\Omega):\psi=0 \text{ on } \partial\Omega\right\}\\ :\;&=\mathring{\mathscr{B}}_p^{1-\frac{1}{p}}(\Omega)\end{aligned}$$

□

Then, we can formulate our main results in the transformed domain as follows.

Theorem 3.6. *Let $f\in L^p((0,1)\times\Omega)$, $\frac{3}{2}<p<\infty$. Then, the following assertions are equivalents*

1. $\Delta_{(\xi,\eta)}w_1$, Δw_0, $\Delta_{(\xi,\eta)}w_0$, $\Delta w_1\in L^p\left(0,1;\mathring{\mathscr{B}}_p^{1-\frac{1}{p}}(\Omega)\right)$.
2. *Problem (2.5)-(2.6) has a unique strict solution*
$$w\in W^{2,p}(0,1;L^p(\Omega))\cap L^p(0,1;W_0^{2,p}(\Omega)),$$
satisfying the maximal regularity property, that is
$$\partial_t^2 w \text{ and } \Delta w\in L^P(0,1;L^p(\Omega)).$$

Using identically a classical perturbation argument as in [10], we conclude that

Theorem 3.7. *Let $f\in L^p((0,1)\times\Omega)$, $\frac{3}{2}<p<\infty$. Then the following assertions are equivalents.*

1. $\Delta_{(\xi,\eta)}w_1,\Delta w_0,\Delta_{(\xi,\eta)}w_0,\Delta w_1\in L^p\left(0,1;\mathring{\mathscr{B}}_p^{1-\frac{1}{p}}(\Pi)\right)$.
2. *Problem (2.4)-(2.6) has a unique strict solution*
$$\begin{aligned}&w\\ \in\;&W^{2,p}(0,1;L^p(\Omega))\cap L^p(0,1;W_0^{2,p}(\Omega)),\end{aligned}$$
satisfying the maximal regularity property, that is
$$\partial_t^2 w \text{ and } \Delta w\in L^p(0,1;L^p(\Omega)).$$

4. Regularity results for problem (1.2)-(1.3)

From [2], the go back to our original domain use the following inverse change of variables

$$\begin{aligned}T^{-1}:\;&]0,1[\times\Omega\to]0,1[\times\Pi,\\ &(t,\xi,\eta,\lambda)\mapsto\left(t,\left(\tfrac{\theta}{\lambda}\right)^{\alpha/\beta}\xi,\left(\tfrac{\theta}{\lambda}\right)^{\alpha/\beta}\eta,\left(\tfrac{\theta}{\lambda}\right)^{1/\beta}\right).\end{aligned}\tag{4.1}$$

Since

$$\begin{aligned}w&=\left(\frac{\theta}{\lambda}\right)^{3\alpha/p\beta}z^{-2\alpha}u\\ &=z^{3\alpha/p}z^{-2\alpha}u,\end{aligned}$$

then, a direct computation leads to

$$\begin{cases}\partial_\xi w=\left(\frac{\theta}{\lambda}\right)^{3\alpha/p\beta}z^{-\alpha}\partial_x u=z^{3\alpha/p}z^{-\alpha}\partial_x u,\\ \partial_\eta w=\left(\frac{\theta}{\lambda}\right)^{3\alpha/p\beta}z^{-\alpha}\partial_y u=z^{3\alpha/p}z^{-\alpha}\partial_y u,\end{cases}$$

and

$$\begin{cases}\partial_\xi^2 w=\left(\frac{\theta}{\lambda}\right)^{3\alpha/p\beta}\partial_x^2 u=z^{3\alpha/p}\partial_x^2 u,\\ \partial_\eta^2 w=\left(\frac{\theta}{\lambda}\right)^{3\alpha/p\beta}\partial_y^2 u=z^{3\alpha/p}\partial_y^2 u,\\ \partial_{\xi\eta}^2 w=\left(\frac{\theta}{\lambda}\right)^{3\alpha/p\beta}\partial_{xy}^2 u=z^{3\alpha/p}\partial_{xy}^2 u.\end{cases}$$

As an immediate consequence of the preceding maximal regularity results with respect to (t,ξ,η,λ), we deduce then that

$$\begin{aligned}&z^{-2\alpha}u,z^{-\alpha}\partial_x u,z^{-\alpha}\partial_y u,\partial_{xy}^2 u,\partial_x^2 u,\partial_y^2 u\\ \in\;&L^p(0,1;L^p(\Pi)),\ \frac{3}{2}<p<\infty\end{aligned}$$

On the other hand, since $\partial_\lambda^2 w=\Delta w-\Delta_{(\xi,\eta)}w$, then

$$\partial_z^2 u\in L^p(0,1;L^p(\Pi)).$$

Regarding now the cross derivatives $\partial_{\xi\lambda}^2 w$ and $\partial_{\eta\lambda}^2 w$, one has

$$\partial_{\xi\lambda}^2 w=\begin{cases}\left(\frac{\theta}{\lambda}\right)^{3\alpha/p\beta}\left\{\alpha\left(2\alpha-\dfrac{3\alpha}{p}\right)\frac{\theta}{\lambda}z^{-\alpha}\partial_x u\right.\\ \left.-\alpha\frac{\theta}{\lambda}\left(\partial_x^2 u+\partial_{xy}^2 u\right)-\partial_{xz}^2 u\right\},\\ \text{and}\\ \left(\frac{\theta}{\lambda}\right)^{3\alpha/p\beta}\left\{\alpha\left(2\alpha-\frac{3\alpha}{p}\right)\frac{\theta}{\lambda}z^{-\alpha}\partial_y u\right.\\ \left.-\alpha\frac{\theta}{\lambda}\left(\partial_y^2 u+\partial_{xy}^2 u\right)-\partial_{xz}^2 u\right\},\end{cases}$$

from which, we deduce that

$$\partial_{xz}u,\partial_{yz}u\in L^p(0,1;L^p(\Pi)),\ \frac{3}{2}<p<\infty.$$

Finally, one has

$$\begin{aligned}\partial_\lambda w\;=\;&\left(\frac{\theta}{\lambda}\right)^{3\alpha/p\beta}\left\{\left(2\alpha-\frac{3\alpha}{p}\right)\frac{\theta}{\lambda}z^{-2\alpha}u\right\}\\ &-\left(\frac{\theta}{\lambda}\right)^{3\alpha/p\beta}\left\{\alpha\frac{\theta}{\lambda}z^{-\alpha}(\partial_x u+\partial_y u)-z^{-\alpha}\partial_z u\right\},\end{aligned}$$

it follows then that

$$z^{-\alpha}\partial_z u\in L^p(0,1;L^p(\Pi)),\ \frac{3}{2}<p<\infty$$

Summing up, one has

$$\begin{aligned}&z^{-2\alpha}u,z^{-\alpha}\partial_x u,z^{-\alpha}\partial_y u,z^{-\alpha}\partial_z u,\\ &\partial_{xy}^2 u,\partial_{xz}u,\partial_{yz}u,\partial_x^2 u,\partial_y^2 u,\partial_z^2 u\\ \in\;&L^p(0,1;L^p(\Pi)).\end{aligned}$$

Similarly, the concrete version of the compatibility conditions on u_0 and u_1 are given by

$$\begin{cases} z^{-2\alpha}u_0, z^{-\alpha}\partial_x u_0, z^{-\alpha}\partial_y u_0, z^{-\alpha}\partial_z u_0, z\partial_{xy}^2 u_0, \partial_x^2 u_0, \partial_y^2 u_0 \\ \in \mathring{\mathscr{B}}_{p,\#}^{2-\frac{1}{p},p}(\Pi), \\ \text{and} \\ z^{-2\alpha}u_1, z^{-\alpha}\partial_x u_1, z^{-\alpha}\partial_y u_1, z^{-\alpha}\partial_z u_1, \partial_{xy}^2 u_1, \partial_x^2 u_1, \partial_y^2 u_1 \\ \in \mathring{\mathscr{B}}_{p,\#}^{2-\frac{1}{p},p}(\Pi). \end{cases}$$

where

$$\mathring{\mathscr{B}}_{p,\#}^{2-\frac{1}{p},p}(\Pi) := \left\{ \psi : z^{3\frac{\alpha}{p}}\psi \in \mathring{\mathscr{B}}_{p}^{2-\frac{1}{p},p}(\Pi) \right\}.$$

Summing up, we are in position to state our main results

Theorem 4.1. *Let $h \in L^p(0,1;L^p(\Pi))$, $\frac{3}{2} < p < \infty$. Assume that*

$$\begin{cases} z^{-2\alpha}u_0, z^{-\alpha}\partial_x u_0, z^{-\alpha}\partial_y u_0, z^{-\alpha}\partial_z u_0, z\partial_{xy}^2 u_0, \partial_x^2 u_0, \partial_y^2 u_0 \\ \in \mathring{\mathscr{B}}_{p,\#}^{2-\frac{1}{p},p}(\Pi), \\ \textit{and} \\ z^{-2\alpha}u_1, z^{-\alpha}\partial_x u_1, z^{-\alpha}\partial_y u_1, z^{-\alpha}\partial_z u_1, \partial_{xy}^2 u_1, \partial_x^2 u_1, \partial_y^2 u_1 \\ \in \mathring{\mathscr{B}}_{p,\#}^{2-\frac{1}{p},p}(\Pi). \end{cases}$$

Then problem (1.2)-(1.3) has unique strict solution

$$u \in W^{2,p}(0,1;L^p(\Pi)) \cap L^p(0,1;W_0^{2,p}(\Pi)),$$

such that

$$\begin{cases} \partial_t^2 u \\ \in L^p(0,1;L^p(\Pi)), \\ \textit{and} \\ z^{-2\alpha}u, z^{-\alpha}\partial_x u, z^{-\alpha}\partial_y u, z^{-\alpha}\partial_z u, \partial_{xy}^2 u, \partial_{xz} u, \partial_{yz} u, \partial_x^2 u, \partial_y^2 u, \partial_z^2 u \\ \in L^p(0,1;L^p(\Pi)). \end{cases}$$

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