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# Inequalities for Tricomi functions

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**Abstract**
In this study, we establish new two-sided inequalities for Tricomi functions. Some special and confluent cases of our main aim are established with the help of the inequalities for hypergeometric functions ${}_0F_1\left(-;c;z\right)$, $c>0$.

**Keywords**
Inequalities for hypergeometric functions, Bessel functions, Modified Bessel functions, Tricomi functions.

**AMS Subject Classification** 26D15, 30A10, 26D07.

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## Contents



## 1. Introduction, Motivation and Preliminaries

Two-sided inequalities for the theory of special functions appear in the literature of many areas. A good number of such two-sided inequalities are motivated by different problems in mathematics, sciences and engineering that involve inequalities for the theory of special functions in [2–5, 7, 16, 19–21, 24–26]. Joshi and Arya [9, 10] are devoted to analogous questions. Recently, Joshi and Bissu [11, 13] introduced the concept of the inequalities for confluent hypergeometric function. The inequalities of Bessel functions of the first kind are important in several problems of applied mathematics, mathematical physics, and engineering. Because of their importance, there is an extensive literature on various properties of the inequalities of Bessel functions of the first kind, and they were investigated by famous researchers such as Watson [30], Joshi and Bissu [13], Joshi and Bissu [11], Laforgia [14, 15], Nasell [22]. For more details, the author [6, 27, 28] has earlier introduced two-sided inequalities of the special functions.

Our main motivation for this paper is to complement and improve the results of Luke [18–21], and Joshi and Bissu [11, 13]. In this present paper, we introduce some new results for the inequalities of Tricomi functions by using inequalities for the hypergeometric functions under certain additional conditions in section 3. By using a similar technique as two-sided inequalities for the modified Bessel and Bessel functions, which may be of interest in themselves, have been discussed briefly for presenting research in section 4.

Here, we use the lemma and theorem in [12, 18] are needed throughout in this paper to obtain our main results. The lemma and theorem bring about numerous applications two-sided inequalities in the theory of special functions.

**Lemma 1.1.** *(i) Let $c>0$ and $z>0$. Then (see Joshi and Bissu [12] eq.(1.1))*

$$1-\frac{z}{c}<{}_0F_1\left(-;c;-z\right)<1-\frac{z}{c}+\frac{z^2}{2(c)(c+1)}. \tag{1.1}$$

*(ii) Let $c>0$ and $0<z<1$. Then (see Joshi and Bissu [12] eq.(1.2))*

$$1+\frac{z}{c}<{}_0F_1\left(-;c;z\right)<1+\frac{2z}{c} \tag{1.2}$$

*where the confluent hypergeometric function is defined by the power series:*

$${}_0F_1\left(-;c;z\right)=\sum_{k\geq 0}\frac{z^k}{k!(c)_k} \tag{1.3}$$

*and $(c)_k=c(c+1)\dots(c+k-1)$ for $k>1$ and $(c)_0=1$.*

Now, we recall the inequalities of Luke [18] which will be used in the follows investigation:

**Theorem 1.2.** *(i) If $c \geq a > 0$ and $z > 0$, then (see Luke [18], eq. (5.3) )*

$$\begin{aligned} &-1+2\left[1+\frac{a}{2c}z\right]^{-1} < {}_1F_1\left(a;c;-z\right) \\ &< 1-\frac{a(c+1)}{c(a+1)}+\frac{a(c+1)}{c(a+1)}\left[1+\frac{a+1}{c+1}z\right]^{-1}. \end{aligned} \tag{1.4}$$

*(ii) If $c > 0$, $a < 1$ and $z > 0$, then (see Luke [18], eq. (5.4))*

$$\begin{aligned} &-\frac{1}{c}+\frac{c+1}{2}\left[1+\frac{a}{c+1}z\right]^{-1} < {}_1F_1\left(a;c;-z\right) \\ &< \frac{1-a}{1+a}+\frac{2a}{a+1}\left[1+\frac{(a+1)}{2c}z\right]^{-1}, \end{aligned} \tag{1.5}$$

*where the confluent hypergeometric functions ${}_1F_1(a;c;z)$ is formally defined as*

$${}_1F_1\left(a;c;z\right) = \sum_{k\geq 0}\frac{(a)_k x^k}{k!(c)_k}.$$

In fact, we give the known hypergeometric functions whose product is also a hypergeometric function leads to new two-sided inequalities for Tricomi functions. For these, we use Kummer's transformations in the form

$${}_0F_0\left(-;-;\frac{1}{2}z\right)\,{}_1F_1\left(\mu;2\mu;-z\right) = {}_0F_1\left(-;\mu+\frac{1}{2};\frac{z^2}{16}\right) \tag{1.6}$$

and the product identities listed is later given by Preece and Bailey [1, 23]:

$$\begin{aligned} &{}_0F_1\left(-;\mu;z\right)\,{}_0F_1\left(-;\mu;z\right) \\ &= {}_0F_3\left(-;\mu,\frac{1}{2}\mu,\frac{1}{2}(\mu+1);-\frac{1}{4}z^2\right). \end{aligned} \tag{1.7}$$

## 2. Inequalities for confluent hypergeometric functions ${}_0F_1(-;c;z)$

In this section, we mention new and known interesting special cases of two-sided inequalities for confluent hypergeometric functions ${}_0F_1(-;c;z)$. In Theorem 1.1, in the other limiting process (replacing $z$ by $\frac{z}{a}$ and letting $a \longrightarrow \infty$), become the following inequalities for limitless confluent hypergeometric function [8]:

$${}_0F_1\left(-;c;z\right) = \lim_{a\to\infty} {}_1F_1\left(a;c;\frac{z}{a}\right) \tag{2.1}$$

or by transforming appropriately (2.1), we have the following theorem:

**Theorem 2.1.** *(i) If $c > 0$, and $z > 0$, then*

$$\begin{aligned} &-1+2\left[1+\frac{1}{2c}z\right]^{-1} < {}_0F_1\left(-;c;-z\right) \\ &< 1-\frac{(c+1)}{c}+\frac{(c+1)}{c}\left[1+\frac{1}{c+1}z\right]^{-1}. \end{aligned} \tag{2.2}$$

*(ii) If $c > 0$ and $z > 0$, then*

$$\begin{aligned} &-\frac{1}{c}+\frac{c+1}{2}\left[1+\frac{1}{c+1}z\right]^{-1} < {}_0F_1\left(-;c;-z\right) \\ &< -1+2\left[1+\frac{1}{2c}z\right]^{-1}. \end{aligned} \tag{2.3}$$

**Theorem 2.2.** *If $\nu \geq -\frac{1}{2}$ and $z > 0$, then*

$$\begin{aligned} &-1+2\left[1+\frac{1}{2}z\right]^{-1} < e^{-z}\,{}_0F_1\left(-;\nu+1;\frac{z^2}{4}\right) \\ &< \frac{1}{(2\nu+3)}+\frac{2(\nu+1)}{(2\nu+3)}\left[1+\frac{2\nu+3}{2(\nu+1)}z\right]^{-1}. \end{aligned} \tag{2.4}$$

*Proof.* From (1.4) and (1.6), we obtain (2.4). □

**Lemma 2.3.** *(i) Let $a > 0$, $b > 0$, $c > 0$ and $z > 0$. Then*

$$\begin{aligned} &1-\frac{z}{abc} < {}_0F_3\left(-;a,b,c;-z\right) \\ &< 1-\frac{z}{abc}+\frac{z^2}{2abc(a+1)(b+1)(c+1)}. \end{aligned} \tag{2.5}$$

*(ii) Let $a > 0$, $b > 0$, $c > 0$ and $0 < z < 1$. Then*

$$1+\frac{z}{abc} < {}_0F_3\left(-;a,b,c;z\right) < 1+\frac{2z}{abc}, \tag{2.6}$$

*where*

$${}_0F_3\left(-;a,b,c;z\right) = \sum_{k\geq 0}\frac{z^k}{k!(a)_k(b)_k(c)_k}.$$

*Proof.* To obtain another important integral representation of hypergeometric functions ${}_0F_3$, we start from the formula for the reciprocal gamma function

$$\frac{1}{\Gamma(b+k)} = \frac{1}{2\pi i}\int_C e^t t^{-(b+k)}dt, \tag{2.7}$$

where $C$ is the contour (see [17], p. 115, No. (5.10.5)). Substituting (2.7) into (1.3), we find that hypergeometric functions ${}_0F_2$

$${}_0F_2(-;b,c;-z) = \frac{\Gamma(b)}{2\pi i}\int_C e^t t^{-b}\,{}_0F_1\left(-;c;-\frac{z}{t}\right)dt, \tag{2.8}$$

and again using the following integral representation, we get

$${}_0F_3(-;a,b,c;-z) = \frac{\Gamma(b)}{2\pi i}\int_C e^t t^{-a}\,{}_0F_2\left(-;b,c;-\frac{z}{t}\right)dt. \tag{2.9}$$

Appropriately applying the inequalities (1.1), (1.2), (1.7), (2.7) and (2.9), we easily obtain (2.5) and (2.6). □

## 3. Inequalities for Tricomi functions

In this section, we present the inequalities for Tricomi functions which are obtained from inequalities for hypergeometric functions ${}_0F_1$ and we just take them as examples and the others can be derived in the same manner. The Tricomi functions of of the first kind is defined as (see Tricomi [29])

$$\mathbf{C}_\nu(z) = \frac{1}{\Gamma(\nu+1)}\, {}_0F_1\Big(-;\nu+1;-z\Big). \tag{3.1}$$

From (2.2)-(2.6) and (3.1), we get the respective inequalities:

**Theorem 3.1.** *(i) If $\nu \geq 0$ and $z > 0$, then the inequality for the Tricomi function is held:*

$$\begin{aligned}&\frac{1}{\Gamma(\nu+1)}\left[-1+2\Big(1+\frac{z}{2(\nu+1)}\Big)^{-1}\right] < \boldsymbol{C}_\nu(z)\\ &< \frac{1}{\Gamma(\nu+1)}\left[-\frac{1}{\nu+1}+\frac{(\nu+2)}{\nu+1}\Big(1+\frac{z}{\nu+2}\Big)^{-1}\right].\end{aligned} \tag{3.2}$$

*(ii) If $\nu > -1$ and $z > 0$, then*

$$\begin{aligned}&\frac{1}{\Gamma(\nu+1)}\left[-\frac{1}{\nu+1}+\frac{\nu+2}{2}\Big(1+\frac{z}{\nu+2}\Big)^{-1}\right]\\ &< \boldsymbol{C}_\nu(z) < \frac{1}{\Gamma(\nu+1)}\left[-1+2\Big(1+\frac{z}{2(\nu+1)}\Big)^{-1}\right].\end{aligned} \tag{3.3}$$

**Lemma 3.2.** *(i) If $\nu > -1$ and $z > 0$, then the following assertion is true:*

$$\begin{aligned}&\frac{1}{\Gamma(\nu+1)}\left[1-\frac{z}{\nu+1}\right] < \boldsymbol{C}_\nu(z)\\ &< \frac{1}{\Gamma(\nu+1)}\left[1-\frac{z}{\nu+1}+\frac{z^2}{2(\nu+1)(\nu+2)}\right].\end{aligned} \tag{3.4}$$

*(ii) Let $\nu > -1$ and $0 < z < 1$. Then*

$$\begin{aligned}&\frac{1}{\Gamma(\nu+1)}\left[1+\frac{z}{\nu+1}\right] < \boldsymbol{C}_\nu(z)\\ &< \frac{1}{\Gamma(\nu+1)}\left[1+\frac{2z}{\nu+1}\right].\end{aligned} \tag{3.5}$$

*Proof.* If we replace $c$ by $\nu+1$ and from Lemma 1.1, we obtain (3.4) and (3.5). □

**Lemma 3.3.** *(i) Let $\nu > -1$ and $z > 0$. Then the Tricomi function satisfy the inequality*

$$\begin{aligned}&\frac{1}{\Big(\Gamma(\nu+1)\Big)^2}\left[1-\frac{4z}{(\nu+1)^2(\nu+2)}\right] < \boldsymbol{C}_\nu^2(z)\\ &< \frac{1}{\Big(\Gamma(\nu+1)\Big)^2}\left[1-\frac{4z}{(\nu+1)^2(\nu+2)}\right.\\ &\left.+\frac{8z^2}{(\nu+1)^2(\nu+2)^2(\nu+3)(\nu+4)}\right].\end{aligned} \tag{3.6}$$

*(ii) If $\nu > -1$ and $0 < z < 1$, then*

$$\begin{aligned}&\frac{1}{\Big(\Gamma(\nu+1)\Big)^2}\left[1+\frac{4z}{(\nu+1)^2(\nu+2)}\right] < \boldsymbol{C}_\nu^2(z)\\ &< \frac{1}{\Big(\Gamma(\nu+1)\Big)^2}\left[1+\frac{8z}{(\nu+1)^2(\nu+2)}\right].\end{aligned} \tag{3.7}$$

*Proof.* By iteration Lemma 2.1, we prove the Lemma 3.2. □

In conclusion we observe that on repeated application of the two-sided inequalities for Tricomi functions $\mathbf{C}_\nu(z)$, more two-sided inequalities could be obtained, but the details are omitted for reasons of brevity. In the next section, we will be applied to the study of similar inequalities for Tricomi functions including two-sided inequalities for modified Bessel and Bessel functions.

## 4. Inequalities for modified Bessel and Bessel functions

Bessel function $J_\nu(z)$ is connected with Tricomi functions by the relation (see Luke , [19], p. 311, eq. 2 and [20], p. 39, eq. 10 and p.120, eq. 6)

$$\begin{aligned}J_\nu(z) &= \Big(\frac{z}{2}\Big)^\nu \mathbf{C}_\nu\Big(\frac{z^2}{4}\Big)\\ &= \frac{1}{\Gamma(\nu+1)}\Big(\frac{z}{2}\Big)^\nu {}_0F_1\Big(-;\nu+1;-\frac{1}{4}z^2\Big).\end{aligned} \tag{4.1}$$

Here, we give the theorem and two lemmas to inequalities for Bessel functions in this first part:

**Theorem 4.1.** *(i) If $\nu \geq 0$ and $z > 0$, then the Bessel functions satisfy the following inequality*

$$\begin{aligned}&\frac{1}{\Gamma(\nu+1)}\Big(\frac{z}{2}\Big)^\nu\left[-1+2\Big(1+\frac{z^2}{8(\nu+1)}\Big)^{-1}\right]\\ &< J_\nu(z) < \frac{1}{\Gamma(\nu+1)}\Big(\frac{z}{2}\Big)^\nu\\ &\left[1-\frac{\nu+2}{\nu+1}+\frac{\nu+2}{\nu+1}\Big(1+\frac{z^2}{4(\nu+2)}\Big)^{-1}\right].\end{aligned} \tag{4.2}$$

*(ii) If $\nu > -1$ and $z > 0$, then*

$$\begin{aligned}&\frac{1}{\Gamma(\nu+1)}\Big(\frac{z}{2}\Big)^\nu\left[-\frac{1}{\nu+1}+\frac{\nu+2}{2}\Big(1+\frac{z^2}{4(\nu+2)}\Big)^{-1}\right]\\ &< J_\nu(z) < \frac{1}{\Gamma(\nu+1)}\Big(\frac{z}{2}\Big)^\nu\left[-1+2\Big(1+\frac{z^2}{8(\nu+1)}\Big)^{-1}\right].\end{aligned} \tag{4.3}$$

*Proof.* Starting with the inequalities for hypergeometric functions (2.1) and (2.2) with (4.1), we obtain (4.2) and (4.3). □

**Lemma 4.2.** *For $\nu > -1$ and $z > 0$, then the following inequality holds:*

$$\frac{1}{\Big(\Gamma(\nu+1)\Big)^2}\left(\frac{z}{2}\right)^{2\nu}\left[1-\frac{z^2}{(\nu+1)^2(\nu+2)}\right] < J_\nu^2(z) < \frac{1}{\Big(\Gamma(\nu+1)\Big)^2}\left(\frac{z}{2}\right)^{2\nu}\left[1-\frac{z^2}{(\nu+1)^2(\nu+2)} + \frac{8z^4}{(\nu+1)^2(\nu+2)^2(\nu+3)(\nu+4)}\right]. \tag{4.4}$$

*Proof.* If we start with the inequalities for hypergeometric functions (2.1) with (4.1), we get (4.4). □

Next, the connection between modified Bessel function and Tricomi function is given by (see Luke [19], p. 311, eq. 2 and [20], p.120, eq. 6)

$$\mathbf{I}_\nu(z) = \left(\frac{z}{2}\right)^\nu \mathbf{C}_\nu\left(-\frac{z^2}{4}\right) = \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu {}_0F_1\left(-;\nu+1;\frac{z^2}{4}\right). \tag{4.5}$$

**Lemma 4.3.** *For $\nu > -1$ and $0 < z < 1$, then the modified Bessel functions satisfy the following inequality*

$$\frac{1}{\Big(\Gamma(\nu+1)\Big)^2}\left(\frac{z}{2}\right)^{2\nu}\left[1+\frac{z^2}{(\nu+1)^2(\nu+2)}\right] < \boldsymbol{I}_\nu^2(z) < \frac{1}{\Big(\Gamma(\nu+1)\Big)^2}\left(\frac{z}{2}\right)^{2\nu}\left[1+\frac{2z^2}{(\nu+1)^2(\nu+2)}\right]. \tag{4.6}$$

*Proof.* Applying the definition modified Bessel function (4.5) and using the same technique as in Lemma 3.2, we obtain (4.6). □

In general, we use all the procedures related in [6, 12, 13, 18] to extend the domain of validity of two-sided inequalities for Tricomi functions and extend our ideas to get inequalities for these functions. It seems that we have sufficiently elaborated on these points in this article and also in our previous study, and further comment is unnecessary.

## 5. Numerical examples

Here, we conclude remarks with some numerical examples:

**Example 5.1.** *In $\boldsymbol{C}_\nu(z)$, let $\nu = -0.5, 0.1, 0.2, 0.5$ and $z = 0.1, 0.2, 1.5$. From (3.2),(3.3), (3.4), (3.5), (3.6) and (3.7), we have*

| *Equations* | $z$ | $\nu$ | *L.H.S.* | *Function* | *R.H.S* |
|---|---|---|---|---|---|
| *Eq.3.2* | $z=0.1$ | $\nu=0.1$ | *0.959733788189015* | $\boldsymbol{C}_{0.1}(0.1)$ | *0.959922637812822* |
| | $z=0.1$ | $\nu=0.2$ | *1.001994467373669* | $\boldsymbol{C}_{0.2}(0.1)$ | *1.002310155611658* |
| | $z=0.2$ | $\nu=0.1$ | *0.875947505093148* | $\boldsymbol{C}_{0.1}(0.2)$ | *0.876639953713775* |
| | $z=0.2$ | $\nu=0.2$ | *0.921566817818592* | $\boldsymbol{C}_{0.2}(0.2)$ | *0.922730412285535* |
| | $z=1.5$ | $\nu=0.5$ | *0.376126389031837* | $\boldsymbol{C}_{0.5}(1.5)$ | *0.423142187660817* |
| *Eq.3.3* | $z=0.1$ | $\nu=-0.5$ | *-0.731683366163496* | $\boldsymbol{C}_{-0.5}(0.1)$ | *0.461609659266346* |
| | $z=0.1$ | $\nu=0.1$ | *0.097946857387688* | $\boldsymbol{C}_{0.1}(0.1)$ | *0.959733788189015* |
| | $z=0.1$ | $\nu=0.2$ | *0.238344619680882* | $\boldsymbol{C}_{0.2}(0.1)$ | *1.001994467373669* |
| | $z=0.2$ | $\nu=0.1$ | *0.052141381133213* | $\boldsymbol{C}_{0.1}(0.2)$ | *0.875947505093148* |
| | $z=0.2$ | $\nu=0.2$ | *0.190596773685209* | $\boldsymbol{C}_{0.2}(0.2)$ | *0.921566817818592* |
| | $z=1.5$ | $\nu=0.5$ | *0.129293446229694* | $\boldsymbol{C}_{0.5}(1.5)$ | *0.376126389031837* |
| *Eq.3.4* | $z=0.1$ | $\nu=-0.5$ | *0.451351666838205* | $\boldsymbol{C}_{-0.5}(0.1)$ | *0.455112930728523* |
| | $z=0.1$ | $\nu=0.1$ | *0.955579096465253* | $\boldsymbol{C}_{0.1}(0.1)$ | *0.957854284790170* |
| | $z=0.1$ | $\nu=0.2$ | *0.998364052636808* | $\boldsymbol{C}_{0.2}(0.1)$ | *1.000426788282752* |
| | $z=0.2$ | $\nu=0.1$ | *0.860021186818727* | $\boldsymbol{C}_{0.1}(0.2)$ | *0.869121940118396* |
| | $z=0.2$ | $\nu=0.2$ | *0.907603684215280* | $\boldsymbol{C}_{0.2}(0.2)$ | *0.915854626799055* |
| | $z=1.5$ | $\nu=0.5$ | *0* | $\boldsymbol{C}_{0.5}(1.5)$ | *0.338513750128654* |
| *Eq.3.5* | $z=0.1$ | $\nu=-0.5$ | *0.677027500257308* | $\boldsymbol{C}_{-0.5}(0.1)$ | *0.789865416966859* |
| | $z=0.1$ | $\nu=0.1$ | *1.146694915758303* | $\boldsymbol{C}_{0.1}(0.1)$ | *1.242252825404828* |
| | $z=0.1$ | $\nu=0.2$ | *1.179884789479864* | $\boldsymbol{C}_{0.2}(0.1)$ | *1.270645157901392* |
| | $z=0.2$ | $\nu=0.1$ | *1.242252825404828* | $\boldsymbol{C}_{0.1}(0.2)$ | *1.433368644697879* |
| | $z=0.2$ | $\nu=0.2$ | *1.270645157901392* | $\boldsymbol{C}_{0.2}(0.2)$ | *1.452165894744449* |
| | $z=1.5$ | $\nu=0.5$ | *2.256758334191025* | $\boldsymbol{C}_{0.5}(1.5)$ | *3.385137501286538* |
| *Eq.3.6* | $z=0.1$ | $\nu=-0.5$ | *-0.021220659078919* | $\boldsymbol{C}^2_{-0.5}(0.1)$ | *-0.016046860293964* |
| | $z=0.1$ | $\nu=0.1$ | *0.930959213312613* | $\boldsymbol{C}^2_{0.1}(0.1)$ | *0.932262497508072* |
| | $z=0.1$ | $\nu=0.2$ | *1.036420286799993* | $\boldsymbol{C}^2_{0.2}(0.1)$ | *1.037433353072623* |
| | $z=0.2$ | $\nu=0.1$ | *0.757029421007594* | $\boldsymbol{C}^2_{0.1}(0.2)$ | *0.762242557789431* |
| | $z=0.2$ | $\nu=0.2$ | *0.886648569054329* | $\boldsymbol{C}^2_{0.2}(0.2)$ | *0.890700834144850* |
| | $z=1.5$ | $\nu=0.5$ | *-0.084882636315677* | $\boldsymbol{C}^2_{0.5}(1.5)$ | *0.018593339383434* |
| *Eq.3.7* | $z=0.1$ | $\nu=-0.5$ | *0.657840431446501* | $\boldsymbol{C}^2_{-0.5}(0.1)$ | *0.997370976709211* |
| | $z=0.1$ | $\nu=0.1$ | *1.278818797922650* | $\boldsymbol{C}^2_{0.1}(0.1)$ | *1.452748590227669* |
| | $z=0.1$ | $\nu=0.2$ | *1.335963722291320* | $\boldsymbol{C}^2_{0.2}(0.1)$ | *1.485735440036984* |
| | $z=0.2$ | $\nu=0.1$ | *1.452748590227669* | $\boldsymbol{C}^2_{0.1}(0.2)$ | *1.800608174837707* |
| | $z=0.2$ | $\nu=0.2$ | *1.485735440036984* | $\boldsymbol{C}^2_{0.2}(0.2)$ | *1.785278875528311* |
| | $z=1.5$ | $\nu=0.5$ | *2.631361725786003* | $\boldsymbol{C}^2_{0.5}(1.5)$ | *3.989483906836843* |

**Example 5.2.** *In $J_\nu(z)$, for $\nu = -0.5, 0.1, 0.2, 0.5$ and $z = 0.1, 0.2, 1.5$, and from (4.2), (4.3) and (4.4), we have*

| *Equations* | $z$ | $\nu$ | *L.H.S.* | *Function* | *R.H.S.* |
|---|---|---|---|---|---|
| *Eq.4.2* | $z=0.1$ | $\nu=0.1$ | *0.777265324176332* | $J_{0.1}(0.1)$ | *0.777265419761771* |
| | $z=0.1$ | $\nu=0.2$ | *0.596989532768496* | $J_{0.2}(0.1)$ | *0.596989650534774* |
| | $z=0.2$ | $\nu=0.1$ | *0.827391713807563* | $J_{0.1}(0.2)$ | *0.827393341573335* |
| | $z=0.2$ | $\nu=0.2$ | *0.681488222208131* | $J_{0.2}(0.2)$ | *0.681490372596634* |
| | $z=1.5$ | $\nu=0.5$ | *0.668613963656627* | $J_{0.5}(1.5)$ | *0.678060628763236* |
| *Eq.4.3* | $z=0.1$ | $\nu=-0.5$ | *-3.157064320397937* | $J_{-0.5}(0.1)$ | *2.510548319915321* |
| | $z=0.1$ | $\nu=0.1$ | *0.108800316612373* | $J_{0.1}(0.1)$ | *0.777265324176332* |
| | $z=0.1$ | $\nu=0.2$ | *0.158782271033230* | $J_{0.2}(0.1)$ | *0.596989532768496* |
| | $z=0.2$ | $\nu=0.1$ | *0.113496782289734* | $J_{0.1}(0.2)$ | *0.827391713807563* |
| | $z=0.2$ | $\nu=0.2$ | *0.179830539384815* | $J_{0.2}(0.2)$ | *0.681488222208131* |
| | $z=1.5$ | $\nu=0.5$ | *0.345677967604787* | $J_{0.5}(1.5)$ | *0.668613963656627* |
| *Eq.4.4* | $z=0.1$ | $\nu=-0.5$ | *6.196432451044458* | $J^2_{-0.5}(0.1)$ | *6.197467210801449* |
| | $z=0.1$ | $\nu=0.1$ | *0.604505328062984* | $J^2_{0.1}(0.1)$ | *0.604512486745953* |
| | $z=0.1$ | $\nu=0.2$ | *0.356754900027543* | $J^2_{0.2}(0.1)$ | *0.356757956537808* |
| | $z=0.2$ | $\nu=0.1$ | *0.686163604941986* | $J^2_{0.1}(0.2)$ | *0.686295175619597* |
| | $z=0.2$ | $\nu=0.2$ | *0.466269023184786* | $J^2_{0.2}(0.2)$ | *0.466333552616367* |
| | $z=1.5$ | $\nu=0.5$ | *0.572957795130823* | $J^2_{0.5}(1.5)$ | *0.747573504123074* |

**Example 5.3.** *In $\boldsymbol{I}_\nu^2(z)$, let $\nu = -0.5, 0.1, 0.2, 0.5$ and $z = 0.1, 0.2, 1.5$, and from (4.6), we have*

| *Equations* | $z$ | $\nu$ | *L.H.S.* | *Function* | *R.H.S.* |
|---|---|---|---|---|---|
| *Eq.4.6* | $z=0.1$ | $\nu=-0.5$ | *6.535962996307167* | $\boldsymbol{I}^2_{-0.5}(0.1)$ | *6.70572826893852* |
| | $z=0.1$ | $\nu=0.1$ | *0.609282138241277* | $\boldsymbol{I}^2_{0.1}(0.1)$ | *0.611670543330424* |
| | $z=0.1$ | $\nu=0.2$ | *0.359014272415304* | $\boldsymbol{I}^2_{0.2}(0.1)$ | *0.360143958609185* |
| | $z=0.2$ | $\nu=0.1$ | *0.708112060917739* | $\boldsymbol{I}^2_{0.1}(0.2)$ | *0.719086288905615* |
| | $z=0.2$ | $\nu=0.2$ | *0.478194062140919* | $\boldsymbol{I}^2_{0.2}(0.2)$ | *0.484156581618985* |
| | $z=1.5$ | $\nu=0.5$ | *1.336901521971921* | $\boldsymbol{I}^2_{0.5}(1.5)$ | *1.718873385392470* |

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