Malaya Journal of Matematik, Vol. 6, No. 2, 338-343, 2018

https://doi.org/10.26637/MJM0602/0006

Eigenfunction expansion of the Sturm-Liouville equation with a non-local boundary condition

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Abstract

In this work we study some spectral properties, normalized eigenfunction, Green's function and expansion formula of a nonlocal boundary value problem of the Sturm-Liouville equation.

Keywords

Sturm-Liouville boundary value problem, nonlocal condition, normalized eigenfunction , expansion formula.

AMS Subject Classification

34A55, 34B10, 34B15, 34B18, 34L10, 34L40, 34K10.

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Article History: Received **24** July **2017**; Accepted **09** January**2018**

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1. Introduction

In differential equation theory Many interesting applications appear (see for example [1], [2] and [5], [6] and [10]-[12]). The eigenfunction expansion of non-local boundary value problems can be investigate through the method of Green function.

For the solution of problem (1.1) when $\rho(x) \neq 1$ under different conditions the spectral expansion formula was investigated with different methods in [7]-[9]. In present work we find the eigenfunction expansion formula and prove its convergence for following version of the Sturm-Liouville equation with a non-local boundary condition (1.1)-(1.2).

Consider the following Sturm-Liouville problem

$$-y'' + q(x)y = \lambda^2 y, \qquad x \in (0,\pi), \tag{1.1}$$

$$y(0) = 0, \quad y(\xi) = 0, \qquad \xi \in (0,\pi], \tag{1.2}$$

where the non-negative real function $q(x)$ has a second piecewise derivatives on $(0,\pi)$ and λ is spectral parameter.

In [3] the author proved that the eigenvalues $\lambda_n, n = 0,1,2,..$ of problems (1.1)-(1.2) are real and the corresponding eigenfunctions $\varphi(x,\lambda), \psi(x,\lambda)$ are orthogonal.

In present work we study the eigenfunction, expansion formula.

Let $\varphi(x,\lambda)$ be the solution of the differential equation, which satisfies the conditions

$$\varphi(0,\lambda) = 0 \qquad \varphi'(0,\lambda) = 1 \tag{1.3}$$

and then

$$-\varphi''(x,\lambda) + q(x)\varphi(x,\lambda) = \lambda^2\varphi(x,\lambda), \tag{1.4}$$

taking the complex conjugate we have

$$\overline{-\varphi''(x,\lambda)} + q(x)\overline{\varphi(x,\lambda)} = \lambda^2\overline{\varphi(x,\lambda)}. \tag{1.5}$$

By the aid of the uniqueness theorem, we have $\varphi(x,\lambda) = \overline{\varphi(x,\lambda)}$. In a similar way, we can see that $\psi(x,\lambda) = \overline{\psi(x,\lambda)}$ where $\psi(x,\lambda)$ is the solution of (1.1)-(1.2), way as [3] which is given by

$$\psi(x,\lambda) = \frac{\cos(\xi - x)}{\lambda} + \int_x^\xi \frac{\sin\lambda(x-\tau)}{\lambda} q(\tau)\psi(\tau,\lambda)d\tau, \tag{1.6}$$

where

$$\psi(\xi,\lambda) = 0, \quad \psi'(\xi,\lambda) = 1,$$

that is, the eigenfunctions of the problem (1.1)-(1.2) are real.
As we know from [3], the eigenvalues of problem (1.1)-(1.2) coincide with the roots of the function $\Psi(\lambda) = 0$, where $\Psi(\lambda)$ is the Wronskian of the two solutions $\varphi(x,\lambda), \psi(x,\lambda)$ of (1.1)-(1.2) and we have in [4]

$$\Psi(\lambda) = W[\varphi(x,\lambda), \psi(x,\lambda)] = 0, \tag{1.7}$$

so that $\psi(x,\lambda_n)$ is a constant multiple of $\varphi(x,\lambda_n)$, say

$$\psi(x,\lambda_n) = \beta_n\varphi(x,\lambda_n), \quad \beta_n \neq 0. \tag{1.8}$$

2. Some spectral properties

Definition 2.1. *For every n=1, 2, ... the numbers*

$$a_n = \int_0^\xi \varphi^2(x,\lambda_n)dx = \frac{\xi}{2} + O\left(\frac{1}{n^2}\right), \tag{2.1}$$

are called the normalization numbers of boundary value problem (1.1)-(1.2).

Lemma 2.2. *The eigenvalues of the non-local boundary value problem (1.1)-(1.2) are simple and give by*

$$\dot{\Psi}(\lambda_n) = 2\lambda_n\beta_n a_n, \tag{2.2}$$

where $\dot{\Psi}(\lambda_n) = \frac{d}{d\lambda}W(\lambda)$.

Proof. Since

$$-\varphi''(x,\lambda_n) + q(x)\varphi(x,\lambda_n) = \lambda_n^2\varphi(x,\lambda_n),$$
$$-\psi''(x,\lambda) + q(x)\psi(x,\lambda) = \lambda^2\psi(x,\lambda),$$

we get

$$\frac{d}{dx}W(\lambda) = (\lambda_n^2 - \lambda^2)\varphi(x,\lambda_n)\psi(x,\lambda_n).$$

With the help of (1.2) and using (2.1), (1.8), we get

$$\dot{\Psi}(\lambda) = (\lambda_n - \lambda)(\lambda_n + \lambda)\beta_n\left[\int_0^\xi \varphi^2(x,\lambda_n)dx\right],$$

for $\lambda \to \lambda_n$ we arrive at (2.2).

3. Green's function

We introduce the function $R(x,t,\lambda)$ by

$$R(x,t,\lambda) = -\frac{1}{\Psi}\begin{cases} \varphi(x,\lambda)\psi(t,\lambda), & t \leq x, \\ \varphi(t,\lambda)\psi(x,\lambda), & x \leq t. \end{cases} \tag{3.1}$$

which is called the Green's function of the nonhomogeneous problem

$$-y'' + q(x)y = \lambda^2 y + f(x), \quad 0 \leq x \leq \pi, \tag{3.2}$$

$$y(0) = 0, \quad y(\xi) = 0, \quad \xi \in (0,\pi].$$

Where $f(x) \in D(A)$. The function $R(x,t,\lambda)$ is also, called the kernel of the resolvent $R_\lambda = (A - \lambda^2 I)^{-1}$, where $A \equiv -(d^2/dx^2) + q(x), D(A) = \{y(x) : \exists y'', y(0) = y(\xi) = 0\}$. In the following lemmas, we prove some essential properties of $R(x,t,\lambda)$ which are useful in the forthcoming study of the eigenfunction expansion of the problem (1.1)-(1.2)

Lemma 3.1. *Let $f(x)$ be any function belonging to $L_2(0,\pi)$, then the function*

$$y(x,\lambda) = \int_0^\xi R(x,t,\lambda)f(t)dt \tag{3.3}$$

is the solution of problem (3.2).

proof. By applying the method of variation of parameters.We seek the solution of the nonhomogeneous problem (3.2) in the following

$$y(x,\lambda) = C_1\varphi(x,\lambda) + C_2\psi(x,\lambda), \tag{3.4}$$

and we get the coefficients $C_1(x,\lambda)$ and $C_2(x,\lambda)$ as

$$C_1(x,\lambda) = -\frac{1}{\Psi(\lambda)}\int_0^x \psi(t,\lambda)f(t)dt, \tag{3.5}$$

$$C_2(x,\lambda) = -\frac{1}{\Psi(\lambda)}\int_x^\xi \varphi(t,\lambda)f(t)dt.$$

Substituting (3.5) into (3.4) and keeping in mind (3.1), we get the required formula (3.3).
Now we show that (3.3) satisfies the non-local boundary condition (3.2). From (3.2) by using (1.1)-(1.2), we have

$$y(0) = -\frac{1}{\Psi}\int_0^\xi \varphi(0,\lambda)\psi(t,\lambda)f(t)dt = 0, \tag{3.6}$$

$$y(\xi) = -\frac{1}{\Psi}\int_0^\xi \varphi(t,\lambda)\psi(\xi,\lambda)f(t)dt = 0$$

The proof is completed.

Lemma 3.2. *Under the conditions of Lemma 2.3, the function $R(x,t,\lambda)$ satisfies the following formula*

$$\begin{aligned} Res_{\lambda=\lambda_n} y(x,\lambda) &= Res_{\lambda=\lambda_n}\int_0^\xi R(x,t,\lambda)f(t)dt \\ &= \frac{-1}{2\lambda_n a_n}\varphi(x,\lambda)\int_0^\xi \varphi(t,\lambda_n)f(t)dt. \end{aligned} \tag{3.7}$$

proof. With the help of (3.1) and (3.3), we get

$$\begin{aligned} Res_{\lambda=\lambda_n}\int_0^\xi R(x,t,\lambda)f(t)dt &= -\frac{1}{\Psi(\lambda)}\Big[\psi(x,\lambda_n) \\ \int_0^x \varphi(t,\lambda)dt &+ \varphi(x,\lambda_n)\int_0^\xi \psi(t,\lambda_n)f(t)dt\Big] \\ &= -\frac{1}{\Psi(\lambda)}\Big[\beta_n\varphi(x,\lambda_n)\int_0^\xi \varphi(t,\lambda_n)f(t)dt\Big] \end{aligned} \tag{3.8}$$

by using (2.2), we arrive (3.7).

Lemma 3.3. *Under the conditions of Lemma 3.7 in [3], the resolvent $R(x,t,\lambda)$ satisfies the following inequality:*

$$R(x,t,\lambda) = \begin{cases} O\left(\frac{e^{|\lambda|(t-x)}}{|\lambda|^2}\right), & 0\le x\le t\le\xi\le\pi, \\ O\left(\frac{e^{|\lambda|(x-t)}}{|\lambda|^2}\right), & 0\le x\le t\le\xi\le\pi. \end{cases} \tag{3.9}$$

proof. From [3], we have

$$\begin{aligned} \varphi(x,\lambda) &= O\left(\frac{e^{|\lambda|x}}{|\lambda|}\right),\ 0\le x\le\pi, \\ \psi(x,\lambda) &= O\left(\frac{e^{|\lambda|(\xi-x)}}{|\lambda|}\right),\ 0\le x\le\xi\le\pi. \end{aligned} \tag{3.10}$$

It can be easily seen that,

$$\frac{1}{\Psi(\lambda)} \le CO\left(|\lambda|e^{-|\lambda|\xi}\right),\ C = cont. \tag{3.11}$$

We have two possibilities, one of which for $x\le t$ and the other one for $t\le x$. by direct substitution from (3.10), (3.11) into the first branch of (3.1), we obtain

$$O\left(\frac{e^{|\lambda|(t-x)}}{|\lambda|^2}\right), \qquad 0\le x\le t\le\xi\le\pi. \tag{3.12}$$

In the case of $x\le t$, again by substituting (3.10), (3.11) into the first branch of (3.1), we obtain

$$O\left(\frac{e^{|\lambda|(x-t)}}{|\lambda|^2}\right), \qquad 0\le x\le t\le\xi\le\pi. \tag{3.13}$$

In the following lemma, we prove an integral formula which is satisfied by $R(x,t,\lambda)$ and help in proving the eigenfunction expansion formula

Lemma 3.4. *If the function $f(x)$ on $[0,\pi]$ has a second-order derivatives and satisfies the non-local condition $f(0) = f(\xi) = 0$, then the following integral formula is true*

$$\begin{aligned} \int_0^\xi R(x,t,\lambda)f(t)dt &= -\frac{f(x)}{\lambda^2} \\ &+ \int_0^\xi \frac{R(x,t,\lambda)}{\lambda^2}\Big[-f''(t)+q(t)f(t)\Big]dt \end{aligned} \tag{3.14}$$

where $R(x,t,\lambda)$ is the kernel of the resolvent of the non-homogeneous

proof. By the aid of lemma 3.1, we have

$$\begin{aligned} \int_0^\xi R(x,t,\lambda)f(t)dt = -\frac{1}{\Psi}\Big[\psi(x,\lambda)\int_0^\xi \varphi(t,\lambda)f(t)dt \\ +\varphi(x,\lambda)\int_0^\xi \psi(t,\lambda)f(t)dt\Big] \end{aligned} \tag{3.15}$$

where the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are the solutions of the homogenous (1.1)-(1.2), so that

$$\int_0^\xi R(x,t,\lambda)f(t)dt = -\frac{1}{\Psi}\left[\frac{\psi(x,\lambda)}{\lambda^2}\int_0^\xi [-\varphi''(t,\lambda)+q(t)\varphi(t,\lambda)]f(t)dt + \frac{\varphi(x,\lambda)}{\lambda^2}\int_0^\xi [-\psi''(t,\lambda)+q(t)\psi(t,\lambda)]f(t)dt\right], \tag{3.16}$$

from which we have

$$\int_0^\xi R(x,t,\lambda)f(t)dt = \frac{1}{\Psi}\left[\frac{\psi(x,\lambda)}{\lambda^2}\int_0^x \varphi''(t,\lambda) f(t)dt + \frac{\varphi(x,\lambda)}{\lambda^2}\int_x^\xi \psi''(t,\lambda)f(t)dt\right] + \frac{1}{\lambda^2}\int_0^\xi R(x,t,\lambda)q(t)dt \tag{3.17}$$

Integrating by parts twice (3.17) and then using the boundary conditions $f(0)=f(\xi)=\varphi(0,\lambda)=0$ and $f(0)=f(\xi)=\psi(\xi,\lambda)=0$, respectively, and keeping in mind (3), we get

$$\psi(x,\lambda)\int_0^x \varphi''(t,\lambda)f(t)dt + \varphi(x,\lambda)\int_x^\xi \psi''(t,\lambda)f(t)dt$$

$$= -\Psi(\lambda)f(x) - \Psi(\lambda)\int_0^\xi R(x,t,\lambda)f''(t)dt \tag{3.18}$$

Substituting from (3.18) into (3.17), we get the required result.

4. Expansion formula

Theorem 4.1. *The eigenfunctions $(varphi(x,\lambda_n)_{n\geq 0}$ of the nonlocal boundary value problem (1.1)-(1.2) is complete in $L_2(0,\pi)$.*

proof. Let $f(x)\in L_2(0,\pi)$ and assume

$$\int_0^\xi f(x)\varphi(x,\lambda_n)dx = 0, \quad n\geq 0.$$

Then from (3.7), we have $Res_{\lambda=\lambda_n} y(x,\lambda)=0$ and consequently, for fixed $x\in[0,\pi]$ the function $y(x,\lambda)$ is entire with respect to λ. Let us denote

$$G_\delta := \lambda : |\lambda-\lambda_n^0|\geq\delta, n=0,\pm1,\pm2,...$$

where δ is sufficiently small positive number from (3.11), we have

$$|\Psi(\lambda)|\geq C\frac{e^{|\lambda|\xi}}{|\lambda|},$$

for fixed $\delta>0$ and sufficiently large $\lambda^*>0$

$$|y(x,\lambda)|\leq|\lambda|C_\delta,\;\; \lambda\in G_\delta,\; |\lambda|\geq\lambda^*.$$

Using the maximum principle and liouville theorem we get $y(x,\lambda)\equiv 0$. From this we obtain $f(x)\equiv 0$ a.e. on $(0,\pi)$. Thus we conclude the completeness of the eigenfunctions $\varphi(x,\lambda_n)$ in $L_2(0,\pi)$.

Theorem 4.2. *Let $f(x)$ be a second-order integrable derivatives on $f\in[0,\pi]$ and satisfy the conditions $f(0)=f(\xi)=0$, then the following formula of eigenfunction expansion is true*

$$f(x)=\sum_{k=0}^{\infty} b_k\varphi(x,\lambda_k), \tag{4.1}$$

where $b_k=\frac{1}{2a_k}\int_0^\xi \varphi(t,\lambda_k)f(t)dt$ and the series uniformly converges to $f(x), x\in[0,\pi]$.

proof. We write (3.14) in the form

$$\int_0^\xi R(x,t,\lambda)f(t)dt = \frac{-f(x)}{\lambda^2}+r(x,\lambda) \tag{4.2}$$

where

$$r(x,\lambda)=\int_0^\xi \frac{R(x,t,\lambda)}{\lambda^2}\left[-f''(t)+q(t)f(t)\right]dt. \tag{4.3}$$

from the condition of the theorem imposed on $q(x)$, it can be easily shown that

$$|r(x,\lambda)|\leq\frac{M_0}{|\lambda^2|},\quad \lambda\in\Gamma_n \tag{4.4}$$

where M_o is constant which is independent of x,t,λ and the contour Γ_n, defined for sufficiently large n on the contours

$$\Gamma_n=\left\{\lambda:|\lambda|=|\lambda_n^0|+\frac{\iota}{2}\right\},\quad \inf_{n\neq m}|\lambda_n^0-\lambda_m^0|=\iota>0.$$

We multiply both sides of (4.2) by $\frac{1}{2\pi i}\lambda$ and integrating with respect to λ on the contour Γ_n

$$I_n = \frac{1}{2\pi i}\oint_{\Gamma_n}\lambda y(x,\lambda)d\lambda = \frac{-f(x)}{2\pi i}\oint_{\Gamma_n}\frac{d\lambda}{\lambda} + \frac{1}{2\pi i}\oint_{\Gamma_n}\lambda r(x,\lambda)d\lambda. \tag{4.5}$$

Among the poles of the function $R(x,t,\lambda)$, as a function of λ, lie only $\lambda_0, \lambda_1, ..., \lambda_n$ inside Γ_n. By using the residues formula and (3.7), we have

$$I_n = \frac{1}{2\pi i}\oint_{\Gamma_n}\lambda y(x,\lambda)d\lambda = \sum_{k=0}^{n} Res_{\lambda=\lambda_n}\left[\int_0^{\xi} R(x,t,\lambda)f(t)dt\right] = \sum_{k=0}^{n} b_k\varphi(x,\lambda_k). \tag{4.6}$$

Further

$$\frac{-f(x)}{2\pi i}\oint_{\Gamma_n}\frac{d\lambda}{\lambda} = -f(x) \tag{4.7}$$

By using (4.4), we have

$$\left|\frac{1}{2\pi i}\oint_{\Gamma_n}\lambda r(x,\lambda)d\lambda\right| \leq \frac{M_0}{2\pi}\oint_{\Gamma_n}\frac{d\lambda}{|\lambda|} \leq \frac{constant}{n} \tag{4.8}$$

By substitution from (4.7), (4.8) into (4.5), we get

$$\left|f(x) - \sum_{k=0}^{n} b_k\varphi(x,\lambda_k)\right| \leq \frac{constant}{n} \tag{4.9}$$

which completes the uniform convergence of the series $\sum_{k=0}^{\infty} b_k\varphi(x,\lambda_k)$ to $f(x)$, $x \in [0,\pi]$. That is

$$f(x) = \sum_{k=0}^{\infty} b_k\varphi(x,\lambda_k). \tag{4.10}$$

Since the system of eigenfunctions $\varphi(x,\lambda_n)n \geq 0$ are complete and orthogonal in $L_2(0,\pi)$, the Parseval equality

$$\int_0^{\xi} |f(x)|^2 dx = \sum_{k=0}^{\infty} a_k|b_k|^2.$$

hold.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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