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Coincidence fixed point theorem in a Menger probabilistic metric spaces

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Abstract
In this paper, we discuss the existence and uniqueness of solutions for a class of multi-term time-fractional impulsive integro-differential equations with state dependent delay subject to some fractional order integral boundary conditions. In our consideration, we apply the Banach, and Sadovskii fixed point theorems to obtain our main results under some appropriate assumptions. An example is given at the end to illustrate the applications of the established results.Fixed point theory of nonexpansive type single valued mappings provides techniques for solving a variety of applied problems in mathematical sciences and engineering. The aim of this paper is to prove the existence of coincidence points, coupled points and common coupled fixed points of nonexpansive type conditions satisfied by single valued maps which include both continuous and discontinuous mappings on Menger probabilistic metric spaces.

Keywords
Menger PM space, nonexpansive mappings, compatible mappings, common fixed point, common coupled fixed point, coincidence point, coupled point, weak reciprocal continuity, reciprocal continuity.

AMS Subject Classification
47H10,54H25,55M20.

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1. Introduction

Fixed point theory plays a basic role in applications of many branches of mathematics. The term metric fixed point theory refers to those fixed point theoretic results in which geometric conditions on the underlying spaces and mappings play a crucial role.

In 1922, a Polish mathematician Banach [1] proved a very important result regarding contraction mapping, known as the famous Banach contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical science and engineering. Then after many authors generalizes and extends the Banach contraction principle an different ways.

It is well known that the probabilistic version of the classical Banach contraction principle was proved in 1972 by Sehgal and Bharucha-Reid [19]. In 2010, a probabilistic version of the Banach fixed point principle for general nonlinear contractions was established by Jacek Jachymski [8]. Also, the fixed point theorems in probabilistic metric spaces for other contraction mappings were investigated by many authors, see [3, 5, 7, 17, 18] the references therein.

Our work is arranged as follows: In preliminaries section, we recall some basic definitions and fundamental results of probabilistic metric spaces. In main results section, we try to extend metric space theorems to the Menger probabilistic metric space and establish some fixed point theorems for nonexpansive type single valued mappings.

2. Preliminaries

Definition 2.1. *[7] A function $f : (-\infty,\infty) \to [0,1]$ is called a distribution function, if it is nondecreasing and left continuous* *with* $\inf_{x\in\mathbb{R}} f(x) = 0$*. If in addition* $f(0) = 0$*, then* f *is called a distance distribution function. Furthermore, a distance distribution function* f *satisfying* $\lim_{t\to\infty} f(t) = 1$ *is called a Menger distance distribution function.*

The set of all Menger distance distribution functions is denoted by Λ^+.

Definition 2.2. *[7] A triangular norm (abbreviated, T-norm) is a binary operation $\triangle$ on $[0,1]$, which satisfies the following conditions:*
(a) $\triangle$ is associative and commutative,
(b) $\triangle$ is continuous,
(c) $\triangle(a,1) = a$ for all $a \in [0,1]$,
(d) $\triangle(a,b) \le \triangle(c,d)$ whenever $a \le c$ and $b \le d$, for each $a,b,c,d \in [0,1]$.

Among the important examples of a T-norm we mention the following two T-norms: $\triangle_p(a,b) = ab$ and $\triangle_m(a,b) = \min\{a,b\}$. The T-norm $\triangle_m$ is the strongest T-norm, that is, $\triangle \le \triangle_m$ for every T-norm $\triangle$.

Definition 2.3. *[5] A triangular norm $\triangle$ is said to be of H-type (Hadžić type) if a family of functions $\{\triangle^n(t)\}_{n=1}^{+\infty}$ is equicontinuous at $t = 1$, that is,*

$$\forall \varepsilon \in (0,1), \exists \delta \in (0,1) : t > 1-\delta \Rightarrow \triangle^n(t) > 1-\varepsilon (n \ge 1),$$

where $\triangle^n : [0,1] \to [0,1]$ is defined as follows:

$$\triangle^1(t) = \triangle(t,t), \quad \triangle^n(t) = \triangle(t, \triangle^{n-1}(t)), \ \ n = 2,3,\dots .$$

Obviously, $\triangle^n(t) \le t$ for any $n \in \mathbf{N}$ and $t \in [0,1]$.

Definition 2.4. *[18] A Menger probabilistic metric space (abbreviated, Menger PM space) is a triple $(X,F,\triangle)$ where X is a nonempty set, $\triangle$ is a continuous T-norm and F is a mapping from $X \times X$ into Λ^+ such that, if $F_{p,q}$ denotes the value of F at the pair (p,q),the following conditions hold:*
(PM_1) $F_{p,q}(t) = 1$ for all $t > 0$ if and only if $p = q$ $(p,q \in X)$,
(PM_2) $F_{p,q}(t) = F_{q,p}(t)$ for all $t > 0$ and $p,q \in X$,
(PM_3) $F_{p,r}(s+t) \ge \triangle(F_{p,q}(s), F_{q,r}(t))$ for all $p,q,r \in X$ and every $s > 0, t > 0$.

Definition 2.5. *[18] A sequence $\{x_n\}$ in Menger PM space X is said to converge to a point x in X (written as $x_n \to x$), if for every $\delta > 0$ and $\lambda \in (0,1)$, there is an integer $N(\delta,\lambda) > 0$ such that $F_{x_n,x} > 1-\lambda$, for all $n \ge N(\delta,\lambda)$. The sequence is said to be Cauchy sequence if for each $\delta > 0$ and $\lambda \in (0,1)$, there is an integer $N(\delta,\lambda) > 0$ such that $F_{x_n,x_m} > 1-\lambda$, for all $n,m \ge N(\delta,\lambda)$. A Menger PM space $(X,F,\triangle)$ is said to be complete if every Cauchy sequence in X converges to a point of X.*

Let Φ denote all the functions $\varphi : [0,\infty) \to [0,\infty)$ which satisfy $\varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$.

Lemma 2.6. *[11] Let $(X,F,\triangle)$ be a Menger PM space and $\varphi \in \Phi$. If $F_{p,q}(\varphi(t)) = F_{p,q}(t)$, for all $t > 0$, then $p = q$.*

Lemma 2.7. *[11] Let $n \ge 1$. If $F \in \Lambda^+$, $g_1, g_2, ..., g_n : \mathbb{R} \to [0,1]$ and for some $\varphi \in \Phi$,*

$$F(\varphi(t)) \ge \min\{g_1(t), g_2(t), ..., g_n(t), F(t)\} \ \ for\ all\ t > 0,$$

then $F(\varphi(t)) \ge \min\{g_1(t), g_2(t), ..., g_n(t)\}$, for all $t > 0$.

For $\tilde{a} = (x,y), \tilde{b} = (u,v) \in X^2$, we introduce a distribution function $\tilde{F}$ from X^2 into Λ^+ defined by

$$\tilde{F}_{\tilde{a},\tilde{b}}(t) = \min\{F_{x,u}(t), F_{y,v}(t)\} \ \ for\ all\ t > 0.$$

Lemma 2.8. *[11] If $(X,F,\triangle)$ is a complete Menger PM space, then $(X^2, \tilde{F}, \triangle)$ is also a complete Menger PM space.*

Let (X,d) be a metric space. A map $T : X \to X$ is said to be nonexpansive if $d(Tx,Ty) \le d(x,y)$ for all $x,y \in X$. Ćirić [2] investigated a class of nonexpansive type self maps T of X and established some fixed point theorems for such type of mappings.
Recently, Jhade et al. [10] gave the following nonexpansive type condition. Let $f,T : X \to X$ and

$$\begin{aligned} &d(Tx,Ty) \le a(x,y)d(fx,fy) + b(x,y)\max\{\\ &d(fx,Tx), d(fy,Ty)\} + c(x,y)\max\{d(fx,fy),\\ &d(fx,Tx), d(fy,Ty)\} + e(x,y)\max\{d(fx,fy),\\ &d(fx,Tx), d(fy,Ty), d(fx,Ty)\}. \end{aligned} \tag{2.1}$$

where $a(x,y), b(x,y), c(x,y), e(x,y) \ge 0$ and $\beta = \inf_{x,y\in X} e(x,y) > 0$ whit $\sup_{x,y\in X}(a(x,y)+b(x,y)+c(x,y)+2e(x,y)) = 1$.

Jhade et al. [10] proved that a compatible pair of maps on complete metric space satisfying (2.1) will have a coincidence point if f is surjective or continuous. After that, in 2016, Jhade et al. [9] extended the study of nonexpansive type condition to the class of mappings which include both continuous and discontinuous mappings by the condition of weak reciprocal continuity.
In this paper, we use the following nonexpansive type condition to the class of two self mappings f,T on a Menger PM space $(X,F,\triangle)$ by continuity and weakly reciprocal continuity.

$$\begin{aligned} F_{Tx,Ty}(\varphi(t)) \ge\ & a(x,y)F_{fx,fy}(t) + b(x,y)\min\{F_{fx,Tx}(t),\\ & F_{fy,Ty}(t)\} + c(x,y)\min\{F_{fx,fy}(t),\\ & F_{fx,Tx}(t), F_{fy,Ty}(t)\}, \end{aligned} \tag{2.2}$$

where $a(x,y), b(x,y), c(x,y) \ge 0$ with $\inf_{x,y\in X}(a(x,y)+b(x,y)+c(x,y)) = 1$.

Definition 2.9. *Let f and g be two maps from X into Y. We say f and g have a coincidence point, if there exists a point x in X such that $fx = gx$.*

Definition 2.10. *Let f and g be two self maps on X. We say $x \in X$ is a common fixed point of f and g, if $fx = gx = x$.*

Definition 2.11. *An element $(x,y) \in X \times X$ is called a coupled point of the mapping $T : X \times X \to X$, if $T(x,y) = x$ and $T(y,x) = y$.*

Definition 2.12. *An element $(x,y) \in X \times X$ is called a coupled coincidence point of the mappings $T : X \times X \to X$ and $g : X \to X$ if $T(x,y) = gx$ and $T(y,x) = gy$.*

Definition 2.13. *An element $(x,y) \in X \times X$ is called a common coupled fixed point of the mappings $T : X \times X \to X$ and $g : X \to X$ if $T(x,y) = gx = x$ and $T(y,x) = gy = y$.*

Definition 2.14. *Let f and g be two self maps of a Menger PM space $(X,F,\triangle)$.Then f and g are said to be Menger compatible if $\lim_{n\to\infty} F_{fgx_n,gfx_n}(t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x \in X$.*

In 1982, Sessa [20] introduced the notion of weak commutativity condition for a pair of single valued maps. Later, Jungek [12] generalized the concept of weak commutativity by introducing the notion of compatibility of maps. Pant [13] introduced point wise R-weakly commutativity of maps for noncompatible maps.

Two self mappings f and g of a metric space (X,d) are called R-weakly commuting of type-(A_g) [16], if there exists some positive real number R such that $d(ffx,gfx) \le Rd(fx,gx)$ for all $x \in X$. Similarly, two self mappings f and g of a metric space (X,d) are called R-weakly commuting of type-(A_f) [16], if there exists some positive real number R such that $d(fgx,ggx) \le Rd(fx,gx)$ for all $x \in X$.

We introduced the concept of weak commutativity condition for a pair of single valued maps in a Menger PM space $(X,F,\triangle)$, as follows[4];

Definition 2.15. *Two self mappings f and g of a Menger PM space $(X,F,\triangle)$ are called R-weakly commuting of type-(MA_g), if there exists some real number $R \ge 1$ such that $F_{ffx,gfx}(t) \ge RF_{fx,gx}(t)$ for all $t > 0$ and $x \in X$.*

In 1998, Pant [14] introduced the concept of reciprocal continuity for the pair of single valued maps. In the following, we have the same definition but in a Menger PM space X.

Definition 2.16. *Two self mappings f and g of a Menger PM space X are called reciprocal continuous , if $\lim_{n\to\infty} gfx_n = gx$ and $\lim_{n\to\infty} fgx_n = fx$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some $x \in X$.*

Note that a pair of mappings which is reciprocal continuous need not be continuous even on their common fixed point (see for example [14]).

Recently, Pant et al. [16] generalized reciprocal continuity by introducing the notion of weakly reciprocal continuity for a pair of single valued maps as follows but in metric space (X,d).

Definition 2.17. *Two self mappings f and g of a Menger PM space X are called weakly reciprocally continuous , if $\lim_{n\to\infty} gfx_n = gx$ or $\lim_{n\to\infty} fgx_n = fx$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some $x \in X$.*

It seems important to note that reciprocal continuity implies weak reciprocal continuity, but the converse is not true as shown below;

Example 2.18. *[4] Let $X = [2,20]$ and d be a usual metric in X. Define $f,g : X \to X$ as follows:*

$$f(x) = \begin{cases} 2, & if\ x = 2\ or\ x > 5 \\ 6, & if\ 2 < x \le 5 \end{cases}$$

and

$$g(x) = \begin{cases} 2, & if\ x = 2 \\ 12, & if\ 2 < x \le 5 \\ \frac{x+1}{3}, & if\ x > 5 \end{cases}$$

Let H and D denote a Menger distribution functions defined by:

$$H(x) = \begin{cases} 0, & if\ t \le 0 \\ 1, & if\ t > 0 \end{cases}$$

and

$$D(x) = \begin{cases} 0, & if\ t \le 0 \\ 1 - e^{-t}, & if\ t > 0 \end{cases}$$

For any $t > 0$, define a function $F : X \times X \to \Lambda^+$ by

$$F_{x,y}(t) = \begin{cases} H(t), & if\ x = y \\ D(\frac{t}{d(x,y)}), & otherwise \end{cases}$$

Set $\triangle(a,b) = \min\{a,b\}$. Then $(X,F,\triangle)$ is a Menger PM space. Then clearly f and g are weakly reciprocally continuous on $(X,F,\triangle)$, but not reciprocally continuous.

It seems to be noted that only weakly reciprocal continuity does not guarantee the existence of common fixed point or even coincidence point. For example see [9] on Menger PM space of above example.

Theorem 2.19. *[4] Let $(X,F,\triangle)$ be a complete Menger PM space with a T-norm $\triangle$ of H-type, T,f are two weakly reciprocally continuous self maps of X satisfying (2.2) for some $\varphi \in \Phi$ with $T(X) \subseteq f(X)$, then T and f have a common fixed point in X if either*
(a) T and f are Menger compatible; or
(b) T and f are R-weakly commuting of type-(MA_f); or
(c) T and f are R-weakly commuting of type-(MA_T).

3. Main Results

Definition 3.1. *Let $(X,F,\triangle)$ be a Menger PM space and $T: X\times X \to X$ and $g: X \to X$. Then T and g are Menger compatible if*

$$\lim_{n\to\infty} F_{gT(x_n,y_n),T(gx_n,gy_n)}(t) = 1 \ for\ all\ t>0$$

and

$$\lim_{n\to\infty} F_{gT(y_n,x_n),T(gy_n,gx_n)}(t) = 1 \ for\ all\ t>0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that

$$\lim_{n\to\infty} T(x_n,y_n) = \lim_{n\to\infty} gx_n = x$$

and

$$\lim_{n\to\infty} T(y_n,x_n) = \lim_{n\to\infty} gy_n = y$$

for all $x,y\in X$.

Theorem 3.2. *Let $(X,F,\triangle)$ be a Menger PM space with a T-norm $\triangle$ of H-type, T,f are two self maps of X satisfying (2.2) for some $\varphi\in\Phi$ with $T(X)\subseteq f(X)$, then T and f have a coincidence point in X if either*
(a) X is complete and f is surjective ; or
$(b$ X is complete and f is continuous and T and f are Menger compatible; or
(c) $f(X)$ is complete; or
(d) $T(X)$ is complete.
Furthermore, the coincidence value is unique, i.e. $fp = fq$ whenever $fp = Tp$ and $fq = Tq$ $(p,q\in X)$.

Proof. Let $x_0 \in X$. Since $T(X)\subseteq f(X)$, choose x_1 so that $y_1 = fx_1 = Tx_0$. In general, choose x_{n+1} such that $y_{n+1} = fx_{n+1} = Tx_n$.
From (2.2), we have

$$\begin{aligned} F_{Tx_n,Tx_{n+1}}(\phi(t)) &\geq aF_{fx_n,fx_{n+1}}(t) + b\min\{ \\ &F_{fx_n,Tx_n}(t), F_{fx_{n+1},Tx_{n+1}}(t)\} + c\min \\ &\{F_{fx_n,fx_{n+1}}(t), F_{fx_n,Tx_n}(t), F_{fx_{n+1},Tx_{n+1}}(t)\} \\ &\geq aF_{fx_n,Tx_n}(t) + b\min\{F_{fx_n,Tx_n}(t), \\ &F_{fx_{n+1},Tx_{n+1}}(t)\} + c\min\{F_{fx_n,Tx_n}(t), \\ &F_{fx_n,Tx_n}(t), F_{fx_{n+1},Tx_{n+1}}(t)\}, \end{aligned}$$

where a,b,c are evaluated at (x_n,x_{n+1}). Suppose that for some n , $F_{fx_{n+1},Tx_{n+1}}(t) < F_{fx_n,Tx_n}(t)$ for some $t>0$. Then substituting in the above inequality we have

$$\begin{aligned} F_{Tx_n,Tx_{n+1}}(\phi(t)) \geq aF_{fx_n,Tx_n}(t) &+ (b+c) \\ F_{fx_{n+1},Tx_{n+1}}(t) &> (a+b+c) \\ F_{fx_{n+1},Tx_{n+1}}(t) &\geq F_{Tx_n,Tx_{n+1}}(t), \end{aligned}$$

a contradiction, because $\varphi(t) < t$ for all $t>0$. Therefore, for all n we have

$$F_{fx_{n+1},Tx_{n+1}}(t) \geq F_{fx_n,Tx_n}(t), \quad for\ all\ \ t>0. \tag{3.1}$$

Also, Following the inequality (2.2), we see that

$$F_{y_{n+1},y_{n+2}}(\phi(t)) \geq (a+b+c)\min\{F_{y_n,y_{n+1}}(t), F_{y_{n+1},y_{n+2}}(t)\}.$$

It follows from $\varphi\in\phi$ and Lemma (2.7) that for all $t>0$,

$$F_{y_{n+1},y_{n+2}}(\phi(t)) \geq (a+b+c)F_{y_n,y_{n+1}}(t) \geq F_{y_n,y_{n+1}}(t).$$

Thus we have

$$F_{y_{n+1},y_{n+2}}(\phi^{n+1}(t)) \geq F_{y_0,y_1}(t,) \ \ for\ all\ t>0.$$

For $\delta>0$ and $\varepsilon\in(0,1)$, since $\lim_{t\to\infty} F_{y_0,y_1}(t) = 1$, there is a t_0 such that $F_{y_0,y_1}(t_0) > 1-\varepsilon$. Also, by $\lim_{n\to\infty}\varphi^n(t_0) = 0$, there is a N_0 such that $\varphi^n(t_0) < \delta$ for $n\geq N_0$. Thus for $n>N_0$ we obtain

$$F_{y_{n+1},y_{n+2}}(\delta) \geq F_{y_{n+1},y_{n+2}}(\varphi^{n+1}(t_0))) \geq F_{y_0,y_1}(t_0) > 1-\varepsilon.$$

This means $\lim_{n\to\infty} F_{y_{n+1},y_{n+2}}(t) = 1$ for all $t>0$.
Next we should prove that the sequence $\{y_n\}$ is a Cauchy sequence in X. It is necessary to prove that, for any $\delta>0$ and $\varepsilon\in(0,1)$, there is $N(\varepsilon,\delta)$ such that

$$F_{y_n,y_m}(\delta) > 1-\varepsilon \ \ for\ all\ m>n\geq N(\varepsilon,\delta).$$

To this end, firstly, we can show the following inequality by mathematical induction:

$$F_{y_{n+k},y_n}(\delta) \geq \triangle^k(F_{y_{n+1},y_n}(\delta-\varphi(\delta))) \ \ for\ all\ k\geq 1. \tag{3.2}$$

As $k=1$,

$$\begin{aligned} F_{y_{n+1},y_n}(\delta) &\geq F_{y_{n+1},y_n}(\delta-\varphi(\delta)) \\ &= \triangle(F_{y_{n+1},y_n}(\delta-\varphi(\delta)),1) \\ &\geq \triangle(F_{y_{n+1},y_n}(\delta-\varphi(\delta)), F_{y_{n+1},y_n}(\delta-\varphi(\delta))) \\ &= \triangle^1(F_{y_{n+1},y_n}(\delta-\varphi(\delta))). \end{aligned}$$

Now we assume (3.2) holds for $1\leq k\leq p$. When $k = p+1$,

$$F_{y_{n+p+1},y_n}(\delta) \geq \triangle(F_{y_{n+1},y_n}(\delta-\varphi(\delta)), F_{y_{n+1},y_{n+p+1}}(\varphi(\delta))).$$

By the formulation (3.1), inequality

$$F_{y_{n+p+1},y_{n+p+2}}(\delta) \geq F_{y_n,y_{n+1}}(\delta)$$

holds for all n. Then we have

$$\begin{aligned} F_{y_{n+1},y_{n+p+1}}(\varphi(\delta)) &= F_{Tx_n,Tx_{n+p}}(\varphi(\delta)) \\ &\geq aF_{fx_n,fx_{n+p}}(\delta) + b\min\{F_{fx_n,Tx_n}(\delta), \\ &F_{fx_{n+p},Tx_{n+p}}(\delta)\} + c\min\{F_{fx_n,fx_{n+p}}(\delta), \\ &F_{fx_n,Tx_n}(\delta), F_{fx_{n+p},Tx_{n+p}}(\delta)\} = aF_{y_n,y_{n+p}}(\delta) \\ &+ b\min\{F_{y_n,y_{n+1}}(\delta), F_{y_{n+p},y_{n+p+1}}(\delta)\} + c \\ &\min\{F_{y_n,y_{n+p}}(\delta), F_{y_n,y_{n+1}}(\delta), F_{y_{n+p},y_{n+p+1}}(\delta)\} \geq a \\ &\triangle^p(F_{y_n,y_{n+1}}(\delta-\varphi(\delta))) + bF_{y_n,y_{n+1}}(\delta-\varphi(\delta)) + c \\ &\min\{\triangle^p(F_{y_n,y_{n+1}}(\delta-\varphi(\delta))), F_{y_n,y_{n+1}}(\delta-\varphi(\delta))\} \\ &\geq (a+b+c)\triangle^p(F_{y_n,y_{n+1}}(\delta-\varphi(\delta)) \\ &\geq \triangle^p(F_{y_n,y_{n+1}}(\delta-\varphi(\delta))). \end{aligned}$$

Then

$$F_{y_n,y_{n+p+1}}(\delta) \geq \triangle(F_{y_n,y_{n+1}}(\delta-\varphi(\delta)),\\ F_{y_{n+1},y_{n+p+1}}(\varphi(\delta))) \geq \triangle(F_{y_n,y_{n+1}}(\delta-\varphi(\delta)),\\ \triangle^p(F_{y_n,y_{n+1}}(\delta-\varphi(\delta))))\\ = \triangle^{p+1}(F_{y_n,y_{n+1}}(\delta-\varphi(\delta))).$$

Thus

$$F_{y_{n+k},y_n}(\delta) \geq \triangle^k(F_{y_{n+1},y_n}(\delta-\varphi(\delta))) \;\; for\ all\ k \geq 1.$$

Noting the T-norm $\triangle$ of H-type, for a given $\varepsilon \in (0,1)$, there exists $\lambda \in (0,1)$ such that $\triangle^n(t) > 1-\varepsilon$ for all $n \geq 1$ and when $t > 1-\lambda$. On the other hand, by $\lim_{n\to\infty} F_{y_n,y_{n+1}}(\delta - \varphi(\delta)) = 1$, there is a $N_1(\varepsilon,\delta)$ such that

$$F_{y_n,y_{n+1}}(\delta-\varphi(\delta)) > 1-\lambda, \;\; for\ all\ n > N_1(\varepsilon,\delta).$$

Thus

$$F_{y_n,y_{n+1}}(\delta) > 1-\varepsilon, \;\; for\ all\ k \geq 1\ and\ n > N_1(\varepsilon,\delta).$$

This implies that the sequence $\{y_n\}$ is a Cauchy sequence in X.

Case(a) : Let X is complete and f is surjective. So, by the completeness of X, $\{y_n\}$ converges to a point p in X. So, $\lim_{n\to\infty} fx_{n+1} = \lim_{n\to\infty} Tx_n = p$. Hence there exists a point z in X such that $p = fz$.

From (2.2), we have

$$F_{Tz,Tx_n}(\varphi(t)) \geq aF_{fz,fx_n}(t) + b\min\{F_{fz,Tz}(t),\\ F_{fx_n,Tx_n}(t)\} + c\min\{F_{fz,fx_n}(t),\\ F_{fz,Tz}(t), F_{fx_n,Tx_n}(t)\}.$$

Taking limit as $n \to \infty$, we get

$$F_{Tz,fz}(\varphi(t)) \geq a + (b+c)F_{Tz,fz}(t)\\ \geq (a+b+c)F_{Tz,fz}(t) \geq F_{Tz,fz}(t),$$

by Lemma (2.6) and $\varphi(t) < t$ for all $t > 0$,implies that $fz = Tz$.

Case(b) : Since X is complete,$\{y_n\}$ converges to a point p in X. Suppose f is continuous and f and T are Menger compatible. Then since $\lim_{n\to\infty} y_n = p$, we have $\lim_{n\to\infty} fy_n = fp$. Note that since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n$ and f and T are Menger compatible,
$\lim_{n\to\infty} F_{fTx_n,Tfx_n}(t) = 1$.
From (2.2), we have

$$F_{Tp,Tfx_n}(\varphi(t)) \geq aF_{fp,ffx_n}(t) + b\min\{F_{fp,Tp}(t),\\ F_{ffx_n,Tfx_n}(t)\} + c\min\{F_{fp,ffx_n}(t), F_{fp,Tp}(t),\\ F_{ffx_n,Tfx_n}(t)\} \geq aF_{fp,ffx_n}(t) + (b+c)\min\\ \{\min\{F_{fp,Tp}(t), F_{ffx_n,Tfx_n}(t)\},\\ \min\{F_{fp,ffx_n}(t), F_{fp,Tp}(t), F_{ffx_n,Tfx_n}(t)\}\}.$$

Note that

$$F_{ffx_n,Tfx_n}(\delta) \geq \triangle(F_{ffx_n,fTx_n}(\varphi(\delta)), F_{fTx_n,Tfx_n}(\delta-\varphi(\delta)))$$

for all $\delta > 0$. Using the continuity of f and compatibility of f and T, it follows that $\lim_{n\to\infty} F_{ffx_n,Tfx_n}(\delta) = 1$ for all $\delta > 0$. Since $\lim_{n\to\infty} ffx_n = fp$, it follows that $\lim_{n\to\infty} Tfx_n = fp$.

Taking limit as $n \to \infty$, we get

$$F_{Tp,fp}(\varphi(t)) \geq a + (b+c)F_{Tp,fp}(t) \geq F_{Tp,fp}(t),$$

implies that $fp = Tp$.

Case(c) : In this case since $\{fx_n\}$ is a sequence in $f(X)$ and (X) is complete, $\lim_{n\to\infty} y_n = fx_n = p$ for some $p \in f(X)$. Let $p = fz$ for some $z \in f^{-1}p$ and the proof is complete by case (a).

Case(d) : In this case $p \in T(X) \subseteq f(X)$ and the proof is complete by case (a).

Uniqueness : Let q be another coincidence point of f and T, then by (2.2) with a,b,c evaluated at (p,q),

$$F_{Tp,Tq}(\varphi(t)) \geq aF_{fp,fq}(t) + b\min\{F_{fp,Tp}(t),\\ F_{fq,Tq}(t)\} + c\min\{F_{fp,fq}(t), F_{fp,Tp}(t),\\ F_{fq,Tq}(t)\} \geq (a+c)F_{Tp,Tq}(t) + b\\ \geq F_{Tp,Tq}(t).$$

This implies that $Tp = Tq$ and $fp = fq$.

□

Corollary 3.3. *Let $(X,F,\triangle)$ be a complete Menger PM space with a T-norm $\triangle$ of H-type and T a self mapping of X satisfying (2.2) for some $\varphi \in \Phi$ with $f = I$, the identity map on X. Then T has a unique fixed point and at this fixed point T is continuous.*

Proof. The existence and uniqueness of the fixed point comes from Theorem (3.2) by setting $f = I$. To prove continuity, let $\{y_n\} \subset X$ with $\lim_{n\to\infty} y_n = p$, p the unique fixed point of T.

Using (2.2), we have

$$F_{Tp,Ty_n}(\varphi(t)) \geq aF_{p,y_n}(t) + b\min\{F_{p,Tp}(t),\\ F_{y_n,Ty_n}(t)\} + c\min\{F_{p,y_n}(t), F_{p,Tp}(t), F_{y_n,Ty_n}(t)\}\\ \geq aF_{p,y_n}(t) + bF_{y_n,Ty_n}(t) + c\min\{F_{p,y_n}(t),\\ F_{y_n,Ty_n}(t)\} \geq (a+c)F_{p,y_n}(t) + bF_{y_n,Ty_n}(t)\\ \geq (a+c)F_{p,y_n}(t) + b\triangle(F_{y_n,p}(t-\varphi(t)),\\ F_{p,Ty_n}(\varphi(t))) \geq (a+c)F_{p,y_n}(t) + bF_{p,Ty_n}(\varphi(t)).$$

Hence

$$F_{p,Ty_n}(\varphi(t)) \geq \frac{a+c}{1-b}F_{p,y_n}(t).$$

Taking limit $n \to \infty$, we get $\lim_{n\to\infty} Ty_n = p = Tp$. □

Corollary 3.4. *Let $(X,F,\triangle)$ be a Menger PM space with a T-norm $\triangle$ of H-type, $T : X\times X \to X$ and $f : X \to X$ are two mappings such that for some $\varphi \in \Phi$*

$$F_{T(x,y),T(u,v)}(\varphi(t)) \geq a\min\{F_{fx,fu}(t),F_{fy,fv}(t)\} + b\min\{F_{fx,T(x,y)}(t),F_{fy,T(y,x)}(t), F_{fu,T(u,v)}(t),F_{fv,T(v,u)}(t)\} + c\min\{F_{fx,fu}(t), F_{fy,fv}(t),F_{fx,T(x,y)}(t),F_{fy,T(y,x)}(t),F_{fu,T(u,v)}(t), F_{fv,T(v,u)}(t)\}, \tag{3.3}$$

where $a,b,c \geq 0$ and evaluated at $(x,y),(u,v)$ with $\inf_{x,y,u,v\in X}(a+b+c) = 1$, with $T(X\times X) \subseteq f(X)$. Then T and f have a coupled coincidence point if either one of the conditions (a) or (b) or (c) in Theorem (3.2) holds, or $T(X\times X)$ is complete. Furthermore, the coupled coincidence value is unique.

Proof. Let $\tilde{X} = X\times X$. It follows from Lemma (2.8) that $(\tilde{X},\tilde{F},\triangle)$ is also a Menger PM space, where

$$\tilde{F}_{\tilde{a},\tilde{b}}(t) := \min\{F_{x,u}(t),F_{y,v}(t)\},$$

for $\tilde{a} = (x,y),\tilde{b} = (u,v) \in \tilde{X}$.

The self mappings $G,h : \tilde{X} \to \tilde{X}$ are given by

$$G\tilde{a} = (T(x,y),T(y,x)) \text{ for all } \tilde{a} = (x,y) \in \tilde{X},$$

and

$$h\tilde{a} = (fx, fy) \text{ for all } \tilde{a} = (x,y) \in \tilde{X}.$$

Then a coupled coincidence point of T and f is a coincidence point of G and h in $X\times X$ and vice versa.
On the other hand, for all $t > 0$ and $\tilde{a} = (x,y),\tilde{b} = (u,v) \in \tilde{X}$, we have

$$\begin{aligned}
&F_{T(x,y),T(u,v)}(\varphi(t)) \geq a\min\{F_{fx,fu}(t),F_{fy,fv}(t)\}\\
&+b\min\{F_{fx,T(x,y)}(t),F_{fy,T(y,x)}(t),F_{fu,T(u,v)}(t),\\
&F_{fv,T(v,u)}(t)\}+c\min\{F_{fx,fu}(t),F_{fy,fv}(t)+\\
&F_{fx,T(x,y)}(t),F_{fy,T(y,x)}(t),F_{fu,T(u,v)}(t),F_{fv,T(v,u)}(t)\}\\
&=a\tilde{F}_{h\tilde{a},h\tilde{b}}(t)+b\min\{\min\{F_{fx,T(x,y)}(t),F_{fy,T(y,x)}(t)\},\\
&\min\{F_{fu,T(u,v)}(t),F_{fv,T(v,u)}(t)\}\}+c\min\{\min\{\\
&F_{fx,fu}(t),F_{fy,fv}(t)\},\min\{F_{fx,T(x,y)}(t),F_{fy,T(y,x)}(t)\},\\
&\min\{F_{fu,T(u,v)}(t),F_{fv,T(v,u)}(t)\}\}\\
&=a\tilde{F}_{h\tilde{a},h\tilde{b}}(t)+b\min\{\tilde{F}_{h\tilde{a},G\tilde{a}}(t),\tilde{F}_{h\tilde{b},G\tilde{b}}(t)\}+\\
&c\min\{\tilde{F}_{h\tilde{a},h\tilde{b}}(t),\tilde{F}_{h\tilde{a},G\tilde{a}}(t),\tilde{F}_{h\tilde{b},G\tilde{b}}(t)\}.
\end{aligned}$$

Similarly

$$F_{T(y,x),T(v,u)}(\varphi(t)) \geq a\tilde{F}_{h\tilde{a},h\tilde{b}}(t)+b\min\{\tilde{F}_{h\tilde{a},G\tilde{a}}(t), \tilde{F}_{h\tilde{b},G\tilde{b}}(t)\}+c\min\{\tilde{F}_{h\tilde{a},h\tilde{b}}(t),\tilde{F}_{h\tilde{a},G\tilde{a}}(t),\tilde{F}_{h\tilde{b},G\tilde{b}}(t)\}.$$

Thus

$$\tilde{F}_{G\tilde{a},G\tilde{b}}(\varphi(t)) \geq a\tilde{F}_{h\tilde{a},h\tilde{b}}(t)+b\min\{\tilde{F}_{h\tilde{a},G\tilde{a}}(t), \tilde{F}_{h\tilde{b},G\tilde{b}}(t)\}+c\min\{\tilde{F}_{h\tilde{a},h\tilde{b}}(t),\tilde{F}_{h\tilde{a},G\tilde{a}}(t),\tilde{F}_{h\tilde{b},G\tilde{b}}(t)\}.$$

If X is complete, it follows from Lemma (2.8) that $(\tilde{X},\tilde{F},\triangle)$ is also a complete Menger PM space. Also, it is easy to see that all conditions in Theorem (3.2), hold for two self mappings G and h on $X\times X$. Thus, following Theorem (3.2), we see that T and f have a coupled coincidence point, that is, there exist $p,q \in X$ such that $T(p,q) = fp$ and $T(q,p) = fq$. □

Following similar arguments as in proof of Corollary (3.3) and (3.4), we can deduce the next result. we omit the details of the proof.

Corollary 3.5. *Let $(X,F,\triangle)$ be a complete Menger PM space with a T-norm $\triangle$ of H-type, $T : X\times X \to X$ is a mapping such that for some $\varphi \in \Phi$,*

$$\begin{aligned}
&F_{T(x,y),T(u,v)}(\varphi(t)) \geq a\min\{F_{x,u}(t),F_{y,v}(t)\}+\\
&b\min\{F_{x,T(x,y)}(t),F_{y,T(y,x)}(t),F_{u,T(u,v)}(t),\\
&F_{v,T(v,u)}(t)\}+c\min\{F_{x,u}(t),F_{y,v}(t),F_{x,T(x,y)}(t),\\
&F_{y,T(y,x)}(t),F_{u,T(u,v)}(t),F_{v,T(v,u)}(t)\},
\end{aligned}$$

where $a,b,c \geq 0$ and evaluated at $(x,y),(u,v)$ with $\inf_{x,y,u,v\in X}(a+b+c) = 1$. Then T has a unique coupled point and at this coupled point T is continuous.

Now, we introduce the new concept of weakly commuting of two mappings $T : X\times X \to X$ and $g : X \to X$ on a Menger PM space X.

Definition 3.6. *Let $(X,F,\triangle)$ be a Menger PM space and $T : X\times X \to X$ and $g : X \to X$. Then T and g are called R-weakly commuting of type-(MA_g), if there exists some real number $R \geq 1$ such that*

$$F_{T(T(x,y),T(y,x)),gT(x,y)}(t) \geq RF_{T(x,y),gx}(t)$$

and

$$F_{T(T(y,x),T(x,y)),gT(y,x)}(t) \geq RF_{T(y,x),gy}(t)$$

for all $t > 0$ and $(x,y) \in X\times X$.

Definition 3.7. *Let $(X,F,\triangle)$ be a Menger PM space and $T : X\times X \to X$ and $g : X \to X$. Then T and g are called R-weakly commuting of type-(MA_T), if there exists some real number $R \geq 1$ such that*

$$F_{T(gx,gy),ggx}(t) \geq RF_{T(x,y),gx}(t)$$

and

$$F_{T(gy,gx),ggy}(t) \geq RF_{T(y,x),gy}(t)$$

for all $t > 0$ and $(x,y) \in X\times X$.

Definition 3.8. *Let $(X,F,\triangle)$ be a Menger PM space and $T : X\times X \to X$ and $g : X \to X$. Then T and f are called reciprocal continuous , if $\lim_{n\to\infty} fT(x_n,y_n) = fx$, $\lim_{n\to\infty} fT(y_n,x_n) = fy$ and $\lim_{n\to\infty} T(fx_n, fy_n) = T(x,y)$, whenever $\{(x_n,y_n)\}$ is a sequence in $X\times X$ such that $\lim_{n\to\infty} T(x_n,y_n) = \lim_{n\to\infty} fx_n = x$ and $\lim_{n\to\infty} T(y_n,x_n) = \lim_{n\to\infty} fy_n = y$ for some $(x,y) \in X\times X$.*

Note that a pair of mappings which is reciprocal continuous need not be continuous even on their common fixed point (see for example [14]).

We generalize reciprocal continuity by introducing the notion of weakly reciprocal continuity for a pair of single valued maps as follows.

Definition 3.9. *Let $(X,F,\triangle)$ be a Menger PM space and $T : X \times X \to X$ and $g : X \to X$. Then T and f are called weakly reciprocal continuous , if $\lim_{n\to\infty} fT(x_n,y_n) = fx$ and $\lim_{n\to\infty} fT(y_n,x_n) = fy$ or $\lim_{n\to\infty} T(fx_n, fy_n) = T(x,y)$, whenever $\{(x_n,y_n)\}$ is a sequence in $X \times X$ such that $\lim_{n\to\infty} T(x_n,y_n) = \lim_{n\to\infty} fx_n = x$ and $\lim_{n\to\infty} T(y_n,x_n) = \lim_{n\to\infty} fy_n = y$ for some $(x,y) \in X \times X$.*

Theorem 3.10. *Let $(X,F,\triangle)$ be a complete Menger PM space with a T-norm $\triangle$ of H-type, $T : X \times X \to X$ and $f : X \to X$ are two weakly reciprocally continuous mappings satisfying (3.3) for some $\varphi \in \Phi$ with $T(X \times X) \subseteq f(X)$, then T and f have a common coupled fixed point in $X \times X$ if either*
(a) T and f are Menger compatible; or
(b) T and f are R-weakly commuting of type-(MA_f); or
(c) T and f are R-weakly commuting of type-(MA_T).

Proof. It follows from proof of Corollary (3.4), that $G,h : \tilde{X} \to \tilde{X}$ are two self mappings on $\tilde{X} = X \times X$ such that common coupled fixed point of T and f is a common fixed point G and h in $X \times X$ and vice versa.

On the other hand, following similar argument as in proof of corollary (3.4), we have

$$\begin{aligned}\tilde{F}_{G\tilde{a},G\tilde{b}}(\varphi(t)) &\geq a\tilde{F}_{h\tilde{a},h\tilde{b}}(t) + b\min\{\tilde{F}_{h\tilde{a},G\tilde{a}}(t),\\ &\tilde{F}_{h\tilde{b},G\tilde{b}}(t)\} + c\min\{\tilde{F}_{h\tilde{a},h\tilde{b}}(t), \tilde{F}_{h\tilde{a},G\tilde{a}}(t), \tilde{F}_{h\tilde{b},G\tilde{b}}(t)\}.\end{aligned}$$

for all $t > 0$ and $\tilde{a} = (x,y), \tilde{b} = (u,v) \in \tilde{X}$.

It is easy to see that G and h are weakly reciprocally continuous self maps of $\tilde{X}$ (and Menger compatible), if T and f are weakly reciprocally continuous (and Menger compatible).

Also, if T and f are R-weakly commuting of type-(MA_f) (or type-(MA_T)), we can prove that G and h are R-weakly commuting of type-(MA_h) (or type-(MA_G)).

Thus, from Theorem (2.19)[4], we see that T and f have common coupled fixed point. i.e., $T(p,q) = fp = p$ and $T(q,p) = fq = q$ for some $(p,q) \in X \times X$. □

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