Malaya Journal of Matematik, Vol. 6, No. 3, 485-491, 2018

https://doi.org/10.26637/MJM0603/0005

On existence and uniqueness of fractional integrodifferential equations with an integral fractional boundary condition

V. V. Kharat [1]*

Abstract
The aim of the present paper is to establish the existence and uniqueness of solutions of fractional integrodifferential equations with an integral fractional boundary condition in Banach spaces.

Keywords
Fractional integrodifferential equations, fractional integral boundary value conditions, existence of solution.

AMS Subject Classification
34A08, 26A33, 34A60.

[1] *Department of Mathematics, N. B. Navale Sinhgad college of Enggineering, Kegaon, Solapur-413255, India.*
***Corresponding author**: [1]* vvkvinod9@gmail.com;
Article History: Received **04** February **2018**; Accepted **18** May **2018**

Contents

1. Introduction

Let $C(J,R)$ be the Banach space of all continuous functions from $J = [0,1]$ into R endowed with the norm

$$\|x\|_\infty = \sup\{\|x(t)\| : t \in J\}.$$

In this paper we consider the fractional integrodifferential equations with an integral fractional boundary condition of the type

$$^{C}D^{\alpha}x(t) = f\left(t, x(t), \int_0^1 k(t,s)x(s)\,ds\right), \quad 0 < \alpha \leq 1, \tag{1.1}$$

$$x(0) = \eta I^{\beta}x(\tau), \quad 0 < \tau < 1, \tag{1.2}$$

where $^{C}D^{\alpha}$ is the Caputo fractional derivative of order α, $f : J \times R \times R \to R$ is a continuous function, $\eta \in R$ is such that $\eta \neq \frac{\Gamma(\beta+1)}{\tau^{\beta}}$, Γ is the Euler gamma function and I^{β}, $0 < \beta < 1$ is the Riemann-Liouville fractional integral of order β.

The theory of fractional calculus has been available and applicable to various fields of study. The investigation of the theory of fractional differential and integral equations has started quite recently. One can see the monographs of Kilbas et.al. [11], Podlubny [15]. Integrodifferential equations arise in many engineering and scientific disciplines, often as approximation to partial differential equations, which represent much of the continuum phenomena. Integrodifferential equation is an equation that the unknown function appears under the sign of integration and it also contains the derivatives of the unknown function. It can be classified into Fredholm equations and Volterra equations. The upper bound of the region for integral part of Volterra type is variable, while it is a fixed number for that of Fredholm type. However, in this paper, we focus on Fredholm integrodifferential equations.

Integral boundary conditions are encountered in population dynamics, blood flow models, chemical engineering, cellular systems, heat transmission, plasma physics, thermoelasticity, etc. They come up when values of the function on the boundary are connected to its values inside the domain, they have physical significations such as total mass, moments, etc. Sometimes it is better to impose integral conditions because they lead to more precise measures than those proposed by a local condition.

Many recent papers have dealt with the existence, uniqueness and other properties of solutions of special forms of the equations (1.1) - (1.2), see [1–4, 7–10, 12, 14] and some of the references cited therein. Recently, in an interesting papers [5], A. Ahmadkhanlu have investigated the existence and uniqueness of solutions of special form of (1.1) - (1.2). The aim of the present paper is to prove the existence and uniqueness of solution of nonlinear fractional integrodifferential equations (1.1) - (1.2). The main tools employed in our analysis are based on the theory of fractional calculus and fixed point theorems.

The paper is organized as follows: Section 2, presents the preliminaries. Section 3 deals with the main results. Finally, in section 4, we discuss example to illustrate the theory.

2. Preliminaries

Before proceeding to the statement of our main results, we setforth definitions, preliminaries and hypotheses that will be used in our subsequent discussion. For more details see [6, 11, 13].

For measurable function $m: J \to R$, define the norm

$$\|m\|_{L^p(J,R)} = \begin{cases} \left(\int_J |m(t)|^p dt\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \inf_{\mu(\bar{J})=0} \{\sup_{J-\bar{J}} |m(t)|\}, & p = \infty, \end{cases}$$

where $\mu(\bar{J})$ is the Lebesgue measure of $\bar{J}$. Let $L^p(J,R)$ be the space of all Lebesgue measurable functions $m: J \to R$, with $\|m\|_{L^p(J,R)} < \infty$.

Definition 2.1. *The Riemann-Liouville fractional integral of order q, is defined by*

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds \ \ ,q > 0, \tag{2.1}$$

provided the right hand side is pointwise defined on $(0,\infty)$.

Definition 2.2. *For an at least $n-$times differentiable function $f: [0,\infty) \to R$, the Caputo derivative of fractional of order q is defined by*

$$^C D^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{1+q-n}} ds, \tag{2.2}$$
$$n-1 < q < n, \ \ n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

We need the following results in our subsequent discussion.

Theorem 2.3. *(Schaefer's fixed point theorem) Let $J = [t_0; t_0 + T]$ and $F: C(J,R) \to C(J,R)$ be a completely continuous operator. If the set $E(F) = \{x \in C(J,R) : x = \lambda F x forsome \lambda \in [0,1]\}$ is bounded, then F has at least a fixed point.*

Lemma 2.4. *[14] Let $\alpha \neq \frac{\Gamma(p+1)}{\eta^p}$. Then for a given $f \in C([0,1],R)$, the solution of the fractional differential equation*

$$^C D^q x(t) = f(t), \ \ 0 < q \le 1$$

subject to the boundary condition

$$x(0) = \alpha I^p x(\eta)$$

is given by

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds + \frac{\alpha\Gamma(p+1)}{\Gamma(p+1) - \alpha\eta^p} \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} f(s) ds, \ \ t \in J.$$

Lemma 2.5. *[5] Let $x \in C$ satisfy the following inequality*

$$|x(t)| \le a + b\int_0^t (t-s)^{\alpha-1} |x(s)|^\lambda ds + c\int_0^\tau (\tau-s)^{\alpha+\beta-1} |x(s)|^\lambda ds, \tag{2.3}$$

where $\alpha, \beta \in (0,1), \lambda \in [0, \frac{1}{1-p}]$ for some $1 < p < \frac{1}{1-\alpha-\beta}, a, b, c \ge 0$ are constants. Then there exists a constant $M^* > 0$ such that

$$|x(t)| \le M^*.$$

For the convenience, we list the following hypotheses used in our further discussion.

(H_1) The function $f: J \times R \times R \to R$ is Lebesgue measurable with respect to t on J.

(H_2) There exists a constant $p \in [0,\alpha)$ and real valued function $h(t) \in L^{\frac{1}{p}}(J,R_+)$ such that $\|f(t,x,y)\| \le h(t)$ for each $t \in J$ and all $x,y \in R$

(H_3) There exists a constant $q \in [0,\alpha)$ and real valued function $m(t) \in L^{\frac{1}{q}}(J,R_+)$ such that

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \le m(t)[\ \|x_1 - x_2\| + \|y_1 - y_2\|\ ],$$

for all $x_1, x_2, y_1, y_2 \in R$.

(H_4) There exists constant $\lambda \in [0, 1 - \frac{1}{p}]$ for some $1 < p < \frac{1}{1-\alpha}$ and $N > 0$ such that

$$\|f(t,x,y)\| \le N[1 + \|x\|^\lambda + \|y\|^\lambda\ ]$$

for each $t \in J$ and all $x,y \in R$.

3. Fractional integrodifferential equation of order $0 < q < 1$

In this section we state and prove results related to existence and uniqueness of solutions of fractional integrodifferential equation of order $q \in (0,1]$ with fractional integral boundary condition in Banach spaces.

Theorem 3.1. *Suppose that the hypothesis $(H_1) - (H_3)$ is satisfied. If*

$$\Omega_{\alpha,q}(t) = \frac{M(1+K_1)}{\Gamma(\alpha)\left(\frac{\alpha-q}{1-q}\right)^{1-q}} + \frac{M(1+K_1)|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|\Gamma(\alpha+\beta)\left(\frac{\alpha+\beta-q}{1-q}\right)^{1-q}} < 1, \tag{3.1}$$

then the system (1.1) - (1.2) has a unique solution, where

$$M = \|m\|_{L^{\frac{1}{p}}(J,R)} \quad \text{and} \quad K_1 = \sup\{|k(t,s)| : t,s \in J\}.$$

Proof. For each $t \in J$, we have

$$\begin{aligned}
&\int_0^t |(t-s)^{\alpha-1} f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)|ds \\
&\leq \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-p}}ds\right)^{1-p}\left(\int_0^t (h(s))^{\frac{1}{p}}ds\right)^{p} \\
&\leq \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-p}}ds\right)^{1-p}\left(\int_0^1 (h(s))^{\frac{1}{p}}ds\right)^{p} \\
&\leq \frac{H}{\left(\frac{\alpha-p}{1-p}\right)^{(1-p)}}.
\end{aligned}$$

Thus $|(t-s)^{\alpha-1} f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)|$ is lebesgue integrable with respect to $s \in [0,t]$ for all $t \in J$ and $x \in C$. Then $(t-s)^{\alpha-1} f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)$ is a Bouchner integrable with respect to $s \in [0,t]$ for all $t \in J$. Hence the fractional integrodifferential equation (1.1) - (1.2) is equivalent to the following integral euqation

$$\begin{aligned}
&x(t) \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)ds \\
&+ \frac{\eta\Gamma(\beta+1)}{\Gamma(\beta+1)-\eta\tau^{\beta}} \times \\
&\quad \int_0^{\tau} \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)ds.
\end{aligned}$$

Let

$$r \geq \frac{H}{\Gamma(\alpha)\left(\frac{\alpha-p}{1-p}\right)^{1-p}} + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\frac{H}{\Gamma(\alpha+\beta)\left(\frac{\alpha+\beta-p}{1-p}\right)^{1-p}},$$

where $H = \|h\|_{L^{\frac{1}{q}}(J,R)}$.

Now, we define an operator $F : B_r \to B_r$ by

$$\begin{aligned}
&(Fx)(t) \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)ds \\
&+ \frac{\eta\Gamma(\beta+1)}{\Gamma(\beta+1)-\eta\tau^{\beta}} \times \\
&\quad \int_0^{\tau} \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)ds.
\end{aligned} \tag{3.2}$$

First we show that $F(B_r) \subset B_r$, where F is defined by (3.2) and $B_r = \{x \in C : \|x\|_{\infty} \leq r\}$. For $x \in B_r$, $t \in J$, we have

$$\begin{aligned}
&\|(Fx)(t)\| \\
&\leq \sup_{t\in[0,1]}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)ds\right. \\
&\quad + \frac{\eta\Gamma(\beta+1)}{\Gamma(\beta+1)-\eta\tau^{\beta}} \times \\
&\quad\left. \int_0^{\tau} \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)ds\right\} \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + \frac{\eta\Gamma(\beta+1)}{\Gamma(\beta+1)-\eta\tau^{\beta}} \times \\
&\quad \int_0^{\tau} \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}h(s)ds \\
&\leq \frac{H}{\Gamma(\alpha)\left(\frac{\alpha-p}{1-p}\right)^{1-p}} + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|} \times \\
&\quad \frac{H}{\Gamma(\alpha+\beta)\left(\frac{\alpha+\beta-p}{1-p}\right)^{1-p}} \\
&\leq r.
\end{aligned}$$

This shows that $F(B_r) \subset B_r$.
Now, for $x, y \in B_r$ and $t \in J$, we obtain

$$
\begin{aligned}
&\|(Fx)-(Fy)\| \\
&\leq \sup_{t\in[0,1]}\left\{\frac{1}{\Gamma(\alpha)}\times \right.\\
&\quad \int_0^t (t-s)^{\alpha-1}\left\| f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)\right.\\
&\left. -f\left(s,y(s),\int_0^1 k(s,\tau)y(\tau)\,d\tau\right)\right\| ds\\
&+\frac{\eta\Gamma(\beta+1)}{\Gamma(\beta+1)-\eta\tau^{\beta}}\\
&\int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left\| f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)\right.\\
&\left.\left. -f\left(s,y(s),\int_0^1 k(s,\tau)y(\tau)\,d\tau\right)\right\| ds\right\}\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)[1+K_1]\,\|x(s)-y(s)\|\,ds\\
&+\frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\times\\
&\quad \int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}m(s)[1+K_1]\,\|x(s)-y(s)\|\,ds\\
&\leq \frac{[1+K_1]\,\|x-y\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)ds\\
&+\frac{[1+K_1]\,\|x-y\|_\infty\,|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}m(s)ds\\
&\leq \frac{[1+K_1]\,\|x-y\|_\infty}{\Gamma(\alpha)}\times\\
&\quad \left[\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-q}}ds\right)^{1-q}\left(\int_0^t (m(s))^{\frac{1}{q}}ds\right)^{q}\right]\\
&+\frac{[1+K_1]\,\|x-y\|_\infty\,|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|\Gamma(\alpha+\beta)}\times\\
&\quad \left[\left(\int_0^{\tau} (\tau-s)^{\frac{\alpha+\beta-1}{1-q}}ds\right)^{1-q}\left(\int_0^{\tau} (m(s))^{\frac{1}{q}}ds\right)^{q}\right]\\
&\leq \frac{[1+K_1]\,\|x-y\|_\infty}{\Gamma(\alpha)\left(\frac{\alpha-q}{1-q}\right)^{1-q}}\|m\|_{L^{\frac{1}{q}}(J,R)}\\
&+\frac{[1+K_1]\,\|x-y\|_\infty\,|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|\Gamma(\alpha+\beta)\left(\frac{\alpha+\beta-q}{1-q}\right)^{1-q}}\|m\|_{L^{\frac{1}{q}}(J,R)}\\
&\leq \left[\frac{M(1+K_1)}{\Gamma(\alpha)\left(\frac{\alpha-q}{1-q}\right)^{1-q}}\right.\\
&\left.+\frac{M(1+K_1)|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|\Gamma(\alpha+\beta)\left(\frac{\alpha+\beta-q}{1-q}\right)^{1-q}}\right]\|x-y\|_\infty\\
&\leq \Omega_{\alpha,q}(t)\,\|x-y\|_\infty
\end{aligned}
$$

As $\Omega_{\alpha,q}(t)<1$, F is a contraction mapping. So, by the contraction mapping principle, F has a unique fixed point x. That is, x is the unique solution of (1.1) - (1.2). This completes the proof of theorem. □

Theorem 3.2. *Assume that hypotheses $(H_3)-(H_4)$ are satisfied. Then the system (1.1) - (1.2) has at least one solution on J.*

Proof. Observe that the operator $F: C(J,R)\to C(J,R)$ defined by (3.2) is well defined due to (H_4). First we show that F is continuous operator.

Let $\{x_n\}$ be a sequence such that $x_n\to x$ in C. Then for each $t\in J$, we have

$$
\begin{aligned}
&\|(Fx_n)(t)-(Fx)(t)\|\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\| f\left(s,x_n(s),\int_0^1 k(s,\tau)x_n(\tau)\,d\tau\right)\right.\\
&\left. -f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)\right\| ds\\
&+\frac{\eta\Gamma(\beta+1)}{\Gamma(\beta+1)-\eta\tau^{\beta}}\times\\
&\quad \int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left\| f\left(s,x_n(s),\int_0^1 k(s,\tau)x_n(\tau)\,d\tau\right)\right.\\
&\qquad \left. -f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)\right\| ds\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)[1+K_1]\,\|x_n(s)-x(s)\|\,ds\\
&+\frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\times\\
&\quad \int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}m(s)[1+K_1]\,\|x_n(s)-x(s)\|\,ds\\
&\leq \frac{[1+K_1]\,\|x_n-x\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)ds\\
&+\frac{[1+K_1]\,\|x_n-x\|_\infty\,|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}m(s)ds\\
&\leq \frac{[1+K_1]\,\|x_n-x\|_\infty}{\Gamma(\alpha)}\left[\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-q}}ds\right)^{1-q}\left(\int_0^t (m(s))^{\frac{1}{q}}ds\right)^{q}\right]\\
&+\frac{[1+K_1]\,\|x_n-x\|_\infty\,|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|\Gamma(\alpha+\beta)}\times\\
&\quad \left[\left(\int_0^{\tau} (\tau-s)^{\frac{\alpha+\beta-1}{1-q}}ds\right)^{1-q}\left(\int_0^{\tau} (m(s))^{\frac{1}{q}}ds\right)^{q}\right]\\
&\leq \left[\frac{M(1+K_1)}{\Gamma(\alpha)\left(\frac{\alpha-q}{1-q}\right)^{1-q}}+\frac{M(1+K_1)|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|\Gamma(\alpha+\beta)\left(\frac{\alpha+\beta-q}{1-q}\right)^{1-q}}\right]\\
&\quad \times\|x_n-x\|_\infty
\end{aligned}
$$

Since $x_n \to x$, we have

$$\begin{aligned}
&\|(Fx_n)(t)-(Fx)(t)\| \\
&\leq \left[\frac{M(1+K_1)}{\Gamma(\alpha)\left(\frac{\alpha-q}{1-q}\right)^{1-q}} + \frac{M(1+K_1)|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^\beta|\Gamma(\alpha+\beta)\left(\frac{\alpha+\beta-q}{1-q}\right)^{1-q}}\right] \\
&\quad \times \|x_n-x\|_\infty \to 0
\end{aligned}$$

as $n\to\infty$. Now we show that F maps bounded sets into bounded sets in $C(J,R)$. For a positive number ρ, let $B_\rho = \{x\in C(J,R) : \|x\|\leq\rho\}$ be a bounded ball in $C(J,R)$. Then we have

$$\begin{aligned}
&\|(Fx)(t)\| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)\right\|ds \\
&+\frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^\beta|}\times \\
&\int_0^\tau \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left\|f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)\right\|ds \\
&\leq \frac{N}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[1+|x(s)|^\lambda+{K_1}^\lambda|x(s)|^\lambda]ds \\
&+\frac{N|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^\beta|}\times \\
&\int_0^\tau \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[1+|x(s)|^\lambda+{K_1}^\lambda|x(s)|^\lambda]ds \\
&\leq \frac{N}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&+\frac{N|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^\beta|}\int_0^\tau \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}ds \\
&+\frac{N[1+{K_1}^\lambda]}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x(s)|^\lambda ds \\
&+\frac{N[1+{K_1}^\lambda]|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^\beta|}\int_0^\tau \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|x(s)|^\lambda ds \\
&\leq \frac{N[1+\rho^\lambda+{K_1}^\lambda\rho^\lambda]}{\Gamma(\alpha+1)} \\
&+\frac{N[1+\rho^\lambda+{K_1}^\lambda\rho^\lambda]|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^\beta|\Gamma(\alpha+\beta+1)}
\end{aligned}$$

which implies that

$$\begin{aligned}
\|Fx\|_\infty \leq &\frac{N[1+\rho^\lambda+{K_1}^\lambda\rho^\lambda]}{\Gamma(\alpha+1)} \\
&+\frac{N[1+\rho^\lambda+{K_1}^\lambda\rho^\lambda]|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^\beta|\Gamma(\alpha+\beta+1)} = l
\end{aligned}$$

Now we show that F maps bounded sets into equicontinuous sets of $C(J,R)$. Let $t_1,t_2\in J$ with $t_1<t_2$ and $x\in B_\rho$. Then

$$\begin{aligned}
&\|(Fx)(t_2)-(Fx)(t_1)\| \\
&=\left\|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)ds\right. \\
&\left.-\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)ds\right\| \\
&\leq\left\|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\times\right. \\
&\qquad\left. f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)ds\right\| \\
&+\left\|\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)ds\right\| \\
&\leq\left\|\frac{N}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right][1+\rho^\lambda+{K_1}^\lambda\rho^\lambda]ds\right\| \\
&+\left\|\frac{N}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}[1+\rho^\lambda+{K_1}^\lambda\rho^\lambda]ds\right\| \\
&\leq\left\|\frac{N[1+\rho^\lambda+{K_1}^\lambda\rho^\lambda]}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]ds\right\| \\
&+\left\|\frac{N[1+\rho^\lambda+{K_1}^\lambda\rho^\lambda]}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right\| \\
&\leq\frac{N[1+\rho^\lambda+{K_1}^\lambda\rho^\lambda]}{\Gamma(\alpha+1)}[|(t_1)^\alpha-(t_2)^\alpha|+2(t_2-t_1)^\alpha] \\
&\leq\frac{3N[1+\rho^\lambda+{K_1}^\lambda\rho^\lambda](t_2-t_1)^\alpha}{\Gamma(\alpha+1)}
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x\in B_\rho$ as $t_2-t_1\to 0$. Therefore it follows by the Ascoli-Arzela theorem that $F : C(J,R)\to C(J,R)$ is completely continuous.

Now it remains to show that the set $E(F)=\{x\in C : x=\mu Fx, \mu\in(0,1)\}$. Then for $t\in J$ we have

$$\begin{aligned}
x(t)=\mu&\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)ds\right. \\
&+\frac{\eta\Gamma(\beta+1)}{\Gamma(\beta+1)-\eta\tau^\beta}\times \\
&\left.\int_0^\tau\frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)ds\right].
\end{aligned}$$

For each $t \in J$, we have

$$\begin{aligned}
&\|(Fx)(t)\| \\
&\le \frac{\mu}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\| f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)\right\| ds \\
&+ \frac{\mu|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^\beta|}\times \\
&\quad \int_0^\tau \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left\| f\left(s,x(s),\int_0^1 k(s,\tau)x(\tau)\,d\tau\right)\right\| ds \\
&\le \frac{\mu N}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[1+|x(s)|^\lambda + {K_1}^\lambda |x(s)|^\lambda] ds \\
&+ \frac{\mu N|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^\beta|}\times \\
&\quad \int_0^\tau \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[1+|x(s)|^\lambda + {K_1}^\lambda |x(s)|^\lambda] ds \\
&\le \frac{\mu N}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} ds \\
&+ \frac{\mu N|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^\beta|}\int_0^\tau \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} ds \\
&+ \frac{\mu N[1+{K_1}^\lambda]}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x(s)|^\lambda ds \\
&+ \frac{\mu N[1+{K_1}^\lambda]|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^\beta|}\int_0^\tau \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|x(s)|^\lambda ds
\end{aligned}$$

By Lemma2.5, there exists a $M^* > 0$ such that

$$\|x(t)\| \le M^*, \quad t \in J.$$

Thus for every $t \in J$, we have

$$\|x\|_\infty \le M^*.$$

This shows that the set $E(F)$ is bounded. Hence by Shaefer's fixed point theorem, we deduce that F has a fixed point $x \in \bar{U}$ which is a solution of the problem (1.1) - (1.2). This proves the Theorem.

□

4. Application

In this section we give the application of our main results established in previous section.

Example 4.1. *Consider the following nonlinear fractional integrodifferential equations with an integral fractional boundary condition*

$$\begin{aligned}
{}^C D^{\frac{1}{2}} x(t) &= \frac{|x(t)|}{2\sqrt{\left(t-\frac{1}{2}\right)}(1+|x(t)|)} - 1 + \cos^2 t \\
&\quad + \frac{1}{8}\int_0^1 \frac{e^{-t}}{2\sqrt{\left(t-\frac{1}{2}\right)}}\frac{x(s)}{1+x(s)} ds, \quad t \in J = [0,1],
\end{aligned} \tag{4.1}$$

$$x(0) = \sqrt{3} I^{\frac{1}{2}} x\left(\frac{1}{3}\right), \tag{4.2}$$

Problem (4.1)-(4.2) is of the form (1.1) - (1.2) with $\alpha = \beta = \frac{1}{2}, \eta = \sqrt{3}, \tau = \frac{1}{3}$ *and*

$$\begin{aligned}
f(t,x(t),Kx(t)) &= \frac{|x(t)|}{2\sqrt{\left(t-\frac{1}{2}\right)}(1+|x(t)|)(1+|x(t)|)} \\
&\quad + 1 + \cos^2 t + \frac{1}{8}Kx(t),
\end{aligned} \tag{4.3}$$

where

$$Kx(t) = \int_0^1 \frac{e^{-t}}{2\sqrt{\left(t-\frac{1}{2}\right)}}\frac{x(s)}{1+x(s)} ds. \tag{4.4}$$

For $x_1, x_2 \in C(J,R)$ *and* $t \in J$*, we have,*

$$\begin{aligned}
&\|Kx_1 - Kx_2\| \\
&\le \left|\frac{e^{-t}}{2\sqrt{\left(t-\frac{1}{2}\right)}}\right| \int_0^1 \left\|\frac{x_1(s)}{1+x_1(s)} - \frac{x_2(s)}{1+x_2(s)}\right\| ds \\
&\le \frac{e^{-t}}{2\sqrt{\left(t-\frac{1}{2}\right)}}\|x_1 - x_2\|
\end{aligned} \tag{4.5}$$

and hence

$$\begin{aligned}
&\|f(t,x_1,Kx_1) - f(t,x_2,Kx_2)\| \\
&\le \frac{e^{-t}}{2\sqrt{\left(t-\frac{1}{2}\right)}}[\|x_1 - x_2\| + \|Kx_1 - Kx_2\|].
\end{aligned} \tag{4.6}$$

For all $x \in [0,\infty)$ *and each* $t \in J$*,*

$$\begin{aligned}
&\|f(t,x(t),Kx(t))\| \\
&= \left\|\frac{|x(t)|}{2\sqrt{\left(t-\frac{1}{2}\right)}(1+|x(t)|)(1+|x(t)|)} + 1 + \cos^2 t + \frac{1}{8}Kx(t)\right\| \\
&\le \left\|\frac{|x(t)|}{2\sqrt{\left(t-\frac{1}{2}\right)}(1+|x(t)|)(1+|x(t)|)} + \frac{1}{8}Kx(t)\right\| \\
&\le \frac{1}{\sqrt{\left(t-\frac{1}{2}\right)}}
\end{aligned}$$

For $t \in J, q \in (0,\alpha)$, *let* $m(t) = \frac{1}{2\sqrt{(t-\frac{1}{2})}}$, $h(t) = \frac{1}{\sqrt{(t-\frac{1}{2})}} \in L^{\frac{1}{q}}(J,R_+)$, $M = \left\| \frac{1}{2\sqrt{(t-\frac{1}{2})}} \right\|_{L^{\frac{1}{q}}(J,R_+)}$. *By suitable choice of* $q \in (0,\alpha)$, *we have*

$$\Omega_{\alpha,q} = \left[\frac{2^{1-q}}{\sqrt{\pi}\left(\frac{1-2q}{1-q}\right)^{1-q}} + \frac{\sqrt{3}\sqrt{\pi}}{(\sqrt{\pi}-2)} \right] 1.70\, M \; < 1.$$

Thus, all the assumptions of the Theorem 3.1 are satisfied, problem (4.1)-(4.2) has a unique solution.

Acknowledgment

The author is grateful to the referee for his valuable suggestions to improve the paper.

References

[1] R. P. Agarwal, B. Andrade and C. Cuevas; Weighted pseudo-almost periodic solutions of a class of semilinear fractional differential equations, *Nonlinear Anal. Real World Appl.* 11 (2010), 3532 - 3554.

[2] B. Ahmad; Existence of solutionsfor irregular boundary value problems of nonlinear fractional differential equations, *Appl. Math. Lett.,* 23 (2010), 390 - 394.

[3] B. Ahmad, A. A. Alsaedi and B. Alghandi; Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions, *Nonlinear Anal. Real World Appl.* 9 (2008), 1727 - 1740.

[4] B. Ahmad, T. Hayat and S. Asghar; Diffraction of a plane wave by an elastic knife-edge adjacent to a strip, *Cacad. Appl. Math. Quart.*, 9 (2001), 303 - 316.

[5] A. Ahmadkhanlu; Existence and uniqueness results for a class of Fractional differential Equations with an Integral Fractional Boundary Condition, *Filomat*, 31:5 (2017), 1241 - 1249.

[6] A. Granas, J. Dugundji; Fixed Point Theory, *Springer-Verlag, New York,* (2005).

[7] T. B. Jagtap and V. V. Kharat; On Existence of Solution to Nonlinear fractional Integrodifferential System, *Journal of Trajectory,* 22 (1), 2014.

[8] S. D. Kendre, T. B. Jagtap and V. V. Kharat; On nonlinear Fractional integrodifferential equations with nonlocal condition in Banach spaces, *Nonl. Anal. Diff. Eq.,* 1 (3) 2013, 129 – 141.

[9] S. D. Kendre, V. V. Kharat and T. B. Jagtap; On Abstract Nonlinear Fractional Integrodifferential Equations with Integral Boundary condition, *Comm. Appl. Nonl. Anal.,* 22 (3), 2015, 93 - 108.

[10] S. D. Kendre, V. V. Kharat and T. B. Jagtap; On Fractional Integrodifferential Equations with Fractional Non-separated Boundary conditions, *Int. Jou. Appl. Math. Sci.,* 13 (3) 2013, 169 – 181.

[11] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo ; Theory and applications of fractional differential equations, *North-Holland Mathematics Studies, vol.204, Elsevier Science B. V., Amsterdam* , (2006).

[12] V. Lakshmikantham and A. S. Vatsala; Basic theory of fractional differential equations, *Nonlinear Analysis: TMA,* 69, 8 (2008), 2677 - 2682.

[13] K. S. Miller, B. Ross; An introduction to the fractional calculus and differential equations, *John Wiley, New York*, (1993).

[14] S. K. Ntouyas; Existence results for first order boundary value problems for fractional differential equations and inclusions with fractional integral boundary conditions, *Journal of Fractional Calculus and Applications,* 3(9) (2012), 1 - 14.

[15] I. Podlubny; Fractional differential equations, *Academic Press, San Diego*, (1999).

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 – 3786
Malaya Journal of Matematik
ISSN(O):2321 – 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆