Malaya Journal of Matematik, Vol. 6, No. 3, 492-498, 2018

https://doi.org/10.26637/MJM0603/0006

Enumeration of disjoint Hamilton cycles in a divisor Cayley graph

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Abstract

Hamilton cycles are cycles of largest length and triangles are cycles of smallest length in a graph. In this paper an enumeration method of determining the number of disjoint Hamilton cycles in the Divisor Cayley graph associated with the arithmetical function, namely the divisor function $d(n), n \geq 1$ is presented.

Keywords

Hamilton cycle, Outer Hamilton cycle, Symmetric set, Cayley graph, Divisor Cayley graph.

AMS Subject Classification

68R10, 05C30, 05C45.

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Article History: Received **21** March **2018**; Accepted **30** May **2018**

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1. Introduction

In 1980 Nathanson [11] introduced the concept of Congruences in Number Theory into Graph Theory and thus paved the way for the emergence of a new class of graphs called Arithmetic Graphs. According to him an arithmetic graph is a simple graph whose vertex set is $V = \{1, 2, ..., n\}$, the set of first n positive integers and two vertices x and y are adjacent if and only if $x + y \equiv z \pmod{n}$ where $z \in S$, a pre-assigned subset of V. Later many researchers [5, 12, 14] followed this trend and studied arithmetical graphs associated with various arithmetical functions.

There is another class of graphs, called Cayley graphs. A Cayley graph is the graph whose vertex set V is the set of elements of a finite group $(X, \cdot)$ and two vertices x and y are adjacent if and only if $x^{-1}y$, or, $y^{-1}x$ are in some symmetric subset S of X (a subset S of a group $(X, \cdot)$ is called a symmetric subset of X if s^{-1} is in S for all $s \in S$). This Cayley graph is denoted by $G(X, S)$ and $|S|$- regular and contains $\frac{|X||S|}{2}$ edges (see pp 15-16, [8]).

The cycle structure of Cayley graphs and Unitary Cayley graphs were studied by Berrizbeitia and Guidici [1, 2] and Detzer and Guidici [6]. Recently Maheswari and Madhavi [8–10] studied the enumeration methods for finding the number of triangles and Hamilton cycles in arithmetic graphs associated with the quadratic residues modulo a prime p and the Euler totient function $\varphi(n)$, $n \geq 1$ an integer. In [4] Chalapathi et al. gave a method of enumeration of triangles in the arithmetic Cayley graph, namely the divisor Cayley graph associated with the divisor function $d(n)$, $n \geq 1$ an integer. The main aim of this paper is to give an enumeration process for counting the number of disjoint Hamilton cycles in the divisor Cayley graph. In this study we have followed Bondy and Murty [3] for graph theory and Apostol [13] for number theory terminology.

2. The Divisor Cayley Graph and its Properties

Definition 2.1. *Let $n \geq 1$ be an integer. Consider the group $(Z_n, \oplus)$, the group of residue classes modulo n with respect to the addition modulo n. The set $D^* = \{d, n-d : d$ divides n and $d \neq n\}$ is a symmetric subset of the group $(Z_n, \oplus)$, which does not contain the identity element 0 of $(Z_n, \oplus)$. The* *divisor Cayley graph $G(Z_n, D^*)$ is the Cayley graph associated with the group $(Z_n, \oplus)$ and its symmetric subset D^*. That is the graph $G(Z_n, D^*)$ is the graph whose vertex set is $Z_n = \{0,1,2,...,n-1\}$ and the edge set is $E = \{(x,y) :$ either $x-y$ or $y-x$ is in $D^*\}$.*

In [4] it is established that $G(Z_n, D^*)$ is $|D^*|$-regular with size $\frac{n|D^*|}{2}$, connected and the degree of each vertex in $G(Z_n, D^*)$ is odd (even) if and only if n is even(odd). Further, it is not bipartite and Eulerian if and only if n is odd.

Lemma 2.2. *The graph $G(Z_n, D^*)$ is Hamiltonian.*

Proof. For $0 \le i \le n-1$, $(i+1)-i = 1$ and $1 \in D^*$ since 1 divides n trivially. So i and $i+1$ are adjacent and thus $(0,1,2,...,n-1,0)$ is a cycle of length n which is a Hamilton cycle in $G(Z_n, D^*)$ so that $G(Z_n, D^*)$ is Hamiltonian. □

Definition 2.3. *The cycle $(0,1,2,...,n-1,0)$ is called the* ***outer Hamilton cycle*** *of the graph $G(Z_n, D^*)$.*

Lemma 2.4. *If n is a prime then the graph $G(Z_n, D^*)$ is the outer Hamilton cycle.*

Proof. Suppose that n is a prime. Then 1 is the only divisor of n other than n so that $D^* = \{1, n-1\}$. Hence the graph $G(Z_n, D^*)$ is 2-regular and the only edges in $G(Z_n, D^*)$ are $(i, i+1)$ for $0 \le i \le n-1$ so that $G(Z_n, D^*)$ is the outer Hamilton cycle. □

Lemma 2.5. *Let $n \ge 1$ be an integer and $d \ne n$ be a divisor of n. Then $G(Z_n, D^*)$ contains exactly d disjoint cycles each of length $\frac{n}{d}$.*

Proof. Let d be a divisor of n and let $k = \frac{n}{d}$.For $1 \le i \le k$, $(i+1)d - id = d \in D^*$ so that $(i, i+1)$ is an edge. Also $kd = (\frac{n}{d})d = n = 0$. So
$C_0 = (0, d, 2d, ..., kd = 0)$
is a cycle of length k in $G(Z_n, D^*)$. Similarly one can see
$C_1 = (1, d+1, 2d+1, ..., kd+1 = 1)$,
$C_2 = (2, d+2, 2d+2, ..., kd+2 = 2)$,
...,
$C_{d-1} = (d-1, d+(d-1), 2d+(d-1), ..., kd+(d-1) = d-1)$
are also cycles of length k. Further no two of these cycles have a vertex in common so that they are disjoint cycles. Moreover the vertex set of $C_0 \cup C_1 \cup C_2 \cup ... \cup C_{d-1}$ is equal to the vertex set of G. Hence corresponding to each divisor d of n there are d disjoint cycles, each of length $k (= \frac{n}{d})$ in $G(Z_n, D^*)$. □

Remark 2.6. *For any divisor d of n the d disjoint cycles $C_0, C_1, C_2, ..., C_{d-1}$ are referred to as the cycles corresponding to the divisor d. These cycles play a crucial role in determining the edge disjoint Hamilton cycles in the divisor Cayley graph $G(Z_n, D^*)$.*

3. Enumeration of disjoint Hamilton cycles in a Divisor Cayley graph

Definition 3.1. *Hamilton cycles H_1 and H_2 are said to be edge disjoint if the edge sets $E(H_1)$ and (H_2) are disjoint.*

Theorem 3.2. *If n is even then the graph $G(Z_n, D^*)$ cannot be decomposed into* ***edge disjoint*** *Hamilton cycles.*

Proof. Let $n \ge 1$ be an even integer. Then by part (a) of Lemma 2.4 of [4] the degree q of each vertex is odd and the number of edges in $G(Z_n, D^*)$ is $\frac{nq}{2}$. If $G(Z_n, D^*)$ is decomposed into the union of k disjoint Hamilton cycles (since each Hamilton cycle contains n edges) then the number of edges in $G(Z_n, D^*)$ is nk so that $nk = \frac{nq}{2}$, or, $k = \frac{q}{2}$. This is not possible since q is odd. So $G(Z_n, D^*)$ cannot be decomposed into edge disjoint Hamilton cycles. □

Theorem 3.3. *The graph $G(Z_n, D^*)$ can be decomposed into edge disjoint Hamilton cycles if, and only if, n is odd.*

Proof. First suppose that the graph $G(Z_n, D^*)$ is a union of k number of edge disjoint Hamilton cycles, say $H_1, H_2, ..., H_k$. Since each H_i, $1 \le i \le k$ contains n edges the number of edges in $G(Z_n, D^*)$ is kn. If $G(Z_n, D^*)$ is r- regular then the number of edges in $G(Z_n, D^*)$ is also equal to $\frac{rn}{2}$ so that $\frac{nr}{2} = nk$. This gives $r = 2k$ so that r is even. That is, each vertex of $G(Z_n, D^*)$ is of even degree so that n is odd.

Conversely assume that n is odd. Adopting the following procedure all the edge disjoint Hamilton cycles of $G(Z_n, D^*)$ can be found. Let $d_1 > d_2 > ... > d_m$ be divisors of n other than 1 and n. Then d_i's are also odd. Choose the outer Hamilton cycle $\mathscr{H}_0 = (0,1,2,...,n-1,0)$ and consider the d disjoint cycles $C_0, C_1, C_2, ..., C_{d-1}$ corresponding to the divisor $d = d_1$. Each of the C_i' s is edge disjoint with $\mathscr{H}_0$. Let the spanning subgraph $\mathscr{C}_0 = C_0 \cup C_1 \cup C_2 \cup ... \cup C_{d-1}$.

Using the following procedure in d stages in which a pair of edges are deleted in each stage from $\mathscr{H}_0$ and adjoined to the spanning sub graph $\mathscr{C}_0$ and on the other hand a pair of edges are deleted from $\mathscr{C}_0$ and adjoined to $\mathscr{H}_0$ and $\mathscr{H}_0$ is transformed into a Hamilton cycle $\mathscr{H}_1$ while the cycles in $\mathscr{C}_0$ are merged into a Hamilton cycle $\mathscr{C}_1$. More specifically at the i^{th} stage

(i) the edges $(i-1, 1)$, $(d+(i-1), d+i)$ are deleted from $H_{012...i-1}$ and these are adjoined to $C_{012...i-1}$ and the edges $(i-1, d+(i-1))$ and $(i, d+i)$ are deleted from $C_{012...i-1}$ and these adjoined to $H_{012...i-1}$ to get $H_{012...i}$ and $C_{012...i}$ **if i is odd** and

(ii) the edges $(i-1, i)$, $((i-1)+d^{-1}, i+d^{-1})$ are deleted from $H_{012...i-1}$ and these are adjoined to $C_{012...i-1}$ and the edges $(i-1, (i-1)+d^{-1})$, $(i, i+d^{-1})$ are deleted from $C_{012...i-1}$ and these are adjoined to $H_{012...i-1}$ to get $H_{012...i}$ and $C_{012...i}$ **if i is even**. Here d^{-1} is the inverse of d in $(Z_n, \oplus)$ and $d^{-1} = (k-1)d$ where $k = \frac{n}{d}$.

We repeat this process by taking the divisor d_2 in place of d_1 and the Hamilton cycle $\mathscr{C}_1$ in place of the outer Hamilton cycle $\mathscr{H}_0$ and the union of disjoint cycles corresponding to d_2 in place of $\mathscr{C}_0$ and obtain a Hamilton cycle $\mathscr{H}_2 = H_{0123...d_2}$ from $\mathscr{C}_2$ and a Hamilton cycle $\mathscr{C}_2 = C_{0123...d_2}$ by merging the d_2 disjoint cycles corresponding to the divisor d_2. Applying this procedure for all the divisors of n other than 1 and n we obtain the required disjoint Hamilton cycles of $G(Z_n, D^*)$. This procedure is illustrated for a divisor d in step 1.

Step 1: Choose the outer Hamilton cycle $\mathscr{H}_0 = (0,1,2,...,n-1,0)$ and the divisor $d(=d_1)$ of n. Let us take its complement $\mathscr{H}_0^C$ of $\mathscr{H}_0$ and the spanning sub graph $\mathscr{C}_0$ that contains all disjoint cycles $C_0, C_1, C_2, ..., C_{d-1}$ generated by the divisor d in $\mathscr{H}_0^C$. It is easy to observe that the vertices $0,1,2,...,(d-1)$ lie in distinct cycles that are generated by d.

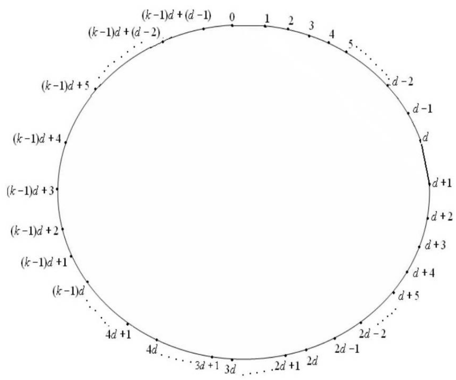

Figure 1. Graph $\mathscr{H}_0$.

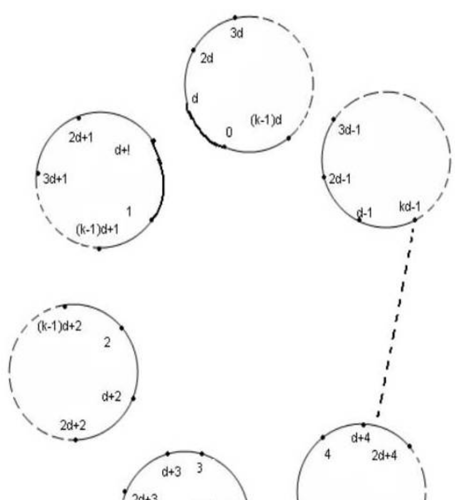

Figure 2. Graph $\mathscr{C}_0$.

Using the procedure outlined above in d stages we construct a pair of Hamilton cycles, one from $\mathscr{H}_0$ and the other by merging the cycles $C_0, C_1, C_2, ..., C_{d-1}$ in the following way.

Stage 1: Delete the edges $(0,1), (d, 1+d)$ from $\mathscr{H}_0$ and adjoin these to $\mathscr{C}_0$ and delete the edges $(0,d), (1,1+d)$ from $\mathscr{C}_0$ and adjoin to $\mathscr{H}_0$. By this $\mathscr{H}_0$ transforms into $H_{01} = (0,d,d-1,d-2,...,3,2,1,d+1,d+2,...,(k-1)d+(d-1),1,0)$ which is a **Hamilton cycle** and $\mathscr{C}_0$ transforms into C_{01}, which is a union of the disjoint cycles $C^1, C_2, ..., C_{d-1}$, where $C^1 = (0,(k-1)d,(k-2)d,...,3d,2d,d,d+1,2d+1,...,(k-1)d+1,1,0)$, which is the cycle got by merging the first two cycles C_0 and C_1 in $\mathscr{C}_0$, by the construction it is clear that H_{01} and C_{01} are disjoint.

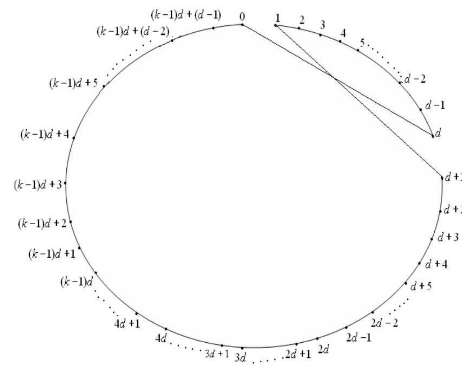

Figure 3. Graph H_{01}.

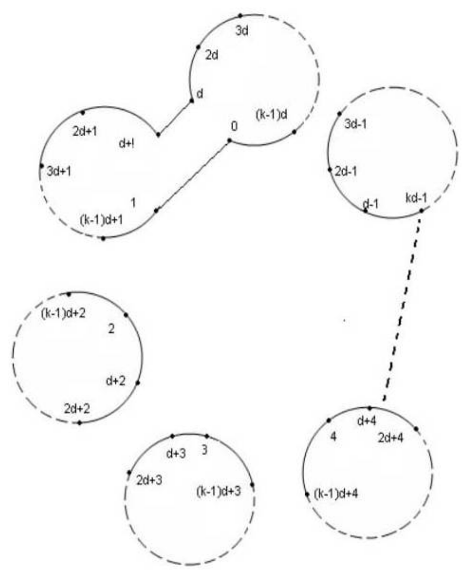

Figure 4. Graph C_{01}.

Stage 2: Delete the edges $(1,2)$ and $(1+(k-1)d, 2+(k-1)d)$ from H_{01} and adjoin these edges to C_{01} and delete the edges $(1,1+(k-1)d)$ and $(2,2+(k-1)d)$ from and C_{01} adjoin these edges to H_{01}. By this H_{01} transforms into H_{012} which is **the disjoint union of the two cycles** $(0,d,d-1,...,2,(k-1)d+2,(k-1)d+3,...,(k-1)d+d-1,0)$ and $(1,d+1,d+2,...,(k-1)d,(k-1)d+1,1)$ while C_{01} transforms into C_{012} which is the disjoint union of $C^2, C_3, C_4, ..., C_{d-1}$, where $C^2 = (0,(k-1)d,(k-1)d-1,...,3d,2d,d,d+1,2d+1,...,(k-1)d+1,(k-1)d+2,...,2d+2,d+2,2,1,0)$. By the construction it is clear that H_{012} and C_{012} are disjoint.

Observe that starting with Hamilton cycle $\mathscr{H}_0$ this process after i^{th} stage transforms the Hamilton cycle $\mathscr{H}_0$ into

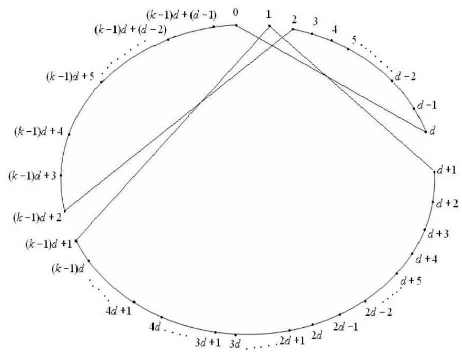

Figure 5. Graph H_{012}.

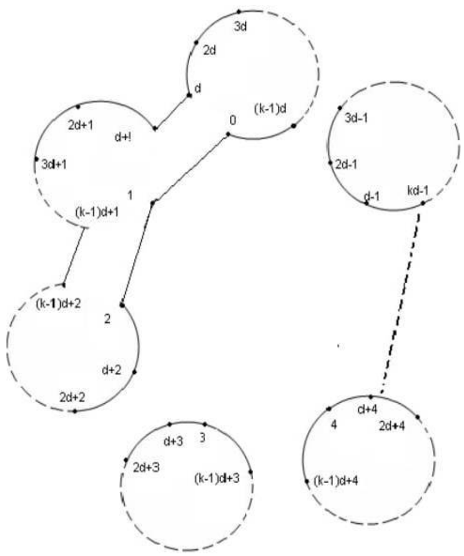

Figure 6. Graph C_{012}.

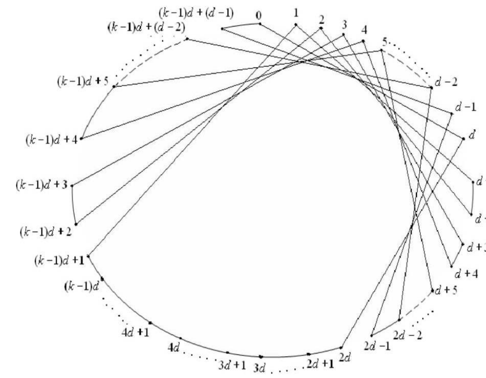

Figure 7. Graph $H_{012...d} = \mathscr{H}_1$.

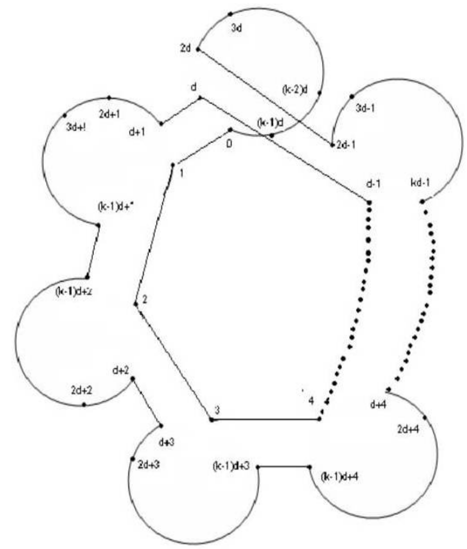

Figure 8. Graph $C_{012...d} = \mathscr{C}_1$.

(i) a Hamilton cycle $H_{0123...i}$ if i is **odd** and
(ii) a subgraph $H_{0123...i}$ which is a union of two disjoint cycles if i is **even**.

Repeating this process successively d times the outer Hamilton cycle $\mathscr{H}_0$ transforms into the **Hamilton cycle** (since d is odd)
$H_{012...d} = (0, d, 2d, ..., 3d, 3d+1, ..., 4d, 4d+1, ..., (k-1)d, (k-1)d+1, 1, d+1, d+2, 2, (k-1)d+2, (k-1)d+3, 3, d+3, d+4, 4, (k-1)d+5, 5, d+5, ..., 2d-2, 2d-1, d-1, (k-1)d+(d-1), 0)$
and the subgraph $\mathscr{C}_0$ is transformed into the Hamilton cycle (which is got by merging the cycles $C_0, C_1, ..., C_{d-1}$)
$C_{012...d} = (0, (k-1)d, ..., 3d, 2d, 2d-1, 3d-1, (k-1)d-1, kd-1, (k-1)d+(d-2), ..., (k-1)d+5, ..., d+4, 2d+4, ..., (k-1)d+4, (k-1)d+3, ..., 2d+3, d+3, d+2, 2d+2, ..., (k-1)d+2, (k-1)d+1, ..., 3d+1, 2d+1, d+1, d, d-1, d-2, ..., 5, 4, 3, 2, 1, 0)$.

Let us take $H_{012...d} = \mathscr{H}_1$ and $C_{012...d} = \mathscr{C}_1$. The construction of the Hamilton cycles $\mathscr{H}_1$ and $\mathscr{C}_1$ shows that they are edge disjoint. The following edges of $\mathscr{H}_1$ are transmitted to $\mathscr{C}_1$.
(i) $(0,1), (1,2), (2,3), ..., (d_1-1, d_1)$
(ii) $(d_1, d_1+1), (d_1+2, d_1+3), ..., (2d_1-1, 2d_1)$ (since $d+d-1 = 2d-1$ and $d+d = 2d$)
(iii) $(n-d_1+1, n-d_1+2), (n-d_1+3, n-d_1+4), ..., (n-2, n-1)$ (since $(n-d)+d-2 = n-2$ and $(n-d)+d-1 = n-1$).

Step 2: Let us take $\mathscr{C}_1$ in place of $\mathscr{H}_0$ and the spanning subgraph consisting of all cycles generated by the divisor d_2 in the complement $\mathscr{C}_1^c$ of $\mathscr{C}_1$. Since d_1 and d_2 are also odd, $d_2 \leq d_1 - 2$. Hence the set as edges $\{(0,1), ..., (d_2-1, d_2)\}$ belong to the set of edges given in (i). Moreover, $d_1, d_1+2, 2d_1-1; n-d_1+1, n-d_1+3, ...$ are all odd and $d_2 < d_1$ so that the set of all edges $\{(d_2, d_2+1), (d_2+2, d_2+3), ..., (2d_2-1, 2d_2), (n-d_2+1, n-d_2+2), ..., (n-2, n-1)\}$ belong to the set of all edges given in (ii) and (iii).

Repeating the procedure given in step1 with d_2 in place of d_1 one obtains the Hamilton cycles $H_{012...d_2} = \mathscr{H}_2$ and $C_{012...d_2} = \mathscr{C}_2$. From the construction of $\mathscr{H}_1, \mathscr{H}_2$ and $\mathscr{C}_2$ it is clear that they are mutually edge disjoint.

Repeating the same procedure as in the step 1 and step 2 for the divisors $d_3, d_4, ..., d_m$, one gets edge disjoint Hamilton cycles $\mathscr{H}_3, \mathscr{H}_4, ..., \mathscr{H}_m$ and $\mathscr{H}_{m+1} = \mathscr{C}_m$ which are edge disjoint with $\mathscr{H}_1$ and $\mathscr{H}_2$.

Further each $\mathscr{H}_i$, $1 \leq i \leq m+1$ being Hamiltonian contains exactly n edges. So the number of edges in $\mathscr{H}_1 \cup \mathscr{H}_2 \cup \ldots \cup \mathscr{H}_{m+1}$ is equal to $n(m+1)$.

Also $d_1, d_2, \ldots, d_m, 1$ are the only divisors of n other than n.So
$D^* = \{d_1, d_2, \ldots, d_m, 1, n-d_1, n-d_2, \ldots, n-d_m, n-1\}$
and $|D^*| = 2(m+1)$. Since $G(Z_n, D^*)$ is $|D^*|$- regular and the number of edges of $G(Z_n, D^*)$ is $\dfrac{n|D^*|}{2}$, or, $n(m+1)$, which is equal to the number of edges in $\mathscr{H}_1 \cup \mathscr{H}_2 \cup \ldots \cup \mathscr{H}_{m+1}$. Thus $G(Z_n, D^*)$ is a disjoint union of the Hamilton cycles $\mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_{m+1}$. □

The following corollary is immediate from the Theorem 3.3.

Corollary 3.4. *If $n \geq 1$ is an odd integer then $G(Z_n, D^*)$ has $m+1$ disjoint Hamilton cycles, where m is the number of divisors of n other than 1 and n.*

Corollary 3.5. *If $n \geq 1$ is an odd integer then the number of edge disjoint Hamilton cycles of $G(Z_n, D^*)$ is $d(n)-1$, where $d(n)$ is the number divisors of n.*

Proof. For any odd positive integer $n \geq 1$, the divisor function $d(n)$ denotes the number of divisors including 1 and n. By the Corollary 3.4 the number of disjoint Hamilton cycles in $G(Z_n, D^*)$ is $m+1$ where m is the number of divisors of n other than 1 and n. So $m = d(n) - 2$ and the number of disjoint Hamilton cycles of $G(Z_n, D^*)$ is $d(n)-1$. □

Example 3.6. *For the divisor Cayley graph the $G(Z_{15}, D^*)$ let us enumerate the edge disjoint Hamilton cycles. The divisors of 15 other than 15 are 1, 3 and 5. So $D = \{1,3,5\}$ and $D^* = \{1,3,5,10,12,14\}$. The graph of $G(Z_{15}, D^*)$ is as follows.*

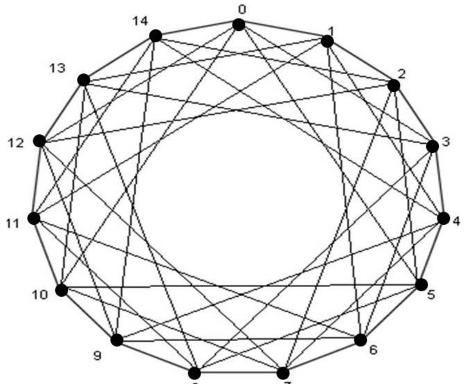

Figure 9. The Graph $G(Z_{15}, D^*)$.

Step 1: *For the divisor $d = 5$, the 5 disjoint cycles of length 3 are $(0,5,10,0)$, $(1,6,11,1)$, $(2,7,12,2)$, $(3,8,13,3)$ and $(4,9,14,4)$.*

The outer Hamilton cycle $\mathscr{H}_0$ and the spanning sub graph $\mathscr{C}_0$ which is the union of the above 5 cycles are as follows.

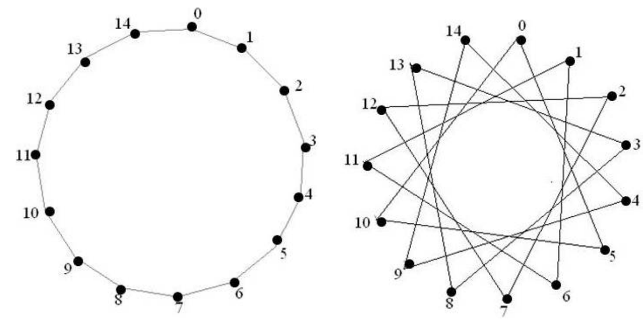

Figure 10. The Graphs of $\mathscr{H}_0$ and $\mathscr{C}_0$ respectively.

(i) Deleting the edges $(0,1),(5,6)$ from $\mathscr{H}_0$ and adjoining these to $\mathscr{C}_0$ and deleting the edges $(0,5),(1,6)$ from $\mathscr{C}_0$ and adjoining these to $\mathscr{H}_0$, $\mathscr{H}_0$ transforms into the Hamilton cycle $H_{01} = (1,2,3,4,5,0,14,13,12,11,10,9,8,7,6,1)$ and $\mathscr{C}_0$ into the subgraph $C_{01} = (0,1,11,6,5,10,0) \cup (3,13,8,3) \cup (2,7,12,2) \cup (4,9,14,4)$.

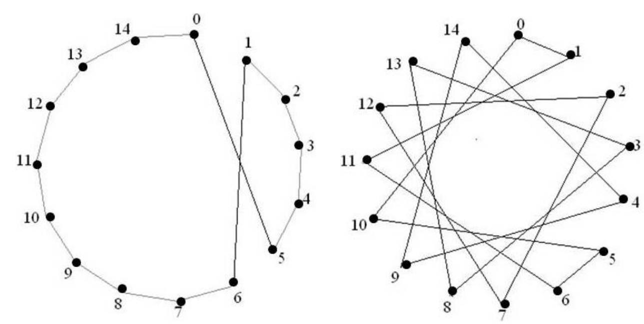

Figure 11. The Graphs of H_{01} and C_{01} respectively.

(ii) Deleting the edges $(1,2),(11,12)$ from H_{01} and adjoining them to C_{01} and deleting the edges $(1,11),(2,12)$ from C_{01} and adjoining them to H_{01}, H_{01} transforms into $H_{012} = (1,11,10,9,8,7,1) \cup (2,3,4,5,0,14,13,12,2)$ and C_{01} transforms into $C_{012} = (0,1,2,7,12,11,6,5,10,0) \cup (3,8,13,3) \cup (4,9,14,4)$.

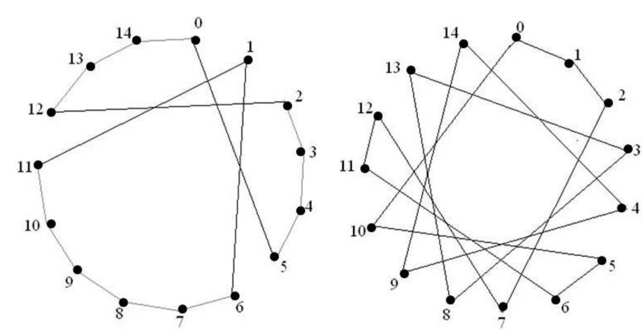

Figure 12. The Graphs of H_{012} and C_{012} respectively.

(iii) Deleting the edges $(2,3),(7,8)$ from H_{012} and adjoining to C_{012} and the Hamilton cycle deleting the edges $(2,7),(3,8)$ from C_{012} and adjoining them to H_{012}, H_{012} transforms into $H_{0123} = (0,5,4,3,8,9,10,11,1,6,7,2,12,13,14,0)$ and the subgraph $C_{0123} = (0,1,2,3,13,8,7,12,11,6,5,10,0) \cup (4,9,14,4)$.
(iv) Deleting the edges $(3,4),(13,14)$ from H_{0123} and adjoining them to C_{0123} and deleting the edges $(3,13),(4,14)$ from

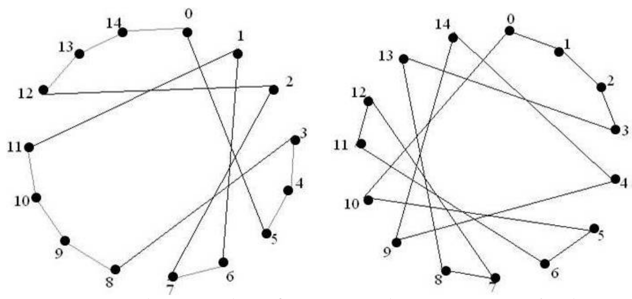

Figure 13. The Graphs of H_{0123} and C_{0123} respectively.

C_{0123} and adjoining them to H_{0123}, H_{0123} transforms into $H_{01234} = (0,5,4,14,0) \cup (1,6,7,2,12,13,3,8,9,10,11,1)$ and the subgraph C_{0123} into $C_{01234} = (0,1,2,3,4,9,14,13,8,7,12,11,6,5,10,0)$.

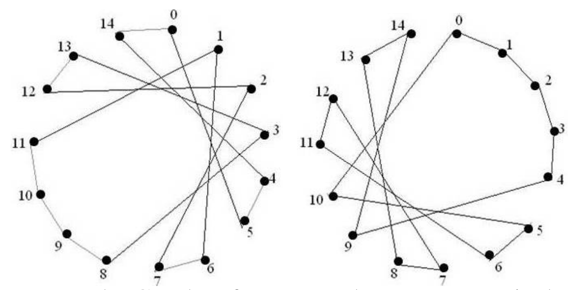

Figure 14. The Graphs of H_{01234} and C_{01234} respectively.

(v) Deleting the edges $(4,5),(9,10)$ from H_{01234} and adjoining them to C_{01234} and deleting the edges $(4,9),(5,10)$ from C_{01234} and adjoining them to H_{01234}, H_{01234} transforms into the Hamilton cycle $H_{012345} = (0,5,10,11,1,6,7,2,12,13,3,8,9,4,14,0)$ and the subgraph C_{01234} into $C_{012345} = (0,1,2,3,4,5,6,11,12,7,8,13,9,10,0)$. We take $H_{012345} = \mathscr{H}_1$ and $C_{012345} = \mathscr{C}_1$.

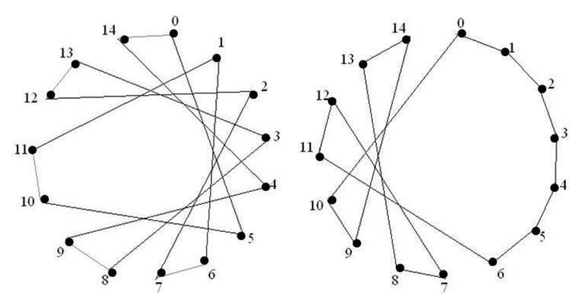

Figure 15. The Hamilton cycles $\mathscr{H}_1$ and $\mathscr{C}_1$ respectively.

Step 2:*Let us take the divisor 3 of 15 in place of the divisor 5 and the Hamilton cycle $\mathscr{C}_1$ in place of the outer Hamilton cycle $\mathscr{H}_0$. Consider the subgraph $\mathscr{C}_0^{'}$ consisting of the disjoint cycles generated by the divisor 5 in the compliment $\mathscr{C}_1^c$. That is $\mathscr{C}_0^{'} = (0,3,6,9,12,0) \cup (1,4,7,10,13,1) \cup (2,5,8,11,14,2)$ corresponding to the divisor 3 of 15 are as follows.*
Let us apply the procedure enumerated in step 1 to $\mathscr{C}_1$ and $\mathscr{C}_0^{'}$ in the following three stages.
(i) Deleting the edges $(0,1),(3,4)$ from $\mathscr{C}_1$ and adjoining to $\mathscr{C}_0^{'}$ and deleting the edges $(0,3),(1,4)$ from $\mathscr{C}_0^{'}$ and adjoining these to $\mathscr{C}_1$,$\mathscr{C}_1$ transforms into $H_{01}^{'} = (0,3,2,1,4,5,6,11,12,7,8,13,14,9,0)$ and $\mathscr{C}_0^{'}$ into the subgraph $C_{01}^{'} = (0,1,13,10,7,4,3,6,9,12,0) \cup (2,5,8,11,14,2)$.
(ii) Deleting the edges $(1,2),(13,14)$ from $H_{01}^{'}$ and adjoining them to $C_{01}^{'}$ and deleting the edges $(1,13),(2,14)$ from $C_{01}^{'}$ and adjoining them to $H_{01}^{'}$, $H_{01}^{'}$ transform into $H_{012}^{'} = (0,3,2,14,9,10,0) \cup (1,4,5,6,11,12,7,8,13,1)$ and $C_{01}^{'}$ into $C_{012}^{1} = (0,1,2,5,8,11,14,13,10,7,4,3,6,9,12,0)$.
(iii) Deleting the edges $(2,3),(5,6)$ from $H_{012}^{'}$ and adjoining them to $C_{012}^{'}$ and deleting the edges $(2,5),(3,6)$ from $C_{012}^{'}$ and adjoining them to $H_{012}^{'}$, $H_{012}^{'}$ transforms into the Hamilton cycle $H_{0123}^{'} = (0,3,6,11,12,7,8,13,1,14,5,2,14,9,10,0)$ and $C_{0123}^{'}$ into the Hamilton cycle $C_{0123}^{'} = (0,1,2,3,4,7,10,13,14,11,8,5,6,9,12,0)$. We denote $H_{0123}^{'} = \mathscr{H}_2$ and $C_{0123}^{'} = \mathscr{H}_3$.

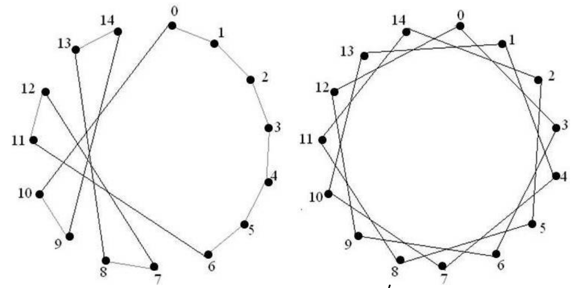

Figure 16. The Graphs of $\mathscr{C}_1$ and $\mathscr{C}_0^{'}$ respectively.

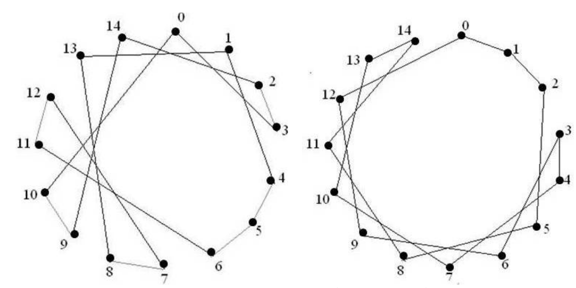

Figure 17. The Hamilton cycles $\mathscr{H}_2$ and $\mathscr{H}_3$ respectively.

Thus the three edge disjoint Hamilton cycles of the divisor Caley graph $G(Z_{15}, D^*)$ are $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}_3$ given in Figure 18.

4. Conclusion

In [4] Madhavi et al. gave a method of enumeration of triangles in a divisor Cayley graph and in [7] they could successfully extend this enumeration process to general Cayley graphs. The authors are working to extend the process of enumeration of disjoint Hamilton cycles in divisor Cayley graphs presented in this paper to general Hamiltonian Cayley graphs and some progress has been obtained in this direction.

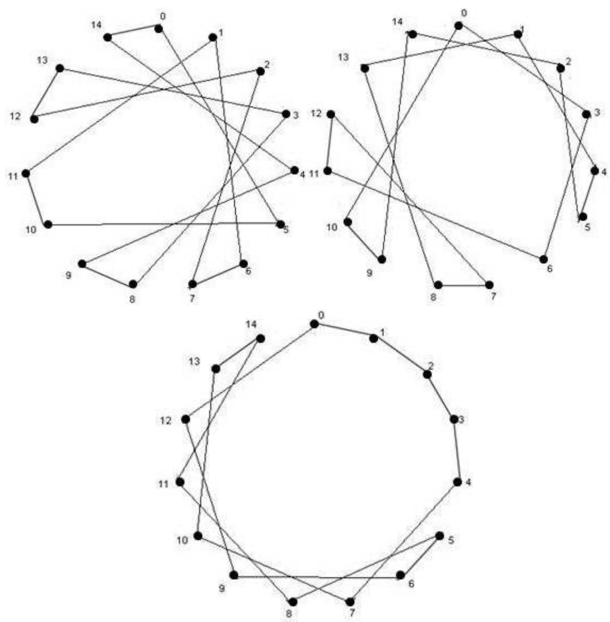

Figure 18. The Hamilton cycles $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}_3$ respectively.

5. Acknowledgement

The authors express their thanks to Prof. L. Nagamuni Reddy for his suggestions during the preparation of this paper.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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