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# Extended Darboux frame field in Minkowski space-time $\mathbb{E}_1^4$

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**Abstract**

In this paper, we extend the Darboux frame field along a non-null curve lying on an orientable non-null hypersurface into Minkowski space-time $\mathbb{E}_1^4$ in two cases which the curvature vector and the normal vector of the hypersurface are linearly independent or dependent. Then the normal curvature, the geodesic curvature(s), and the geodesic torsion(s) of the hypersurface are given when the curve lying on the hypersurface is an asymptotic or geodesic curve.

**Keywords**

Curves on hypersurface, Darboux frame field, curvatures, Minkowski space-time.

**AMS Subject Classification**

53A04, 53A07.

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## 1. Introduction

The most known frame fields of differential geometry are the Frenet frame field and the Darboux frame field, and these frame fields occupy an important place in the study of curves and surfaces. There are many studies about generalization of the Frenet frame in higher dimensional spaces in the literature, but there is no study about generalization of Darboux frame to higher dimensional spaces, except for the article given by Düldül et al., [2].

As we know, in differential geometry the Darboux frame field along a curve lying on a surface denoted by $\{\mathsf{T},\mathsf{V},\mathsf{N}\}$, where $\mathsf{T}$ is the unit tangent vector of the curve, $\mathsf{N}$ is the surface normal restricted to the curve, and $\mathsf{V}=\mathsf{T}\times\mathsf{N}$. The normal curvature, the geodesic curvature and the geodesic torsion of the surface can be calculated by means of the derivative equations of this frame field, [1,5,7,8,10].

In this paper, similar to given in Euclidean 4-space we construct a frame field along a non-null curve lying on an orientable non-null hypersurface in Minkowski space-time $\mathbb{E}_1^4$ and call as "extended Darboux frame field" or shortly "ED-frame field". After, we obtain the derivative equations of this frame field and give the normal curvature, the geodesic curvature(s) and the geodesic torsion(s) of the hypersurface.

We hope that this new frame field will provide the basis for future works to be done in this area.

## 2. Preliminaries

The Minkowski space-time $\mathbb{E}_1^4$ is the Euclidean space $\mathbb{E}^4$ provided with the indefinite flat metric given by

$$\langle , \rangle = -dx_1^2+dx_2^2+dx_3^2+dx_4^2,$$

where $(x_1,x_2,x_3,x_4)$ is a rectangular coordinate system of $\mathbb{E}_1^4$. Since the above metric is an indefinite metric, we know that a vector in $\mathbb{E}_1^4$ can have one of the three causal characters: The arbitrary vector $\mathsf{v}$ is called a spacelike, a timelike, and a null or lightlike vector if $\langle \mathsf{v},\mathsf{v}\rangle>0$ or $\mathsf{v}=0$, $\langle \mathsf{v},\mathsf{v}\rangle<0$, and $\langle \mathsf{v},\mathsf{v}\rangle=0$ for $\mathsf{v}\neq 0$, respectively. The norm of a vector $\mathsf{v}$ is defined by $\|\mathsf{v}\|=\sqrt{|\langle \mathsf{v},\mathsf{v}\rangle|}$ and two vectors $\mathsf{v}$ and $\mathsf{w}$ are called orthogonal if $\langle \mathsf{v},\mathsf{w}\rangle=0$. A vector $\mathsf{v}$ satisfying $\langle \mathsf{v},\mathsf{v}\rangle=\pm1$ is called a unit vector. For an arbitrary curve $\alpha(s)$ in $\mathbb{E}_1^4$, the curve is called a spacelike, a timelike and a null or lightlike curve, if all of its velocity vectors $\alpha'(s)$ are spacelike, timelike and null or lightlike, respectively, [6].

A hypersurface in the Minkowski space-time $\mathbb{E}_1^4$ is called a spacelike or a timelike hypersurface if the induced metric on the hypersurface is a positive definite Riemannian metric or a Lorentzian metric, respectively. The normal vector on the spacelike or the timelike hypersurface is a timelike or a spacelike vector, respectively.

The ternary (or vector) product of the vectors $\mathsf{u} = \sum_{i=1}^{4} u_i \mathsf{e}_i$, $\mathsf{v} = \sum_{i=1}^{4} v_i \mathsf{e}_i$, and $\mathsf{w} = \sum_{i=1}^{4} w_i \mathsf{e}_i$ in $\mathbb{E}_1^4$ is defined by

$$\mathsf{u} \otimes \mathsf{v} \otimes \mathsf{w} = -\begin{vmatrix} -\mathsf{e}_1 & \mathsf{e}_2 & \mathsf{e}_3 & \mathsf{e}_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix},$$

where $\{\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3, \mathsf{e}_4\}$ is the standard basis of $\mathbb{E}_1^4$. The equations;

$$\mathsf{e}_1 \otimes \mathsf{e}_2 \otimes \mathsf{e}_3 = \mathsf{e}_4, \quad \mathsf{e}_2 \otimes \mathsf{e}_3 \otimes \mathsf{e}_4 = \mathsf{e}_1,$$

$$\mathsf{e}_3 \otimes \mathsf{e}_4 \otimes \mathsf{e}_1 = \mathsf{e}_2, \quad \mathsf{e}_4 \otimes \mathsf{e}_1 \otimes \mathsf{e}_2 = -\mathsf{e}_3$$

are satisfied for the vectors $\mathsf{e}_i$, $1 \leq i \leq 4$, [11].

Let $M$ be an orientable non-null hypersurface and $\alpha : I \subset \mathbb{R} \to M$ be a unit speed non-null curve in $\mathbb{E}_1^4$. Let $\{\mathsf{t}, \mathsf{n}, \mathsf{b}_1, \mathsf{b}_2\}$ be the moving Frenet frame along $\alpha$. Then $\mathsf{t}$, $\mathsf{n}$, $\mathsf{b}_1$, and $\mathsf{b}_2$ are the unit tangent, the principal normal, the first binormal, and the second binormal vector fields, respectively. If $k_1$, $k_2$, and $k_3$ are the curvature functions of the unit speed non-null curve $\alpha$, then for the non-null frame vectors we have the following Frenet equations:

$$\begin{cases} \mathsf{t}' &= \varepsilon_{\mathsf{n}} k_1 \mathsf{n}, \\ \mathsf{n}' &= -\varepsilon_{\mathsf{t}} k_1 \mathsf{t} + \varepsilon_{\mathsf{b}_1} k_2 \mathsf{b}_1, \\ \mathsf{b}_1' &= -\varepsilon_{\mathsf{n}} k_2 \mathsf{n} - \varepsilon_{\mathsf{t}} \varepsilon_{\mathsf{n}} \varepsilon_{\mathsf{b}_1} k_3 \mathsf{b}_2, \\ \mathsf{b}_2' &= -\varepsilon_{\mathsf{b}_1} k_3 \mathsf{b}_1, \end{cases}$$

where $\varepsilon_{\mathsf{t}} = \langle \mathsf{t}, \mathsf{t} \rangle$, $\varepsilon_{\mathsf{n}} = \langle \mathsf{n}, \mathsf{n} \rangle$, $\varepsilon_{\mathsf{b}_1} = \langle \mathsf{b}_1, \mathsf{b}_1 \rangle$, $\varepsilon_{\mathsf{b}_2} = \langle \mathsf{b}_2, \mathsf{b}_2 \rangle$ whereby $\varepsilon_{\mathsf{t}}, \varepsilon_{\mathsf{n}}, \varepsilon_{\mathsf{b}_1}, \varepsilon_{\mathsf{b}_2} \in \{-1, 1\}$, $1 \leq i \leq 4$ and $\varepsilon_{\mathsf{t}} \varepsilon_{\mathsf{n}} \varepsilon_{\mathsf{b}_1} \varepsilon_{\mathsf{b}_2} = -1$, [4].

**Definition 2.1.** **i)** *Let $\mathscr{A} = [a_{ij}] \in \mathbb{R}_n^m$ and $\mathscr{B} = [b_{jk}] \in \mathbb{R}_p^n$ be two matrices, where $\mathbb{R}_n^m$ and $\mathbb{R}_p^n$ are the real vector spaces with the matrix addition and the scalar-matrix multiplication. Then, the Lorentzian matrix multiplication or shortly L-multiplication of the matrices $\mathscr{A}$ and $\mathscr{B}$ is defined by*

$$\mathscr{A} \cdot_L \mathscr{B} = \Big[ -a_{i1} b_{1k} + \sum_{j=1}^{n} a_{ij} b_{jk} \Big].$$

*The real vector space $\mathbb{R}_n^m$ with L-multiplication is denoted by $\mathbb{L}_n^m$.*

**ii)** *The matrix*

$$\mathscr{I}_n = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

*is called $n \times n$ L-identity matrix according to L-multiplication. For every $\mathscr{A} \in \mathbb{L}_n^m$, $\mathscr{I}_m \cdot_L \mathscr{A} = \mathscr{A} \cdot_L \mathscr{I}_n = \mathscr{A}$.*

**iii)** *An $n \times n$ matrix $\mathscr{A}$ is called L-invertible if there exists an $n \times n$ matrix $\mathscr{B}$ such that $\mathscr{A} \cdot_L \mathscr{B} = \mathscr{B} \cdot_L \mathscr{A} = \mathscr{I}_n$. Then $\mathscr{B}$ is called the L-inversible of $\mathscr{A}$ and is denoted by $\mathscr{A}^{-1}$.*

**iv)** *The matrix $\mathscr{A}^T = [a_{ji}] \in \mathbb{L}_m^n$ is called the transpose of the matrix $\mathscr{A} = [a_{ij}] \in \mathbb{L}_n^m$.*

**v)** *The matrix $\mathscr{A} \in \mathbb{L}_n^n$ is called L-orthogonal matrix if $\mathscr{A}^{-1} = \mathscr{A}^T$,[3].*

## 3. Extended Darboux frame field in $\mathbb{E}_1^4$

Let $M$ be an orientable non-null hypersurface, $N$ be its non-null unit normal vector field in $\mathbb{E}_1^4$, and $\alpha(s)$ be a non-null Frenet curve parametrized by arc-length parameter $s$ lying on $M$. If the non-null unit tangent vector field of $\alpha$ is denoted by $\mathsf{T}$, and the non-null unit normal vector field of $M$ restricted to $\alpha$ is denoted by $\mathsf{N}$, we have $\alpha'(s) = \mathsf{T}(s)$ and $N(\alpha(s)) = \mathsf{N}(s)$.

As in Euclidean 4-space $\mathbb{E}^4$ [2], the extended Darboux frame can be constructed in two different cases in Minkowski space-time $\mathbb{E}_1^4$ according to whether the set $\{\mathsf{N}, \mathsf{T}, \alpha''\}$ is linearly independent or linearly dependent. Let us denote the ED-frame field is the first kind and the second kind if the set $\{\mathsf{N}, \mathsf{T}, \alpha''\}$ is linearly independent and linearly dependent, respectively.

Now, let us construct the ED-frame field of the first kind in Case 1 and the second kind in Case 2 along the non-null Frenet curve $\alpha$ in $\mathbb{E}_1^4$. As explained in [2], using the Gram-Schmidt orthonormalization method, we have

$$\mathsf{E} = \frac{\alpha'' - \langle \alpha'', \mathsf{N} \rangle \mathsf{N}}{||\alpha'' - \langle \alpha'', \mathsf{N} \rangle \mathsf{N}||}$$

for Case 1 and

$$\mathsf{E} = \frac{\alpha''' - \langle \alpha''', \mathsf{N} \rangle \mathsf{N} - \langle \alpha''', \mathsf{T} \rangle \mathsf{T}}{||\alpha''' - \langle \alpha''', \mathsf{N} \rangle \mathsf{N} - \langle \alpha''', \mathsf{T} \rangle \mathsf{T}||}$$

for Case 2. If we define $-\mathsf{D} = \mathsf{N} \otimes \mathsf{T} \otimes \mathsf{E}$ for both cases, we obtain the orthonormal frame field $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$ another from Frenet frame field $\{\mathsf{T}, \mathsf{n}, \mathsf{b}_1, \mathsf{b}_2\}$ along the curve $\alpha$. With respect to the orthonormal frame $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$, the vector fields $\mathsf{T}', \mathsf{E}', \mathsf{D}', \mathsf{N}'$ have the following decompositions:

$$\begin{aligned} \mathsf{T}' &= \varepsilon_1 \langle \mathsf{T}', \mathsf{T} \rangle \mathsf{T} + \varepsilon_2 \langle \mathsf{T}', \mathsf{E} \rangle \mathsf{E} + \varepsilon_3 \langle \mathsf{T}', \mathsf{D} \rangle \mathsf{D} \\ &\quad + \varepsilon_4 \langle \mathsf{T}', \mathsf{N} \rangle \mathsf{N}, \\ \mathsf{E}' &= \varepsilon_1 \langle \mathsf{E}', \mathsf{T} \rangle \mathsf{T} + \varepsilon_2 \langle \mathsf{E}', \mathsf{E} \rangle \mathsf{E} + \varepsilon_3 \langle \mathsf{E}', \mathsf{D} \rangle \mathsf{D} \\ &\quad + \varepsilon_4 \langle \mathsf{E}', \mathsf{N} \rangle \mathsf{N}, \\ \mathsf{D}' &= \varepsilon_1 \langle \mathsf{D}', \mathsf{T} \rangle \mathsf{T} + \varepsilon_2 \langle \mathsf{D}', \mathsf{E} \rangle \mathsf{E} + \varepsilon_3 \langle \mathsf{D}', \mathsf{D} \rangle \mathsf{D} \\ &\quad + \varepsilon_4 \langle \mathsf{D}', \mathsf{N} \rangle \mathsf{N}, \\ \mathsf{N}' &= \varepsilon_1 \langle \mathsf{N}', \mathsf{T} \rangle \mathsf{T} + \varepsilon_2 \langle \mathsf{N}', \mathsf{E} \rangle \mathsf{E} + \varepsilon_3 \langle \mathsf{N}', \mathsf{D} \rangle \mathsf{D} \\ &\quad + \varepsilon_4 \langle \mathsf{N}', \mathsf{N} \rangle \mathsf{N}, \end{aligned}$$

where $\varepsilon_1 = \langle \mathsf{T}, \mathsf{T} \rangle$, $\varepsilon_2 = \langle \mathsf{E}, \mathsf{E} \rangle$, $\varepsilon_3 = \langle \mathsf{D}, \mathsf{D} \rangle$, $\varepsilon_4 = \langle \mathsf{N}, \mathsf{N} \rangle$ whereby $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{-1, 1\}$. Besides, when $\varepsilon_i = -1$, then $\varepsilon_j = 1$ for all $j \neq i$, $1 \leq i, j \leq 4$ and $\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 = -1$.

Similar operations are performed in [2], we obtain $\langle \mathsf{T}', \mathsf{D}\rangle = 0$ for Case 1 and $\langle \mathsf{T}', \mathsf{E}\rangle = \langle \mathsf{T}', \mathsf{D}\rangle = 0$ and $\langle \mathsf{N}', \mathsf{D}\rangle = 0$ for Case 2. If we use $\langle \mathsf{T}', \mathsf{N}\rangle = \kappa_n$ and denote $\langle \mathsf{E}', \mathsf{N}\rangle = \tau_g^1$, $\langle \mathsf{D}', \mathsf{N}\rangle = \tau_g^2$, $\langle \mathsf{T}', \mathsf{E}\rangle = \kappa_g^1$, $\langle \mathsf{E}', \mathsf{D}\rangle = \kappa_g^2$, where $\kappa_g^i$ and $\tau_g^i$ are the geodesic curvature and the geodesic torsion of order $i$, $(i = 1, 2)$, respectively, then the differential equations for ED-frame field have the form for Case 1:

$$\begin{pmatrix} \mathsf{T}' \\ \mathsf{E}' \\ \mathsf{D}' \\ \mathsf{N}' \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_2\kappa_g^1 & 0 & \varepsilon_4\kappa_n \\ -\varepsilon_1\kappa_g^1 & 0 & \varepsilon_3\kappa_g^2 & \varepsilon_4\tau_g^1 \\ 0 & -\varepsilon_2\kappa_g^2 & 0 & \varepsilon_4\tau_g^2 \\ -\varepsilon_1\kappa_n & -\varepsilon_2\tau_g^1 & -\varepsilon_3\tau_g^2 & 0 \end{pmatrix} \begin{pmatrix} \mathsf{T} \\ \mathsf{E} \\ \mathsf{D} \\ \mathsf{N} \end{pmatrix},$$

and for Case 2:

$$\begin{pmatrix} \mathsf{T}' \\ \mathsf{E}' \\ \mathsf{D}' \\ \mathsf{N}' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \varepsilon_4\kappa_n \\ 0 & 0 & \varepsilon_3\kappa_g^2 & \varepsilon_4\tau_g^1 \\ 0 & -\varepsilon_2\kappa_g^2 & 0 & 0 \\ -\varepsilon_1\kappa_n & -\varepsilon_2\tau_g^1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathsf{T} \\ \mathsf{E} \\ \mathsf{D} \\ \mathsf{N} \end{pmatrix}.$$

Now, let us consider the normal curvature, the geodesic curvatures, and the geodesic torsions of the curve $\alpha$ and let us give the geometrical results of these real valued functions. In both cases, we know that $\kappa_n = \langle \mathsf{T}', \mathsf{N}\rangle$ is the normal curvature of the hypersurface in the direction of the tangent vector $\mathsf{T}$. Therefore, $\alpha$ is an asymptotic curve if and only if $\kappa_n = 0$ along $\alpha$.

**Theorem 3.1.** *Let us consider a unit speed non-null curve $\alpha$ on an orientable non-null hypersurface $M$ in Minkowski space-time $\mathbb{E}_1^4$. Let $M_1$ and $M_2$ be the non-null hyperplanes at $\alpha(s_0) \in M$ determined by $\{\mathsf{T}(s_0), \mathsf{D}(s_0), \mathsf{N}(s_0)\}$ and $\{\mathsf{T}(s_0), \mathsf{E}(s_0), \mathsf{N}(s_0)\}$, respectively. Denoting the transversal intersection curve of $M_1$, $M_2$, and $M$ with $\beta$, then the first curvature $k_1^\beta(s_0)$ of $\beta$ at the point $\beta(s_0)$ is given by $k_1^\beta(s_0) = |\kappa_n(s_0)|$, where $\kappa_n$ is the normal curvature of the hypersurface $M$ in the direction of* $\mathsf{T}$.

*Proof.* According to the transversal intersection of three hypersurfaces, $\mathsf{T}$ is the tangent vector of the intersection curve $\beta$. Using the similar calculations in [9], since the normal vectors of the hypersurfaces are orthogonal at the point $\beta(s_0)$, for the first curvature of $\beta$ at $\beta(s_0)$ we find

$$k_1^\beta(s_0) = \sqrt{|\varepsilon_2(\kappa_n^1)^2(s_0) + \varepsilon_3(\kappa_n^2)^2(s_0) + \varepsilon_4(\kappa_n^3)^2(s_0)|},$$

where $\kappa_n^1 = \langle \mathsf{T}', \mathsf{E}\rangle$, $\kappa_n^2 = \langle \mathsf{T}', \mathsf{D}\rangle$, $\kappa_n^3 = \langle \mathsf{T}', \mathsf{N}\rangle$. Then, we obtain

$$k_1^\beta(s_0) = \sqrt{|\varepsilon_4(\kappa_n)^2(s_0)|}.$$

Since $\varepsilon_4 = \pm 1$, we get $k_1^\beta(s_0) = |\kappa_n(s_0)|$. □

**Theorem 3.2.** *Let $\alpha$ be a unit speed non-null curve on an orientable non-null hypersurface $M$ in $\mathbb{E}_1^4$. Let us denote the non-null orthogonal projection curve of $\alpha$ onto the non-null tangent hyperplane at $\alpha(s_0)$ with $\beta$. Then the first curvature $k_1^\beta(s_0)$ of the projection curve $\beta$ at the point $\beta(s_0)$ is equal to $\varepsilon_\mathsf{n}\kappa_g^1(s_0)$, where $\kappa_g^1$ is the geodesic curvature of order 1 of $M$.*

*Proof.* Since $\beta$ is the orthogonal projection curve onto the tangent hyperplane at $\alpha(s_0)$ of $\alpha$, we can write

$$\beta(s) = \alpha(s) - \langle \alpha(s) - \alpha(s_0), \mathsf{N}(s_0)\rangle\, \mathsf{N}(s_0).$$

If we differentiate both sides of this equation three times according to $s$, we find

$$\beta'(s_0) = \alpha'(s_0) = \mathsf{T}(s_0),$$

$$\beta''(s_0) = \mathsf{T}'(s_0) = \varepsilon_2\kappa_g^1(s_0)\mathsf{E}(s_0),$$

$$\begin{aligned}\beta'''(s_0) &= \left\{-\varepsilon_1\varepsilon_2(\kappa_g^1)^2(s_0) - \varepsilon_1\varepsilon_4(\kappa_n)^2(s_0)\right\}\mathsf{T}(s_0) \\ &\quad + \left\{\varepsilon_2(\kappa_g^1)'(s_0) - \varepsilon_2\varepsilon_4\kappa_n(s_0)\tau_g^1(s_0)\right\}\mathsf{E}(s_0) \\ &\quad + \left\{\varepsilon_2\varepsilon_3\kappa_g^1(s_0)\kappa_g^2(s_0) - \varepsilon_3\varepsilon_4\kappa_n(s_0)\tau_g^2(s_0)\right\}\mathsf{D}(s_0)\end{aligned}$$

at the point $\beta(s_0) = \alpha(s_0)$. Then, we obtain

$$\mathsf{b}_2(s_0) = \varepsilon_{\mathsf{b}_1}\frac{\beta'(s_0)\otimes\beta''(s_0)\otimes\beta'''(s_0)}{||\beta'(s_0)\otimes\beta''(s_0)\otimes\beta'''(s_0)||} = (0,0,0,\varepsilon_{\mathsf{b}_1}\varepsilon_2),$$

$$\mathsf{b}_1(s_0) = -\varepsilon_\mathsf{n}\frac{\mathsf{b}_2(s_0)\otimes\beta'(s_0)\otimes\beta''(s_0)}{||\mathsf{b}_2(s_0)\otimes\beta'(s_0)\otimes\beta''(s_0)||} = (0,0,\varepsilon_\mathsf{n}\varepsilon_{\mathsf{b}_1},0),$$

$$\mathsf{n}(s_0) = \frac{\mathsf{b}_1(s_0)\otimes\mathsf{b}_2(s_0)\otimes\beta'(s_0)}{||\mathsf{b}_1(s_0)\otimes\mathsf{b}_2(s_0)\otimes\beta'(s_0)||} = (0,\varepsilon_\mathsf{n}\varepsilon_2,0,0),$$

and

$$k_1^\beta(s_0) = \frac{\langle \mathsf{n}(s_0), \beta''(s_0)\rangle}{||\beta'(s_0)||^2} = \varepsilon_\mathsf{n}\kappa_g^1(s_0).$$

□

**Theorem 3.3.** *Let $\alpha$ be a unit speed non-null asymptotic curve on an orientable non-null hypersurface $M$ in $\mathbb{E}_1^4$. Let us denote the non-null orthogonal projection curve of $\alpha$ onto the non-null hyperplane determined by $\{\mathsf{T}(s_0), \mathsf{E}(s_0), \mathsf{N}(s_0)\}$ at $\alpha(s_0)$ with $\gamma$. Then the first curvature $k_1^\gamma(s_0)$ of the projection curve $\gamma$ at the point $\gamma(s_0)$ is given by $k_1^\gamma(s_0) = \varepsilon_\mathsf{n}\kappa_g^1(s_0)$.*

*Proof.* If we write

$$\gamma(s) = \alpha(s) - \langle \alpha(s) - \alpha(s_0), \mathsf{D}(s_0)\rangle\, \mathsf{D}(s_0)$$

and do the similar calculations at the proof of Theorem 3.2, we find the desired result. □

Now, let us take into consideration the non-null Frenet frame $\{\mathsf{T}, \mathsf{n}, \mathsf{b}_1, \mathsf{b}_2\}$ along the non-null curve $\alpha$. Since $\mathsf{n}$, $\mathsf{b}_1$, $\mathsf{b}_2$, $\mathsf{E}$, $\mathsf{D}$, $\mathsf{N}$ are orthogonal to $\mathsf{T}$, we can write

$$\mathrm{Y} = \mathscr{A}\,._L\mathrm{X}, \tag{3.1}$$

where

$$\mathrm{Y} = \begin{pmatrix} \mathsf{n} \\ \mathsf{b}_1 \\ \mathsf{b}_2 \end{pmatrix}, \quad \mathrm{X} = \begin{pmatrix} \mathsf{E} \\ \mathsf{D} \\ \mathsf{N} \end{pmatrix},$$

$$\mathscr{A} = \begin{pmatrix} \sinh\phi_1 & \sinh\phi_2 & \sinh\phi_3 \\ \sinh\psi_1 & \sinh\psi_2 & \sinh\psi_3 \\ \sinh\theta_1 & \sinh\theta_2 & \sinh\theta_3 \end{pmatrix} \tag{3.2}$$

Since the matrix $\mathscr{A}$ is an $L-$*orthogonal* matrix, we may write

$$\mathscr{I}_3 \cdot_L \mathrm{X} = \mathscr{A}^T \cdot_L \mathrm{Y},$$

where

$$\mathscr{I}_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3}.$$

Then we have

$$\begin{aligned} \mathsf{E} &= -\sinh\phi_1 \mathsf{n} + \sinh\psi_1 \mathsf{b}_1 + \sinh\theta_1 \mathsf{b}_2, \\ \mathsf{D} &= -\sinh\phi_2 \mathsf{n} + \sinh\psi_2 \mathsf{b}_1 + \sinh\theta_2 \mathsf{b}_2, \\ \mathsf{N} &= -\sinh\phi_3 \mathsf{n} + \sinh\psi_3 \mathsf{b}_1 + \sinh\theta_3 \mathsf{b}_2. \end{aligned} \tag{3.3}$$

Therefore, if we use Frenet formula $\mathsf{T}' = \varepsilon_{\mathsf{n}} k_1 \mathsf{n}$ and (3.3), we get

$$\kappa_g^1 = \langle \mathsf{T}', \mathsf{E} \rangle = -k_1 \sinh\phi_1 \tag{3.4}$$

and

$$\kappa_n = \langle \mathsf{T}', \mathsf{N} \rangle = -k_1 \sinh\phi_3, \tag{3.5}$$

where $k_1$ is the first curvature of $\alpha$.

**Theorem 3.4.** *Let $\alpha$ be a unit speed non-null curve with arc-length parameter $s$ on an orientable non-null hypersurface $M$ in $\mathbb{E}_1^4$. If $\alpha$ is a geodesic curve on $M$, then*

$$\kappa_n = -k_1, \quad \kappa_g^2 = -k_3, \quad \tau_g^1 = -k_2,$$

*where $k_i\,(i=1,2,3)$ denotes the i-th curvature functions of $\alpha$.*

*Proof.* Since $\alpha$ is a geodesic curve, by the proper orientation of the hypersurface with $\mathsf{N}(s) = \mathsf{n}(s)$, Case 2 is valid. In this case, $\mathsf{E}$ and $\mathsf{D}$ coincide with $\mathsf{b}_1$ and $\mathsf{b}_2$, respectively. So, the frame $\{\mathsf{T},\mathsf{E},\mathsf{D},\mathsf{N}\}$ coincides with the frame $\{\mathsf{T},\mathsf{b}_1,\mathsf{b}_2,\mathsf{n}\}$. Using (3.1) and (3.2), since

$$\begin{aligned} &\langle \mathsf{n},\mathsf{E}\rangle = 0, && \langle \mathsf{n},\mathsf{D}\rangle = 0, && \langle \mathsf{n},\mathsf{N}\rangle = \varepsilon_4, \\ &\langle \mathsf{b}_1,\mathsf{E}\rangle = \varepsilon_2, && \langle \mathsf{b}_1,\mathsf{D}\rangle = 0, && \langle \mathsf{b}_1,\mathsf{N}\rangle = 0, \\ &\langle \mathsf{b}_2,\mathsf{E}\rangle = 0, && \langle \mathsf{b}_2,\mathsf{D}\rangle = \varepsilon_3, && \langle \mathsf{b}_2,\mathsf{N}\rangle = 0 \end{aligned}$$

we obtain

$$\phi_1(s) = \phi_2(s) = \psi_2(s) = \psi_3(s) = \theta_1(s) = \theta_3(s) = 0 \tag{3.6}$$

and

$$\sinh\phi_3(s) = -\sinh\psi_1(s) = \sinh\theta_2(s) = 1 \tag{3.7}$$

along $\alpha$. Substituting (3.7) into (3.5), we find

$$\kappa_n = -k_1.$$

On the other hand, since

$$\begin{aligned} \mathsf{E}' = {} & k_1 \varepsilon_{\mathsf{T}} \sinh\phi_1 \mathsf{T} + \left(-\phi_1' \cosh\phi_1 - k_2 \varepsilon_{\mathsf{n}} \sinh\psi_1\right)\mathsf{n} \\ & + \left(\psi_1' \cosh\psi_1 - k_2 \varepsilon_{\mathsf{b}_1} \sinh\phi_1 - k_3 \varepsilon_{\mathsf{b}_1} \sinh\theta_1\right)\mathsf{b}_1 \\ & + \left(\theta_1' \cosh\theta_1 - k_3 \varepsilon_{\mathsf{T}} \varepsilon_{\mathsf{n}} \varepsilon_{\mathsf{b}_1} \sinh\psi_1\right)\mathsf{b}_2 \end{aligned}$$

we obtain

$$\begin{aligned} \kappa_g^2 = \langle \mathsf{E}', \mathsf{D}\rangle = {} & \phi_1' \cosh\phi_1 \sinh\phi_2 \varepsilon_{\mathsf{n}} \\ & + \psi_1' \cosh\psi_1 \sinh\psi_2 \varepsilon_{\mathsf{b}_1} + \theta_1' \cosh\theta_1 \sinh\theta_2 \varepsilon_{\mathsf{b}_2} \\ & + k_2\left(\sinh\psi_1 \sinh\phi_2 - \sinh\phi_1 \sinh\psi_2\right) \\ & + k_3\left(\sinh\psi_1 \sinh\theta_2 - \sinh\theta_1 \sinh\psi_2\right) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \tau_g^1 = \langle \mathsf{E}', \mathsf{N}\rangle = {} & \phi_1' \cosh\phi_1 \sinh\phi_3 \varepsilon_{\mathsf{n}} + \psi_1' \cosh\psi_1 \sinh\psi_3 \varepsilon_{\mathsf{b}_1} \\ & + \theta_1' \cosh\theta_1 \sinh\theta_3 \varepsilon_{\mathsf{b}_2} + k_2\left(\sinh\psi_1 \sinh\phi_3 - \sinh\phi_1 \sinh\psi_3\right) \\ & + k_3\left(\sinh\psi_1 \sinh\theta_3 - \sinh\theta_1 \sinh\psi_3\right). \end{aligned} \tag{3.9}$$

Substituting (3.6) and (3.7) into (3.8) and (3.9), we get

$$\kappa_g^2 = -k_3$$

and

$$\tau_g^1 = -k_2.$$

□

**Theorem 3.5.** *Let $\alpha$ be a unit speed non-null curve with arc-length parameter $s$ on an orientable non-null hypersurface $M$ in $\mathbb{E}_1^4$. If $\alpha$ is an asymptotic curve on $M$, then*

$$\kappa_g^1 = k_1, \qquad \kappa_g^2 = k_2 \sinh\psi_2, \qquad \tau_g^1 = k_2 \sinh\psi_3,$$

$$\begin{aligned} \tau_g^2 = {} & \psi_2' \cosh\psi_2 \sinh\psi_3 \varepsilon_{\mathsf{b}_1} + \theta_2' \cosh\theta_2 \sinh\theta_3 \varepsilon_{\mathsf{b}_2} + \\ & k_3\left(\sinh\psi_2 \sinh\theta_3 - \sinh\theta_2 \sinh\psi_3\right), \end{aligned}$$

*where $k_i\,(i=1,2,3)$ denotes the i-th curvature functions of $\alpha$.*

*Proof.* Since $\alpha$ is an asymptotic curve, then $\kappa_n = 0$. In this case, $\mathsf{T}' = \varepsilon_{\mathsf{n}} k_1 \mathsf{n} = \varepsilon_2 \kappa_g^1 \mathsf{E}$, i.e. $\mathsf{n}$ and $\mathsf{E}$ are linearly dependent. So Case 1 is valid. Using (3.1) and (3.2), since

$$\begin{aligned} &\langle \mathsf{n},\mathsf{E}\rangle = -\varepsilon_2 \sinh\phi_1, && \langle \mathsf{n},\mathsf{D}\rangle = 0, \\ &\langle \mathsf{b}_1,\mathsf{E}\rangle = 0, && \langle \mathsf{b}_1,\mathsf{D}\rangle = \varepsilon_3 \sinh\psi_2, \\ &\langle \mathsf{b}_2,\mathsf{E}\rangle = 0, && \langle \mathsf{b}_2,\mathsf{D}\rangle = \varepsilon_3 \sinh\theta_2, \end{aligned}$$

$$\begin{aligned} &\langle \mathsf{n},\mathsf{N}\rangle = 0, \\ &\langle \mathsf{b}_1,\mathsf{N}\rangle = \varepsilon_4 \sinh\psi_3, \\ &\langle \mathsf{b}_2,\mathsf{N}\rangle = \varepsilon_4 \sinh\theta_3 \end{aligned}$$

we have

$$\sinh\phi_1(s) = -1, \quad \phi_2(s) = \phi_3(s) = \psi_1(s) = \theta_1(s) = 0 \tag{3.10}$$

along $\alpha$. Substituting (3.10) into (3.4), (3.8) and (3.9) yield

$$\kappa_g^1 = k_1,$$

$$\kappa_g^2 = k_2 \sinh\psi_2,$$

and

$$\tau_g^1 = k_2 \sinh \psi_3.$$

Besides, since

$$\begin{aligned} \mathsf{D}' = & k_1 \varepsilon_{\mathsf{T}} \sinh \phi_2 \mathsf{T} + \big( -\phi_2' \cosh \phi_2 - k_2 \varepsilon_{\mathsf{n}} \sinh \psi_2 \big) \mathsf{n} \\ & + \big( \psi_2' \cosh \psi_2 - k_2 \varepsilon_{\mathsf{b}_1} \sinh \phi_2 - k_3 \varepsilon_{\mathsf{b}_1} \sinh \theta_2 \big) \mathsf{b}_1 \\ & + \big( \theta_2' \cosh \theta_2 - k_3 \varepsilon_{\mathsf{T}} \varepsilon_{\mathsf{n}} \varepsilon_{\mathsf{b}_1} \sinh \psi_2 \big) \mathsf{b}_2 \end{aligned}$$

we get

$$\begin{aligned} \tau_g^2 = \langle \mathsf{D}', \mathsf{N} \rangle = & \phi_2' \cosh \phi_2 \sinh \phi_3 \varepsilon_{\mathsf{n}} \\ & + \psi_2' \cosh \psi_2 \sinh \psi_3 \varepsilon_{\mathsf{b}_1} \\ & + \theta_2' \cosh \theta_2 \sinh \theta_3 \varepsilon_{\mathsf{b}_2} \\ & + k_2 \big( \sinh \psi_2 \sinh \phi_3 - \sinh \phi_2 \sinh \psi_3 \big) \\ & + k_3 \big( \sinh \psi_2 \sinh \theta_3 - \sinh \theta_2 \sinh \psi_3 \big). \end{aligned}$$

Using (3.10) yields

$$\begin{aligned} \tau_g^2 = & \psi_2' \cosh \psi_2 \sinh \psi_3 \varepsilon_{\mathsf{b}_1} \\ & + \theta_2' \cosh \theta_2 \sinh \theta_3 \varepsilon_{\mathsf{b}_2} \\ & + k_3 \big( \sinh \psi_2 \sinh \theta_3 - \sinh \theta_2 \sinh \psi_3 \big). \end{aligned}$$

□

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