Malaya Journal of Matematik, Vol. 6, No. 3, 478-482, 2018

https://doi.org/10.26637/MJM0603/0003

On two general nonlocal differential equations problems of fractional orders

Abd El-Salam Sh. A.[1]* and Gaafar F. M.[2]

Abstract

In this paper, we prove some local and global existence theorems for a fractional orders differential equations with nonlocal conditions, also the uniqueness of the solution will be studied.

Keywords

Fractional calculus; fractional order differential equations with nonlocal conditions.

AMS Subject Classification

26A33, 30E25, 34A12, 34A34, 34A37, 37C25, 45J05.

[1,2] *Faculty of Science, Damanhour University, Damanhour, Egypt.*

***Corresponding author**: [1] shrnahmed@sci.dmu.edu.eg; [2]fatmagaafar@sci.dmu.edu.eg

Article History: Received **21** February **2018**; Accepted **26** May **2018**

Contents

1. Introduction

In this work, we consider an arbitrary (fractional) orders differential equation of the form:

$$\frac{du}{dt} = f(t, D^{\alpha} u(t)), \ \ \alpha \in (0,\ 1) \tag{1.1}$$

with the nonlocal conditions

$$I^{\alpha} u(t)|_{t\,=\,\eta} = I^{\alpha} u(t)|_{t\,=\,1}, \ \ \eta \in (0,\ 1) \tag{1.2}$$

or

$$t^{1\,-\,\alpha} u(t)|_{t\,=\,\eta} = t^{1\,-\,\alpha} u(t)|_{t\,=\,1}, \ \ \eta \in (0,\ 1) \tag{1.3}$$

The nonlocal problems have been intensively studied by many authors, for instance in [4], the authors proved the existence of L_1-solution of the nonlocal boundary value problem

$$\begin{cases} D^{\beta} u(t) + f(t, u(\phi(t))) = 0, \beta \in (1,2), t \in (0,1), \\ I^{\gamma} u(t)|_{t=0} = 0, \gamma \in (0,1], \alpha u(\eta) = u(1), 0 < \eta < 1, \\ 0 < \alpha \eta^{\beta - 1} < 1. \end{cases}$$

where the function f satisfies Caratheodory conditions and the growth condition.

And, in [3], the authors proved by using the Banach contraction fixed point theorem, the existence of a unique solution of the fractional-order differential equation:

$${}_C D^{\alpha}\, x(t) = c(t)\, f(x(t)) + b(t),$$

with the nonlocal condition:

$$x(0) + \sum_{k=1}^{m} a_k\, x(t_k) = x_0,$$

where $x_0 \in \Re$ and $0 < t_1 < t_2 < \cdots < t_m < 1$, and $a_k \neq 0$ for all $k = 1, 2, \cdots, m$.

(Where ${}_C D^{\alpha}$ is the Caputo derivative).

Also, the nonlocal problems is studied in [5] - [7].

2. Preliminaries

Define $L_1(I)$ as the class of Lebesgue integrable functions on the interval $I = [a,b]$, where $0 \leq a < b < \infty$ and let $\Gamma(.)$ be the gamma function. Let $C(U,X)$ be The set of all compact operators from the subspace $U \subset X$ into the Banach space X and let $B_r = \{u \in L_1(I) : ||u|| < r, r > 0\}$.

Definition 1.1 The fractional integral of the function $f(.) \in L_1(I)$ of order $\beta \in R^+$ is defined by (see [8] - [11])

$$I_a^{\beta}\, f(t) = \int_a^t \frac{(t\, -\, s)^{\beta\, -\, 1}}{\Gamma(\beta)}\, f(s)\, ds.$$

Definition 1.2 The Riemann-Liouville fractional-order derivative of $f(t)$ of order $\alpha \in (0,1)$ is defined as (see [8] - [11])

$$D_a^{\alpha} f(t) = \frac{d}{dt} I_a^{1-\alpha} f(t), \quad t \in [a,b].$$

In this paper, we prove the existence of L_1-solutions for problems (1.1) - (1.2) and (1.1) - (1.3). Also, we will study the uniqueness of the solution.

Now, let us state the theorems which will be needed in the paper.

Theorem 2.1. (Rothe Fixed Point Theorem) [1]

Let U be an open and bounded subset of a Banach space E, let $T \in C(\bar{U}, E)$. Then T has a fixed point if the following condition holds

$$T(\partial U) \subseteq \bar{U}.$$

Theorem 2.2. (Nonlinear alternative of Laray-Schauder type) [1]

Let U be an open subset of a convex set D in a Banach space E. Assume $0 \in U$ and $T \in C(\bar{U}, E)$. Then either

(A1) T has a fixed point in $\bar{U}$, or

(A2) there exists $\gamma \in (0,1)$ and $x \in \partial U$ such that $x = \gamma Tx$.

Theorem 2.3. (Kolmogorov compactness criterion) [2]

Let $\Omega \subseteq L^p(0,1)$, $1 \leq p < \infty$. If

(i) Ω is bounded in $L^p(0,1)$ and

(ii) $x_h \to x$ as $h \to 0$ uniformly with respect to $x \in \Omega$, then Ω is relatively compact in $L^p(0,1)$, where

$$x_h(t) = \frac{1}{h} \int_t^{t+h} x(s)\, ds.$$

3. Main Results

Firstly, we will prove the equivalence of equation (1.1) with the corresponding Volterra integral equation:

$$y(t) = \frac{u_0\, t^{-\alpha}}{\Gamma(1-\alpha)} + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s, y(s))\, ds, \ t \in (0,1). \tag{3.1}$$

Indeed: integrate both sides of (1.1), we get

$$u(t) - u_0 = I\, f(t, D^{\alpha} u(t)), \tag{3.2}$$

Now, operating by $I^{1-\alpha}$ on both sides of (3.2), then

$$I^{1-\alpha} u(t) - I^{1-\alpha} u_0 = I^{2-\alpha} f(t, D^{\alpha} u(t)). \tag{3.3}$$

Differentiating both sides we get

$$D^{\alpha} u(t) - \frac{u_0\, t^{-\alpha}}{\Gamma(1-\alpha)} = I^{1-\alpha} f(t, D^{\alpha} u(t)).$$

Take $y(t) = D^{\alpha} u(t)$, we get (3.1)
Conversely, operate by I^{α} on both sides of (3.3), and differentiate twice we obtain (1.1).

Now define the operator T as

$$Ty(t) = \frac{u_0\, t^{-\alpha}}{\Gamma(1-\alpha)} + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s, y(s))\, ds, \ t \in (0,1).$$

To solve equation (3.1), we must prove that the operator T has a fixed point.

Consider the following assumptions:

(a) $f : (0,1) \times R \to R$ be a function with the following properties:

  (i) for each $t \in (0,1)$, $f(t,.)$ is continuous,

  (ii) for each $y \in R$, $f(.,y)$ is measurable,

  (iii) there exist two real functions $t \to a(t)$, $t \to b(t)$ such that

$$|f(t,y)| \leq a(t) + b(t)\,|y|, \ \text{for each } t \in (0,1), \ y \in R,$$

  where $a(.) \in L_1(0,1)$ and $b(.)$ is measurable and bounded.

Now, for the local existence of the solutions we have the following theorem:

Theorem 3.1.

If assumptions (i) - (iii) are satisfied, such that

$$\frac{\sup |\, b(t)\, |}{\Gamma(2-\alpha)} < 1, \tag{3.4}$$

then the fractional order integral equation (3.1) has a solution $y \in B_r$, where

$$r \leq \frac{\frac{u_0}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(2-\alpha)} \,||\, a \,||}{1 - \frac{\sup |\, b(t)\, |}{\Gamma(2-\alpha)}}.$$

Proof. Let u be an arbitrary element in B_r. Then from the assumptions (i) - (iii), we have

$$\begin{aligned}
||Ty|| &= \int_0^1 |Ty(t)|\,dt \\
&\leq \int_0^1 \left|\frac{u_0}{\Gamma(1-\alpha)}t^{-\alpha}\right| dt \\
&+ \int_0^1 |\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s,y(s))\,ds|\,dt \\
&\leq \frac{u_0\, t^{1-\alpha}}{\Gamma(2-\alpha)}\Big|_0^1 \\
&+ \int_0^1 \int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\,dt\,|f(s,y(s))|\,ds \\
&\leq \frac{u_0}{\Gamma(2-\alpha)} \\
&+ \int_0^1 \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)}\Big|_s^1 \Big(|a(s)|+|b(s)|\,|y(s)|\Big)\,ds \\
&\leq \frac{u_0}{\Gamma(2-\alpha)} \\
&+ \int_0^1 \frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)} \Big(|a(s)|+|b(s)|\,|y(s)|\Big)\,ds \\
&\leq \frac{u_0}{\Gamma(2-\alpha)} \\
&+ \frac{1}{\Gamma(2-\alpha)}\int_0^1 \Big(|a(s)|+|b(s)|\,|y(s)|\Big)\,ds \\
&\leq \frac{u_0}{\Gamma(2-\alpha)} \\
&+ \frac{1}{\Gamma(2-\alpha)}||a|| + \frac{1}{\Gamma(2-\alpha)}\sup|b(t)|\,||y||.
\end{aligned}$$

therefore the operator T maps L_1 into itself. Now, let $y \in \partial B_r$, that is, $||y|| = r$, then the last inequality implies

$$||Ty|| \leq \frac{u_0}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(2-\alpha)}||a|| + \frac{1}{\Gamma(2-\alpha)}\sup|b(t)|\,r.$$

Then $T(\partial B_r) \subset \bar{B}_r$ (closure of B_r) if

$$r \leq \frac{u_0}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(2-\alpha)}||a|| + \frac{1}{\Gamma(2-\alpha)}\sup|b(t)|\,r.$$

Therefore

$$r \leq \frac{\frac{u_0}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(2-\alpha)}||a||}{1-\frac{\sup|b(t)|}{\Gamma(2-\alpha)}}.$$

From inequality (3.4) we deduce that $r > 0$. Also, since

$$\begin{aligned}
||f|| &= \int_0^1 |f(s,y(s))|\,ds \\
&\leq \int_0^1 \Big(|a(s)|+|b(s)|\,|y(s)|\Big)\,ds \\
&\leq ||a|| + \sup|b(t)|\,||y||.
\end{aligned}$$

Then f in $L_1(0,1)$.
Further, from (assumption (i)) f is continuous in y and since I^α maps $L_1(0,1)$ continuously into itself, then $I^\alpha f(t,y(t))$ is continuous in y. Since y is an arbitrary element in B_r, then T maps B_r into $L_1(0,1)$ continuously.
Now, we will show that T is compact, by using Theorem 2.3. So, let Ω be a bounded subset of B_r. Then $T(\Omega)$ is bounded in $L_1(0,1)$, i.e. condition (i) of Theorem 2.3 is satisfied. It remains to show that $(Ty)_h \to Ty$ in $L_1(0,1)$ when $h \to 0$, uniformly.

$$\begin{aligned}
||(Ty)_h - Ty|| &= \int_0^1 |(Ty)_h(t) - (Ty)(t)|\,dt \\
&= \int_0^1 \left|\frac{1}{h}\int_t^{t+h}(Ty)(s)ds - (Ty)(t)\right| dt \\
&\leq \int_0^1 \left(\frac{1}{h}\int_t^{t+h}|(Ty)(s)-(Ty)(t)|ds\right)dt \\
&\leq \int_0^1 \frac{1}{h}\int_t^{t+h}\Big|\frac{u_0}{\Gamma(1-\alpha)}s^{-\alpha} \\
&\qquad - \frac{u_0}{\Gamma(1-\alpha)}t^{-\alpha}\Big|\,ds\,dt \\
&+ \int_0^1 \frac{1}{h}\int_t^{t+h} |I^{1-\alpha} f(s,y(s)) \\
&\qquad - I^{1-\alpha} f(t,y(t))|\,ds\,dt.
\end{aligned}$$

Since $f \in L_1(0,1)$, then $I^{1-\alpha}f(.) \in L_1(0,1)$. Moreover, since $t^{-\alpha} \in L_1(0,1)$. Then, we have (see [12])

$$\frac{1}{h}\int_t^{t+h}\left|\frac{u_0}{\Gamma(1-\alpha)}s^{-\alpha} - \frac{u_0}{\Gamma(1-\alpha)}t^{-\alpha}\right| ds \to 0$$

and

$$\frac{1}{h}\int_t^{t+h}|I^{1-\alpha} f(s,y(s)) - I^{1-\alpha} f(t,y(t))|\,ds \to 0$$

for a.e. $t \in (0,1)$. Therefore, by Theorem 2.3, we have that $T(\Omega)$ is relatively compact, that is, T is a compact operator. Therefore, Theorem 2.1 with $U = B_r$ and $E = L_1(0,1)$ implies that T has a fixed point. This completes the proof.

Now, for the existence of global solution, we will prove the following theorem :

Theorem 3.2.

Let the conditions (i) - (iii) be satisfied in addition to the following condition:

(b) Assume that every solution $y(.) \in L_1(0,1)$ to the equation

$$y(t) = \gamma\left(\frac{u_o}{\Gamma(1-\alpha)}t^{-\alpha} + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s,y(s))\,ds\right)$$

a.e. on $(0,1)$, $0 < \alpha < 1$

satisfies $||y|| \neq r$ (r is arbitrary but fixed).

Then the fractional order integral equation (3.1) has at least one solution $y \in L_1(0,1)$.
Proof. Let y be an arbitrary element in the open set $B_r = \{y : ||y|| < r, r > 0\}$. Then from the assumptions (i) - (iii), we have

$$||Ty|| \leq \frac{u_0}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(2-\alpha)}||a|| + \frac{1}{\Gamma(2-\alpha)} \sup|b(t)|\ ||y||.$$

The above inequality means that the operator T maps B_r into L_1. Moreover, we have

$$||f|| \ \leq \ ||a|| \ + \ \sup|b(t)|\ ||y||.$$

This estimation shows that f in $L_1(0,1)$.
Then from Theorem 3.1 we get that T maps B_r into $L_1(0,1)$ continuously, and the operator T is compact.
Set $U = B_r$ and $D = E = L_1(0,1)$, then from assumption (b), we find that condition $A2$ of Theorem 2.2 does not hold. Therefore, Theorem 2.2 implies that T has a fixed point. This completes the proof.

4. Uniqueness of the solution

Theorem 4.1.

If the function $f : (0,1) \times R \to R$ satisfy assumption (ii) of Theorem 3.1 and satisfy the following assumption

$$|f(t,y) \ - \ f(t,z)| \ \leq \ L\,|y \ - \ z|, \tag{4.1}$$

then the fractional order integral equation (3.1) has a unique solution.
Proof. From assumption (4.1), we get

$$|f(t,y) \ - \ f(t,0)| \ \leq \ L\,|y|,$$

but since

$$|f(t,y)| \ - \ |f(t,0)| \ \leq \ |f(t,y) \ - \ f(t,0)| \ \leq \ L\,|y|,$$

therefore

$$|f(t,y)| \ \leq \ |f(t,0)| \ + \ L\,|y|,$$

i.e. assumptions (i) and (iii) of theorem 3.1 are satisfied.
Now, let $y_1(t)$ and $y_2(t)$ be any two solutions of equation (3.1), then

$$|y_2(t) \ - \ y_1(t)| \quad \leq \quad L \int_0^t \frac{(t \ - \ s)^{-\,\alpha}}{\Gamma(1 \ - \ \alpha)}\, |y_2(s) - y_1(s)|\, ds.$$

Therefore

$$\begin{aligned}
\int_0^1 |y_2(t) - y_1(t)|dt \quad &\leq \quad L\int_0^1 \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} |y_2(s) - y_1(s)|dsdt,\\
||y_2 - y_1|| \quad &\leq \quad L\int_0^1 \int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} dt|y_2(s) - y_1(s)|ds\\
&\leq \quad \frac{L}{\Gamma(2-\alpha)}||y_2 - y_1||.
\end{aligned}$$

which implies that

$$y_1(t) \ = \ y_2(t).$$

Now for the existence and uniqueness of the solution of problems (1.1) - (1.2) and (1.1) - (1.3), we have the following two theorems:

Theorem 4.2.

If the assumptions of theorem 4.1 are satisfied, then problem (1.1) - (1.2) has a unique solution.
Proof. Since

$$u(t) \ = \ u_0 \ + \ I\, f(t,y(t)) \qquad \text{from (3.2)},$$

then from conditions (1.2), we get

$$\begin{aligned}
u_0\,(\eta^{\alpha} \ - \ 1) \quad &= \quad \int_0^1 (1 \ - \ s)^{\alpha}\, f(s,y(s))\, ds\\
&\qquad - \int_0^{\eta} (\eta \ - \ s)^{\alpha}\, f(s,y(s))\, ds,\\
u_0 \quad &= \quad \int_0^1 G(\eta,\ s)\ f(s,y(s))\, ds,
\end{aligned}$$

where

$$G(\eta,s) = \begin{cases} \frac{(1-s)^{\alpha} - (\eta - s)^{\alpha}}{\eta^{\alpha} \ - \ 1} & 0 \ \leq \ s \ \leq \ \eta \ \leq \ 1,\\[2ex] \frac{(1-s)^{\alpha}}{\eta^{\alpha} \ - \ 1} & 0 \ \leq \ \eta \ \leq \ s \ \leq \ 1. \end{cases}$$

Therefore,

$$u(t) \ = \ \int_0^1 G(\eta,\ s)\ f(s,y(s))\, ds \ + \ I\, f(t,y(t)),$$

which completes the proof.

Theorem 4.3.

If the assumptions of theorem 4.1 are satisfied, then problem (1.1) - (1.3) has a solution.
Proof. Since

$$u(t) \ = \ u_0 \ + \ I\, f(t,y(t)) \qquad \text{from (3.2)},$$

then from conditions (1.3), we get

$$\begin{aligned}
u_0(\eta^{1-\alpha} - 1) \quad &= \quad \int_0^1 f(s,y(s))ds - \int_0^{\eta} \eta^{1-\alpha} f(s,y(s))ds,\\
u_0 \quad &= \quad \int_0^1 G(\eta,\ s)\ f(s,y(s))\, ds,
\end{aligned}$$

where

$$G(\eta,s) = \begin{cases} -1 & 0 \ \leq \ s \ \leq \ \eta \ \leq \ 1,\\[2ex] \frac{1}{\eta^{1 \ - \ \alpha} \ - \ 1} & 0 \ \leq \ \eta \ \leq \ s \ \leq \ 1. \end{cases}$$

Therefore,

$$u(t) \ = \ \int_0^1 G(\eta,\ s)\ f(s,y(s))\, ds \ + \ I\, f(t,y(t)),$$

which completes the proof.

References

[1] Deimling, K. *Nonlinear Functional Analysis*, Springer-Verlag (1985).
[2] Dugundji, J. Granas, A. *Fixed Point theory*, Monografie Mathematyczne, PWN, Warsaw (1982).
[3] El-Sayed, A. M. A. and Abd El-Salam, Sh. A. On the stability of a fractional-order differential equation with nonlocal initial condition, *EJQTDE*, 29 (2008) 1-8.
[4] El-Sayed, A. M. A. and Abd El-Salam, Sh. A. Nonlocal boundary value problem of a fractional-order functional diffrential equation, *Inter. J. of Non. Sci.* (2009).
[5] El-Sayed, A. M. A. and Abd El-Salam, Sh. A. Solvability of a nonlocal boundary value problem of a fractional-order diffrential equation, *Math. Sci. Res. J.* (2009).
[6] El-Sayed, A. M. A., Abd-El-Rahman, R. O. and El-Gendy, M. Continuous dependence of the solution of a stochastic differential equation with nonlocal conditions, *Malaya J. Mat.* 4(3)(2016), 488-496.
[7] El-Sayed, A. M. A. and Abd El-Salam, Sh. A. Solvability of some fractional-order three point boundary value problems, *Malaya J. Mat.* 6(2)(2018), 390-395.
[8] Miller, K. S. and Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley, New York (1993).
[9] Podlubny, I. and EL-Sayed, A. M. A. On two definitions of fractional calculus, *Preprint UEF 03-96 (ISBN 80-7099-252-2), Slovak Academy of Science-Institute of Experimental phys.* (1996).
[10] Podlubny, I. *Fractional Differential Equations*, Acad. press, San Diego-New York-London (1999).
[11] Samko, S., Kilbas, A. and Marichev, O. L. *Fractional Integrals and Derivatives*, Gordon and Breach Science Publisher, (1993).
[12] Swartz, C. *Measure, Integration and Function spaces*, World Scientific, Singapore (1994).

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆