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Method of upper lower solutions for nonlinear system of fractional differential equations and applications

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Abstract
Our aim is to develop the method of upper lower solutions and apply it to prove existence and uniqueness of solution of periodic boundary value problems for a system of fractional differential equations involving a Riemann - Liouville fractional derivatives.

Keywords
Periodic boundary value problems, System of fractional differential equations, Riemann-Liouville fractional derivatives, Upper and lower solutions, Existence and uniqueness results.

AMS Subject Classification
26A33;34A08,34B15,34B99.

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Contents

1. Introduction

Now-a-days the theory of fractional differential equations have been occupying an importance place in science and technology.Fractional differential equations have been widely used for modeling various processes in physics, chemistry,biology, aerodynamics of complex medium,polymer rheology, thermo-elasticity and control of dynamical systems(see [3, 5, 17] and the references therein). Recently, many researchers have given attention to the existence and uniqueness of solution of the initial value problems [8, 15], periodic boundary value problems [2, 14, 18], problems with integral boundary conditions [4, 7, 10–13] and with nonlocal integral boundary conditions [1] for fractional differential equations. It is well known that the method of upper and lower solutions [6, 16]coupled with its associated monotone iteration scheme is an interesting, constructive and powerful mechanism which offers existence and uniqueness results for nonlinear problems in a closed set. Recently,Wei et.al.[19] proved existence and uniqueness of the solution of periodic boundary value problem for a fractional differential equation,using the method of upper lower solutions and its associated monotone iterations.In this paper, we extend these results for nonlinear system of Riemann-Liouville fractional differential equations,by removing the bounded demand of $f(t,u(t))$ in [9].

We organize the paper as follows: In Section 2,we consider the periodic boundary value problem for nonlinear system of Riemann - Liouville fractional differential equations and introduce the notion of upper lower solution. Existence and uniqueness results of periodic boundary value problem for system of nonlinear fractional differential equations involving Riemann-Liouville fractional derivatives are proved in the last section.

2. Upper Lower Solutions

In this section,we consider the periodic boundary value problems for a system of nonlinear Riemann - Liouville fractional differential equations and introduce the notion of upper and lower solutions.Consider the following system of nonlinear Riemann-Liouville fractional differential equations

$$D^{\alpha}u_1(t)=f_1\big(t,u_1(t),u_2(t)\big),\qquad t\in\ (0,T],\qquad 0<\alpha\le 1,$$
$$D^{\alpha}u_2(t)=f_2\big(t,u_1(t),u_2(t)\big),\tag{2.1}$$

with periodic boundary conditions

$$\begin{aligned} t^{1-\alpha}u_1(t)|_{t=0}&=t^{1-\alpha}u_1(t)|_{t=T},\\ t^{1-\alpha}u_2(t)|_{t=0}&=t^{1-\alpha}u_2(t)|_{t=T}.\end{aligned}\tag{2.2}$$

It is called periodic boundary value problems(PBVP) for the system of nonlinear Riemann-Liouville fractional differential equations. Assume that $J=[0,T]\subset\mathbb{R}$ is a compact interval and $f_i(t,u_1(t),u_2(t))\in C([0,T]\times\mathbb{R}^2,\mathbb{R}), i=1,2$. Further assume that u_1 and u_2 are measurable Lebesgue functions i.e.$u_1,u_2\in L_1(0,T)$. Suppose

$$C([0,T])=\big\{u_i:u_i(t)\text{is continuous on }[0,T],||u_i||_C=\max_{t\in[0,T]}|u_i(t)|\big\},i=1,2.$$
$$C_{1-\alpha}([0,T])=\big\{u_i\in C[0,T]:t^{1-\alpha}u_i(t)\in C([0,T]),||u_i||_{C_{1-\alpha}}=||t^{1-\alpha}u_i||_C\big\}.$$

Assume that upper and lower solutions satisfy the following order relation

$$(v_1,v_2)\le(w_1,w_2),t\in(0,T]:t^{1-\alpha}v_i(t)|_{t=0}\le t^{1-\alpha}w_i(t)|_{t=0},\ i=1,2.\tag{2.3}$$

Now we define the order interval or (functional interval)sector as follows:

Definition 2.1. *The order interval in a space $C_{1-\alpha}([0,T])\cap L_1(0,T)$ is denoted by S and is defined as*

$$S=\Big\{(u_1,u_2)\in C_{1-\alpha}([0,T])\cap L_1(0,T):\big(v_1(t),v_2(t)\big)\le\big(u_1(t),u_2(t)\big)\le \big(w_1(t),w_2(t)\big),t\in(0,T];t^{1-\alpha}v_i(t)|_{t=0}\le t^{1-\alpha}u_i(t)|_{t=0}\le t^{1-\alpha}w_i(t)|_{t=0}\Big\}.$$

In the following,we define quasimonotonicity and Lipschitz condition of function $f_i(t,u_1,u_2), i=1,2$ as follows.

Definition 2.2. *A function $f_i(t,u_1,u_2)\in C(J\times\mathbb{R}^2,\mathbb{R})$, $i=1,2$ is said to be quasimonotone nondecreasing $\big($nonincreasing $\big)$ if for each $i,u_i\le v_i$ and $u_j=v_j,i\ne j$, then*

$$f_i(t,u_1,u_2)\le f_i(t,v_1,v_2)\big(f_i(t,u_1,u_2)\ge f_i(t,t,v_1,v_2)\big).$$

Definition 2.3. *Let $f_i(t,u_1,u_2):[0,T]\times\mathbb{R}^2\to\mathbb{R}$ be a real valued continuous function. We say that $f_i(t,u_1,u_2)$ satisfies one sided Lipschitz condition if there exists $M_i\ge 0$ such that*

$$\begin{aligned} f_1(t,u_1,u_2)-f_1(t,u_1^*,u_2)&\ge -M_1(u_1-u_1^*)\,\textit{for}\ v_1\le u_1^*\le u_1\le w_1,\\ f_2(t,u_1,u_2)-f_2(t,u_1,u_2^*)&\ge -M_2(u_2-u_2^*)\,\textit{for}\ v_2\le u_2^*\le u_2\le w_2.\end{aligned}\tag{2.4}$$

Further to ensure the uniqueness of solution of PBVP (2.1)-(2.2),we assume that there exists $N_i\ge 0$ such that

$$\begin{aligned} f_1(t,u_1,u_2)-f_1(t,u_1^*,u_2)&\le N_1(u_1-u_1^*)\ \text{for}\ v_1\le u_1^*\le u_1\le w_1,\\ f_2(t,u_1,u_2)-f_2(t,u_1,u_2^*)&\le N_2(u_2-u_2^*)\ \text{for}\ v_2\le u_2^*\le u_2\le w_2.\end{aligned}\tag{2.5}$$

From conditions (2.4) and (2.5), we conclude that function **f**=(f_1,f_2) satisfies Lipschitz condition

$$|f_i(t,u_1,u_2)-f_i(t,u_1^*,u_2^*)|\le K_i(|u_1-u_1^*|+|u_2-u_2^*|),\tag{2.6}$$

with $M_i=N_i=K_i$.

Now we consider the following results of the linear PBVP for a fractional differential equation which are main ingredients in the proof of our existence and uniqueness results of solution of the PBVP (2.1)-(2.2).

Lemma 2.4. *[19] The linear periodic boundary value prob-*

lem

$$D^{\alpha}u(t)+Mu(t)=\sigma(t),$$
$$t^{1-\alpha}u(t)|_{t=0}=t^{1-\alpha}u(t)|_{t=T},$$

where $M>0$ is a constant and $\sigma\in C[0,T]$ has the following integral representation of the solution

$$u=\frac{T^{1-\alpha}\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(-Mt^{\alpha})}{[1-\Gamma(\alpha)]E_{\alpha,\alpha}(-MT^{\alpha})} \tag{2.7}$$
$$(\times)\int_0^T(T-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(T-s)^{\alpha}\big)\sigma(s)ds$$
$$+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(t-s)^{\alpha}\big)\sigma(s)ds, \tag{2.8}$$

where $E_{\alpha,\alpha}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma((k+1)\alpha)}$ is the Mittag-Leffler function (see [5]).

Lemma 2.5. *[19] If $u(t)\in C_{1-\alpha}([0,T])\cap L_1(0,T)$ and satisfies the relations*

$$D^{\alpha}u(t)+Mu(t)\geq 0, t\in(0,T),$$
$$t^{1-\alpha}u(t)|_{t=0}=t^{1-\alpha}u(t)|_{t=T},$$

where $M>0$ is a constant,then $u(t)\geq 0, t\in(0,T]$.

3. Main Results

In this section we develop method of upper lower solutions and construct two monotone convergent sequences,which converge monotonically from above and below to maximal and minimal solutions respectively. As an application of this method,existence and uniqueness results for the PBVP (2.1) – (2.2) are proved when the functions $f_1(t,u_1,u_2)$ and $f_2(t,u_1,u_2)$ are quasimonotone nonincreasing as well as quasimonotone nondecreasing

Theorem 3.1. *Suppose that*

(i) *$v^0=(v_1^0,v_2^0)$ and $w^0=(w_1^0,w_2^0)\in C_{1-\alpha}([0,T])\cap L_1(0,T)$ are lower and upper solutions of the PBVP (2.1) – (2.2), such that order relation (2.3) holds,*

(ii) *function $f_i(t,u_1,u_2)\in C([0,T]\times\mathbb{R}^2,\mathbb{R})$ satisfies one-sided Lipschitz condition (2.4),*

(iii) *functions f_1 and f_2 are quasimonotone nondecreasing.*

Then there exist monotone sequences $\{v_1^n(t),v_2^n(t)\},\{w_1^n(t),w_2^n(t)\}\subset C_{1-\alpha}([0,T])\cap L_1(0,T)$ such that

$$\{v_1^n(t),v_2^n(t)\}\to(v_1,v_2) \text{ and } \{w_1^n(t),w_2^n(t)\}\to(w_1,w_2),$$

as $n\to\infty$ on $(0,T]$, where the functions $(v_1(t),v_2(t))$ and $(w_1(t),w_2(t))$ are minimal and maximal solutions on S for the PBVP (2.1) – (2.2) and satisfy the monotone property

$$v_1^0\leq v_1^1\leq\ldots\leq v_1^n\leq\ldots\leq v_1\leq w_1\leq\ldots\leq w_1^n\ldots\leq w_1^1\leq w_1^0,$$
$$v_2^0\leq v_2^1\leq\ldots\leq v_2^n\leq\ldots\leq v_2\leq w_2\leq\ldots\leq w_2^n\ldots\leq w_2^1\leq w_2^0. \tag{3.1}$$

Also,if the one sided Lipschitz condition (2.5) holds,then the PBVP (2.1) – (2.2) has unique solution on S.

Proof: Consider PBVP for system of linear fractional differential equations

$$D^{\alpha}u_1(t)+M_1u_1=f_1(t,\eta_1,\eta_2)+M_1\eta_1$$
$$=\sigma_1(t,\eta_1,\eta_2),t\in(0,T), \tag{3.2}$$
$$t^{1-\alpha}u_1(t)|_{t=0}=t^{1-\alpha}u_1(t)|_{t=T},$$

$$D^{\alpha}u_2(t)+M_2u_2=f_2(t,\eta_1,\eta_2)+M_2\eta_2$$
$$=\sigma_2(t,\eta_1,\eta_2),t\in(0,T), \tag{3.3}$$
$$t^{1-\alpha}u_2(t)|_{t=0}=t^{1-\alpha}u_2(t)|_{t=T},$$

for any $(\eta_1,\eta_2)\in S$. Clearly,linear problems (3.2) and (3.3) have exactly one solution $u_1(t)$ and $u_2(t)\in C_{1-\alpha}([0,T])\cap L_1(0,T)$ respectively, follows from Lemma 2.1 and whose integral representation is as in (2.7). Now define $A[\eta_1,\mu]=u_1(t)$ as follows:

$$u_1=\frac{T^{1-\alpha}\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(-M_1t^{\alpha})}{[1-\Gamma(\alpha)]E_{\alpha,\alpha}(-M_1T^{\alpha})}$$
$$(\times)\int_0^T(T-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M_1(T-s)^{\alpha}\big)\sigma_1ds$$
$$+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M_1(t-s)^{\alpha}\big)\sigma_1ds. \tag{3.4}$$

Also we define and $A[\eta_2,\mu]=u_2(t)$, as follows:

$$u_2=\frac{T^{1-\alpha}\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(-M_2t^{\alpha})}{[1-\Gamma(\alpha)]E_{\alpha,\alpha}(-M_2T^{\alpha})}$$
$$(\times)\int_0^T(T-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M_2(T-s)^{\alpha}\big)\sigma_2ds$$
$$+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M_2(t-s)^{\alpha}\big)\sigma_2ds. \tag{3.5}$$

An operator A is from $[v_i^0(t),w_i^0(t)]$ into $C_{1-\alpha}([0,T])\cap L_1(0,T)$ and η_i is solution of the PBVP (2.1)- (2.2) iff $\eta_i=A[\eta_i,\mu]$. Now we prove

$$(I)v_i^0(t)\leq A[v^0(t),w^0(t)] \quad\text{and}\quad w_i^0(t)\geq A[w^0(t),v^0(t)],\quad i=1,2, \tag{3.6}$$
$$(II)\quad\text{If}\quad v_i^0\leq\eta_i\leq\mu_i\leq w_i^0\quad\text{then}\quad A[\eta_i,\mu]\leq A[\eta,\mu_i],\quad i=1,2. \tag{3.7}$$

To prove (I), set $A[v^0(t),w^0(t)] = v_i^1(t)$ where $v_i^1(t) = (v_1^1(t),v_2^1(t))$. Note that $v_1^1(t)$ and $v_2^1(t)$ are the unique solutions of linear PBVP (3.2) and (3.3) respectively. The function $v_i^0(t)$ is a lower solution of the PBVP (2.1) − (2.2). Set $p_i(t) = v_i^0(t) - v_i^1(t)$ with $\eta_i = v_i^0(t)$. We observe that

$$\begin{aligned} D^\alpha p_i(t) &= D^\alpha v_i^0(t) - D^\alpha v_i^1(t), \\ &\le f_i(t,v_1^0(t),v_2^0(t)) - f_i(t,v_1^0(t),v_2^0(t)) \\ &\quad - M_i(v_i^0(t) - v_i^1(t)), \\ &\le -M_i(v_i^0(t) - v_i^1(t)), \\ D^\alpha p_i(t) &\le -M_i p_i(t), \end{aligned}$$

and boundary conditions

$$\begin{aligned} t^{1-\alpha}p_i(t)|_{t=0} &= t^{1-\alpha}v_i^0(t)|_{t=0} - t^{1-\alpha}v_i^1(t)|_{t=0} \\ &= t^{1-\alpha}v_i^0(t)|_{t=T} - t^{1-\alpha}v_i^1(t)|_{t=T} \\ &= t^{1-\alpha}p_i(t)|_{t=T}. \end{aligned}$$

Using Lemma 2.2,we get $p_i(t) \le 0$ implies that $v_i^0(t) \le v_i^1(t) = A[v^0(t),w^0(t)]$. To prove that $w_i^0(t) \ge A[w^0(t),v^0(t)]$; we set $A[w^0(t),v^0(t)] = w_i^1(t)$ where $w_i^1(t) = (w_1^1(t),w_2^1(t))$. Note that $w_1^1(t)$ and $w_2^1(t))$ are the unique solutions of linear PBVP (3.2) and (3.3) respectively. The function $w_i^0(t)$ is an upper solution of the PBVP (2.1) − (2.2). Define $p_i(t) = w_i^0(t) - w_i^1(t)$ with $\eta_i = w_i^0(t)$. We observe that

$$\begin{aligned} D^\alpha p_i(t) &= D^\alpha w_i^0(t) - D^\alpha w_i^1(t), \\ D^\alpha p_i(t) &\ge -M_i p_i(t), \end{aligned}$$

and boundary conditions

$$\begin{aligned} t^{1-\alpha}p_i(t)|_{t=0} &= t^{1-\alpha}w_i^0(t)|_{t=0} - t^{1-\alpha}w_i^1(t)|_{t=0} \\ &= t^{1-\alpha}w_i^0(t)|_{t=T} - t^{1-\alpha}w_i^1(t)|_{t=T} \\ &= t^{1-\alpha}p_i(t)|_{t=T}. \end{aligned}$$

Using Lemma 2.2,we get $p_i(t) \ge 0$ implies that

$$w_i^0(t) \ge w_i^1(t) = A[w^0(t),v^0(t)].$$

Now,we prove (II).The operator A is monotone.Let $\bar{\eta} = (\eta_1,\eta_2)$ and $\mu = (\mu_1,\mu_2)$ in $[v^0(t),w^0(t)]$ be such that $\eta_i \le \mu_i$. Suppose that $A[\eta_i,\mu] = u_i = (u_i^1,u_i^2)$ and $A[\eta,\mu_i] = v_i = (v_i^1,v_i^2)$. Consider $p_i(t) = u_i(t) - v_i(t)$ and observe that

$$\begin{aligned} D^\alpha p_i(t) &= D^\alpha u_i(t) - D^\alpha v_i(t) \\ &= f_i(t,\eta_1,\eta_2) - f_i(t,\mu_1,\mu_2) + M_i(\eta_i - u_i) \\ &\quad - M_i(\mu_i - v_i) \\ &\le M_i(\eta_i - u_i) - M_i(\mu_i - v_i) + M_i(\mu_i - \eta_i) \\ &\le -M_i(u_i(t) - v_i(t)), \\ D^\alpha p_i(t) &\le -M_i p_i(t), \end{aligned}$$

and boundary conditions

$$\begin{aligned} t^{1-\alpha}p_i(t)|_{t=0} &= t^{1-\alpha}u_i(t)|_{t=0} - t^{1-\alpha}v_i(t)|_{t=0} \\ &= t^{1-\alpha}u_i|_{t=T} - t^{1-\alpha}v_i(t)|_{t=T} \\ &= t^{1-\alpha}p_i(t)|_{t=T}. \end{aligned}$$

Applying Lemma 2.2,we get $p_i(t) \le 0$ implies that $u_i(t) \le v_i(t)$. Hence $A[\eta_i,\mu] \le A[\eta,\mu_i]$ Thus the operator A possess the monotone property on $[v^0(t),w^0(t)]$. Define the sequences $\{v_i^n\}$ and $\{w_i^n\}$ by $v_i^n = A[v_i^{n-1},w_i^{n-1}]$ and $w_i^n = A[w_i^{n-1},v_i^{n-1}]$. Using (3.6) and (3.7),we obtain

$$v_i^0 \le v_i^1 \le ... \le v_i^n \le ... \le w_i^n ... \le w_i^1 \le w_i^0, \quad i = 1,2. \tag{3.8}$$

Let $P_i = \{v_i^n : n = 1,2,...\}$ and $Q_i = \{w_i^n : n = 1,2,...\}$. We show that the sets P_i and Q_i are relatively compact in $C_{1-\alpha}([0,T]) \cap L_1(0,T)$. For any $\eta_i \in S$ and by definition of lower and upper solution along with one sided Lipschitz condition,we have

$$\begin{aligned} D^\alpha v_i^0 + M_i v_i^0 &\le f_i(t,v_1^0,v_2^0) + M_i v_i^0 \\ &\le f_i(t,\eta_1,\eta_2) + M_i\eta_i \le \\ f_i(t,w_1^0,w_2^0) + M_i w_i^0 &\le D^\alpha w_i^0 + M_i w_i^0. \end{aligned}$$

Let $P_i = \{v_i^n : n = 1,2,...\}, i = 1,2$ and $S \subset C_{1-\alpha}([0,T]) \cap L_1(0,T)$ are bounded sets.Furthermore,the set $\{\sigma_i(t,\eta_1,\eta_2) = f_i(t,\eta_1,\eta_2) + M_i\eta_i | \eta_i \in S\}$ is also a bounded set.Hence there exist constants $B_i, i = 1,2$ such that

$$||\sigma_i(t,v_i^n)|| = \max_{0\le t\le T} |t^{1-\alpha}\sigma_i(t,v_i^n)| \tag{3.9}$$

$$\le B_i \iff |\sigma_i(t,v_i^n)| \le B_i t^{1-\alpha}, t \in (0,T] \tag{3.10}$$

On the other hand $\{v_i^n(t)|n = 1,2,...\}, i = 1,2$ satisfy

$$v_i^n(t) = \Gamma(\alpha)u_{i0}e_\alpha^{(-M_i t)} + \int_0^t e_\alpha^{(-M_i(t-s))}\sigma_i(v_i^{n-1})(s)ds \tag{3.11}$$

where

$$\begin{cases} e_\alpha^{(-M_i t)} = t^{\alpha-1}E_{\alpha,\alpha}(-M_i t^\alpha) \\ u_{i0} = \frac{T^{1-\alpha}}{[1-\Gamma(\alpha)]E_{\alpha,\alpha}(-M_i T^\alpha)} \int_0^t e_\alpha^{(-M_i(t-s))}\sigma_i(v_i^{n-1})(s)ds \end{cases}$$

From the condition (3.9),we have

$$\begin{cases} |\sigma_i(t,\eta_1,\eta_2)| \le B_i t^{1-\alpha}, t \in (0,T], \eta_1,\eta_2 \in P_i \\ |u_{i0}| \le \frac{B_i T^\alpha}{\Gamma(2\alpha)[1-\Gamma(\alpha)E_{\alpha,\alpha}(-M_i T^\alpha)]} \end{cases}$$

Without lose of generality,we assume that $0 \le t_1 \le t_2 \le 1$ and for $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon)$ when $|t_1 - t_2| < \delta$,and since $E_{\alpha,\alpha}(t) \in C[0,T]$, we have

$$|E_{\alpha,\alpha}(-M_i t_1^\alpha) - E_{\alpha,\alpha}(-M_i t_2^\alpha)| < \frac{\varepsilon}{3\Gamma(\alpha)max\{|u_{i0}|, B_i/M_i\}},$$

$$(t_2 - t_1)^\alpha < \frac{\varepsilon\Gamma(2\alpha)}{6B_i\Gamma(\alpha)} \tag{3.12}$$

From above equations,we obtain

$$\begin{aligned}
&|t_1^{1-\alpha} v_i^n(t_1) - t_2^{1-\alpha} v_i^n(t_2)| \qquad (3.13)\\
&= |\Gamma(\alpha) u_{i0}[t_1^{1-\alpha} e_\alpha^{(-M_i t_1)} - t_2^{1-\alpha} e_\alpha^{(-M_i t_2)}]| +\\
&\quad [t_1^{1-\alpha} e_\alpha^{(-M_i t_1)} * \sigma(v_i^{n-1})(t_1) -\\
&\quad t_2^{1-\alpha} e_\alpha^{(-M_i t_1)} * \sigma_i(v_i^{n-1})(t_2)]\\
&\leq \Gamma(\alpha)|u_{i0}||E_{\alpha,\alpha}(-M_i t_1^\alpha) - E_{\alpha,\alpha}(-M_i t_2^\alpha)| +\\
&\quad \frac{L\Gamma(\alpha)}{M_i}|E_{\alpha,\alpha}(-M_i t_1^\alpha) - E_{\alpha,\alpha}(-M_i t_2^\alpha)| +\\
&\quad \frac{2B_i\Gamma(\alpha)}{\Gamma(2\alpha)}(t_2 - t_1)^\alpha\\
&< \varepsilon. \qquad (3.14)
\end{aligned}$$

This implies that P_i is equi-continuous and by the Ascoli-Arzela theorem, we conclude that P_i is relatively compact set of $C_{1-\alpha}([0,T]) \cap L_1(0,T)$. Similarly,we can show that Q_i is relatively compact set of $C_{1-\alpha}([0,T]) \cap L_1(0,T)$.Therefore,the sequences $\{v_1^n, v_2^n\}$ and $\{w_1^n, w_2^n\}$ converge uniformly to (v_1, v_2) and (w_1, w_2) on $[0,T]$ respectively.We have point wise limits

$$\{v_1^n(t), v_2^n(t)\} \to (v_1, v_2) \text{ and } \{w_1^n(t), w_2^n(t)\} \to (w_1, w_2)$$

as $n \to \infty$ on $(0,T]$. Moreover, by (3.8), the limit functions satisfy the following monotone property

$$\begin{aligned}
&v_1^0 \leq v_1^1 \leq ... \leq v_1^n \leq ... \leq v_1 \leq w_1 \leq ... \leq w_1^n ... \leq w_1^1 \leq w_1^0\\
&v_2^0 \leq v_2^1 \leq ... \leq v_2^n \leq ... \leq v_2 \leq w_2 \leq ... \leq w_2^n ... \leq w_2^1 \leq w_2^0
\end{aligned} \qquad (3.15)$$

Now,we prove that (v_1, v_2) and (w_1, w_2) are solutions of PBVP (2.1) - (2.2).We know

$$\sigma_1(t, \eta_1, \eta_2) = f_1(t, \eta_1, \eta_2) + M_1 \eta_1$$

Clearly, the function σ_1 is continuous and monotone non-decreasing and monotone convergence of $\{v_1^n(t)\}$ to $v_1(t)$ as $n \to \infty$ on $(0,T]$ implies that $\sigma_1(v_1^n)(t)$ converges to $\sigma_1(v_1)(t), t \in (0,T]$. Let $n \to \infty$ in (3.11) and apply the dominated convergence theorem,we observe that $v_1(t)$ satisfies the integral equation

$$\begin{aligned}
&v_1(t)\\
&= N_1 \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}\big(-M_1(T-s)^\alpha\big)\sigma_1(v_1)(s)ds\\
&+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-M_1(t-s)^\alpha\big)\sigma_1(v_1)(s)ds,
\end{aligned} \qquad (3.16)$$

where $N_1 = \frac{T^{1-\alpha}\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(-M_1 t^\alpha)}{[1-\Gamma(\alpha)]E_{\alpha,\alpha}(-M_1 T^\alpha)}$. We conclude that $v_1(t)$ is an integral representation of the solution to problem (3.2), i.e.$v_1(t)$ is an integral representation of the solution to problem (2.1)-(2.2). By the assumption of the function f_1 and Lemma 2.1,$v_1(t)$ is the classical solution of the PBVP (2.1)-(2.2). This proves that the lower sequence $\{v_1^n(t)\}$ converges to a solution $v_1(t)$ of problem (2.1)-(2.2). Further,we know

$$\sigma_2(t, \eta_1, \eta_2) = f_2(t, \eta_1, \eta_2) + M_2 \eta_2$$

Clearly, the function σ_2 is continuous and monotone non-decreasing and monotone convergence of $\{v_2^n(t)\}$ to $v_2(t)$ as $n \to \infty$ on $(0,T]$, implies that $\sigma_2(v_2^n)(t)$ converges to $\sigma_2(v_2)(t), t \in (0,T]$. Let $n \to \infty$ in (3.11) and apply the dominated convergence theorem,we observe that $v_2(t)$ satisfies the integral equation

$$\begin{aligned}
&v_2\\
&= N_2 \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}\big(-M_2(T-s)^\alpha\big)\sigma_2(v_2)ds\\
&+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-M_2(t-s)^\alpha\big)\sigma_2(v_2)(s)ds,
\end{aligned} \qquad (3.17)$$

where $N_2 = \frac{T^{1-\alpha}\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(-M_2 t^\alpha)}{[1-\Gamma(\alpha)]E_{\alpha,\alpha}(-M_2 T^\alpha)}$. We conclude that $v_2(t)$ is an integral representation of the solution to problem (3.3), i.e.$v_2(t)$ is an integral representation of the solution to problem (2.1)-(2.2).By the assumption of the function f_2 and lemma 2.1,$v_2(t)$ is the classical solution of the PBVP (2.1)-(2.2).This proves that the lower sequence $\{v_2^n(t)\}$ converges to a solution $v_2(t)$ of the problem (2.1)-(2.2). Similarly,we can prove that upper sequence $\{w_1^n, w_2^n\}$ converge uniformly to a solution (w_1, w_2) of periodic boundary value problems (2.1) - (2.2) and satisfies the relation $v_1(t) \leq w_1(t)$ and $v_2(t) \leq w_2(t) t \in (0,T]$. It follows that relations (3.1) hold as well as (v_1, v_2) and (w_1, w_2) are minimal and maximal solutions of the PBVP (2.1)-(2.2) on the order interval S respectively.

Finally, if condition (2.9) holds, then $v_i(t) = w_i(t), i = 1,2$ is a unique solution of the PBVP (2.1)-(2.2). It is sufficient to prove $v_i(t) \geq w_i(t), t \in (0,T]$, since we have $v_i(t) \leq w_i(t)$. We observe that the function $u_i(t) = v_i(t) - w_i(t)$ satisfies the relations

$$\begin{aligned}
D^\alpha u_i(t) + M_1 u_i(t) &= -[f_i(t, w_1, w_2) - f_i(t, v_i, v_2)]\\
&\quad + M_1(v_i(t) - w_i(t)) \geq 0\\
t^{1-\alpha} u_i(t)|_{t=0} &= t^{1-\alpha} u_i(t)|_{t=T}, \quad i = 1,2, \quad t \in (0,T]
\end{aligned}$$

Then Lemma 2.3 implies that $u_i(t) \geq 0, t \in (0,T]$, which proves $v_i(t) \geq w_i(t), t \in (0,T]$ and hence we obtain that $v_i(t) = w_i(t)$ is a unique solution of the PBVP (2.1)-(2.2). This completes the proof.

Corollary 3.2. *Assume that*

- **(i)** $v^0 = (v_1^0, v_2^0)$ *and* $w^0 = (w_1^0, w_2^0) \in C_{1-\alpha}([0,T]) \cap L_1(0,T)$ *are lower and upper solutions of the PBVP* (2.1)−(2.2), *such that order relation* (2.3) *holds,*
- **(ii)** *function* $f_i(t, u_1, u_2) \in C([0,T) \times \mathbb{R}^2, \mathbb{R})$ *satisfies Lipschitz condition* (2.6) *,*

(iii) *functions f_1 and f_2 are quasimonotone nondecreasing.*

Then the PBVP (2.1) – (2.2) *has unique solution in the order interval.*

Proof. Observe that

$$-K_i(u_i - u_i^*) \leq f_i(t,u_1,u_2) - f_i(t,u_1^*,u_2^*) \qquad (3.18)$$
$$\leq K_i(u_i - u_i^*), \qquad (3.19)$$

for $v_i^0 \leq u_i^* \leq u_i \leq w_i^0$, which follows from (2.6) i.e.Lipschitz conditions (2.4) and (2.5) hold with $K_i = M_i$. Then the Theorem 3.1 implies that the problem (2.1)-(2.2) has one and only one solution in the ordered interval. □

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