Malaya Journal of Matematik 4(1)(2013) 30–43

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Existence of mild solutions for impulsive fractional stochastic equations with infinite delay

Toufik Guendouzi* and Khadem Mehdi

Laboratory of Stochastic Models, Statistic and Applications, Tahar Moulay University PO.Box 138 En-Nasr, 20000 Saida, Algeria.

Abstract

This paper is mainly concerned with the existence of mild solutions for a class of fractional stochastic differential equations with impulses in Hilbert spaces. A new set of sufficient conditions are formulated and proved for the existence of mild solutions by means of Sadovskii's fixed point theorem. An example is given to illustrate the theory.

Keywords: Existence result, fractional stochastic differential equation, fixed point technique, infinite delay, resolvent operators.

2010 MSC: 34K30, 34K50, 26A33.

1 Introduction

The stochastic differential equations have been widely applied in science, engineering, biology, mathematical finance and in almost all applied sciences. In the present literature, there are many papers on the existence and uniqueness of solutions to stochastic differential equations (see [1, 2, 8] and references therein). More recently, Chang et al. [4] investigated the existence of square-mean almost automorphic mild solutions to nonautonomous stochastic differential equations in Hilbert spaces by using semigroup theory and fixed point approach. Fu and Liu [8] discussed the existence and uniqueness of square-mean almost automorphic solutions to some linear and nonlinear stochastic differential equations and in which they studied the asymptotic stability of the unique square-mean almost automorphic solution in the square-mean sense.

Recently, fractional differential equations have found numerous applications in various fields of science and engineering [11]. The existence of solutions for nonlinear fractional stochastic differential equations have been studied by few authors [9, 18].

On the other hand, the theory of impulsive differential equations is emerging as an active area of investigation due to the application in area such as mechanics, electrical engineering, medicine biology, and ecology, see Benchohra and Henderson [3], Hernández et al. [10], Lin and Hu [13], Prato and Zabczyk [14]. As an adequate model, impulsive differential equations are used to study the evolution of processes that are subject to sudden changes in their states. However, to the best of our knowledge, it seems that little is known about impulsive fractional stochastic equations with infinite delay and the aim of this paper is to fill this gap. We refer the interested reader, for instance, to [18] and references therein for impulsive fractional stochastic equations.

Inspired by the mentioned work [18] in this paper, we are interested in studying the existence of mild solutions

*Corresponding author.

E-mail addresses: tf.guendouzi@gmail.com (T. Guendouzi) and m.mehdi1986@gmail.com (K. Mehdi)

of the following impulsive fractional stochastic differential equations with infinite delay in the form

$$\begin{cases} {}^cD_t^\alpha[x(t)+g(t,x_t)] &=& A\Big[x(t)+g(t,x_t)\Big]+f(t,x_t,B_1x(t))+\sigma(t,x_t,B_2x(t))\dfrac{dw(t)}{dt}, \\ & & t\in J:=[0,T],\quad T>0,\quad t\neq t_k, \\ \Delta x(t_k) &=& I_k(x(t_k^-)),\quad k=1,2,\dots,m, \\ x(t) &=& \phi(t),\quad \phi(t)\in\mathcal{B}_h, \end{cases} \tag{1.1}$$

where ${}^cD_t^\alpha$ is the Caputo fractional derivative of order α, $0<\alpha<1$; $x(.)$ takes the value in the separable Hilbert space $\mathcal{H}$; $A:\mathcal{D}(A)\subset\mathcal{H}\to\mathcal{H}$ is the infinitesimal generator of an α-resolvent family $S_\alpha(t)_{t\geq 0}$. The history $x_t:(-\infty,0]\to\mathcal{H}$, $x_t(\theta)=x(t+\theta)$, $\theta\leq 0$, belongs to an abstract phase space $\mathcal{B}_h$, which will be described axiomatically in Section 2; $g:J\times\mathcal{B}_h\to\mathcal{H}$, $f:J\times\mathcal{B}_h\times\mathcal{H}\to\mathcal{H}$ and $\sigma:J\times\mathcal{B}_h\times\mathcal{H}\to\mathcal{L}_2^0$ are appropriate functions to be specified later; $I_k:\mathcal{B}_h\to\mathcal{H}$, $k=1,2,\dots,m$, are appropriate functions. The terms $B_1x(t)$ and $B_2x(t)$ are given by $B_1x(t)=\int_0^t K(t,s)x(s)ds$ and $B_2x(t)=\int_0^t P(t,s)x(s)ds$ respectively, where $K,P\in\mathcal{C}(\mathcal{D},\mathbb{R}^+)$ are the set of all positive continuous functions on $\mathcal{D}=\{(t,s)\in\mathbb{R}^2:0\leq s\leq t\leq T\}$. Here $0=t_0<t_1<\ldots<t_m<t_{m+1}=T$, $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$, $x(t_k^+)=\lim_{h\to 0}x(t_k+h)$ and $x(t_k^-)=\lim_{h\to 0}x(t_k-h)$ represent the right and left limits of $x(t)$ at $t=t_k$, respectively. The initial data $\phi=\{\phi(t),t\in(-\infty,0]\}$ is an $\mathcal{F}_0$-measurable, $\mathcal{B}_h$-valued random variable independent of w with finite second moments.

The paper is organized as follows. In section 2, we briefly present some basic notations and preliminaries. In section 3, is devoted to the development of our main existence results and our basic tool include Sadovskii's fixed point theorem. Finally, the paper is conclude with an example to illustrate the obtained results.

2 Preliminaries and basic properties

Let $\mathcal{H},\mathcal{K}$ be two separable Hilbert spaces and $\mathcal{L}(\mathcal{K},\mathcal{H})$ be the space of bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$. For convenience, we will use the same notation $\|.\|$ to denote the norms in $\mathcal{H},\mathcal{K}$ and $\mathcal{L}(\mathcal{K},\mathcal{H})$, and use $(.,.)$ to denote the inner product of $\mathcal{H}$ and $\mathcal{K}$ without any confusion. Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ be a filtered complete probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. $w=(w_t)_{t\geq 0}$ be a Q-Wiener process defined on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ with the covariance operator Q such that $trQ<\infty$. We assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathcal{K}$, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k=\lambda_k e_k$, $k=1,2,\dots$ and a sequence $\{\beta_k\}_{k\geq 1}$ of independent Brownian motions such that

$$(w(t),e)_{\mathcal{K}}=\sum_{k=1}^{\infty}\sqrt{\lambda_k}(e_k,e)_{\mathcal{K}}\beta_k(t),\quad e\in\mathcal{K},t\in[0,b].$$

Let $\mathcal{L}_2^0=\mathcal{L}_2(Q^{1/2}\mathcal{K},\mathcal{H})$ be the space of all HilbertSchmidt operators from $Q^{1/2}\mathcal{K}$ into $\mathcal{H}$ with the inner product $\langle\psi,\pi\rangle_{\mathcal{L}_2^0}=tr[\psi Q\pi^\star]$.

Assume that $h:(-\infty,0]\to(0,\infty)$ with $l=\int_{-\infty}^0 h(t)dt<\infty$ a continuous function. We define the abstract phase space $\mathcal{B}_h$ by

$$\begin{aligned}\mathcal{B}_h &= \Bigg\{\phi:(-\infty,0]\to\mathcal{H},\text{ for any } a>0, (\mathbb{E}|\phi(\theta)|^2)^{1/2}\text{ is bounded and measurable} \\ &\qquad \text{function on } [-a,0] \text{ with } \phi(0)=0 \text{ and } \int_{-\infty}^0 h(s)\sup_{s\leq\theta\leq 0}(\mathbb{E}|\phi(\theta)|^2)^{1/2}ds<\infty\Bigg\}.\end{aligned}$$

If $\mathcal{B}_h$ is endowed with the norm

$$\|\phi\|_{\mathcal{B}_h}=\int_{-\infty}^0 h(s)\sup_{s\leq\theta\leq 0}(\mathbb{E}|\phi(\theta)|^2)^{1/2}ds,\quad \phi\in\mathcal{B}_h,$$

then $(\mathcal{B}_h,\|.\|_{\mathcal{B}_h})$ is a Banach space [5].

We consider the space

$$\begin{aligned}\mathcal{B}_b &= \Bigg\{x:(-\infty,T]\to\mathcal{H}\text{ such that } x|_{J_k}\in\mathcal{C}(J_k,\mathcal{H})\text{ and there exist} \\ &\qquad x(t_k^+)\text{ and } x(t_k^-)\text{ with } x(t_k)=x(t_k^-),\ x_0=\phi\in\mathcal{B}_h,\ k=1,2,\dots,m\Bigg\},\end{aligned}$$

where $x|_{J_k}$ is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$. the function $\|.\|_{\mathcal{B}_h}$ to be a seminorm in $\mathcal{B}_b$, it is defined by

$$\|x\|_{\mathcal{B}_b} = \|\phi\|_{\mathcal{B}_h} + \sup_{0\leq s\leq T} (\mathbb{E}\|x(s)\|^2)^{1/2}, \quad x \in \mathcal{B}_b$$

Lemma 2.1 ([16]). *Assume that $x \in B_h$; then for $t \in J$, $x_t \in \mathcal{B}_h$. Moreover,*

$$l(\mathbb{E}\|x(t)\|^2)^{1/2} \leq l \sup_{0\leq s\leq T} (\mathbb{E}\|x(s)\|^2)^{1/2} + \|x_0\|_{\mathcal{B}_h},$$

where $l = \int_{-\infty}^{0} h(s)ds < \infty$.

Let us recall the following known definitions. For more details see [12].

Definition 2.1. *The fractional integral of order α with the lower limit 0 for a function f is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \alpha > 0$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function.

Definition 2.2. *Riemann-Liouville derivative of order α with lower limit zero for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$$^{L}D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0, n-1 < \alpha < n. \tag{2.2}$$

Definition 2.3. *The Caputo derivative of order α for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$$^{c}D^\alpha f(t) = {}^{L}D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^k(0) \right), \quad t > 0, n-1 < \alpha < n. \tag{2.3}$$

If $f(t) \in C^n[0, \infty)$, then

$$^{c}D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s) ds = I^{n-\alpha} f^n(s), \quad t > 0, n-1 < \alpha < n$$

Obviously, the Caputo derivative of a constant is equal to zero. The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as

$$L\{^{c}D^\alpha f(t); s\} = s^\alpha \hat{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0); \quad n-1 \leq \alpha < n.$$

Definition 2.4. *A two parameter function of the Mittag-Leffler type is defined by the series expansion*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_C \frac{\mu^{\alpha-\beta} e^\mu}{\mu^\alpha - z} d\mu, \quad \alpha, \beta \in C, \mathcal{R}(\alpha) > 0,$$

where C is a contour which starts and ends at $-\infty$ end encircles the disc $|\mu| \leq |z|^{1/2}$ counter clockwise.

For short, $E_\alpha(z) = E_{\alpha,1}(z)$. It is an entire function which provides a simple generalization of the exponent function: $E_1(z) = e^z$ and the cosine function: $E_2(z^2) = \cos h(z)$, $E_2(-z^2) = \cos(z)$, and plays a vital role in the theory of fractional differential equations. The most interesting properties of the Mittag-Leffler functions are associated with their Laplace integral

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^\alpha) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega}, \quad Re\lambda > \omega^{\frac{1}{\alpha}}, \omega > 0,$$

and for more details see [12].

Definition 2.5 ([23]). *A closed and linear operator A is said to be sectorial if there are constants $\omega \in \mathbb{R}$, $\theta \in [\frac{\pi}{2}, \pi]$, $M > 0$, such that the following two conditions are satisfied:*

i. $\rho(A) \subset \Sigma_{\theta,\omega} = \{\lambda \in C : \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\}$, *ii.* $\|R(\lambda, A)\| = \|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda-\omega|}$, $\lambda \in \Sigma_{\theta,\omega}$.

Definition 2.6. *Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space H. Let $\rho(A)$ be the resolvent set of A. We say that A is the generator of an α-resolvent family if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(H)$, where $L(H)$ is a Banach space of all bounded linear operators from H into H and the corresponding norm is denoted by $\|.\|$, such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and*

$$(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{\lambda t} S_\alpha(t)x dt, \quad Re\lambda > \omega, x \in H, \tag{2.4}$$

where $S_\alpha(t)$ is called the α-resolvent family generated by A.

Definition 2.7. *Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space H and $\alpha > 0$. We say that A is the generator of a solution operator if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(H)$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and*

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{\lambda t} S_\alpha(t)x dt, \quad Re\lambda > \omega, x \in H, \tag{2.5}$$

where $S_\alpha(t)$ is called the solution operator generated by A.

The concept of the solution operator is closely related to the concept of a resolvent family. For more details on α-resolvent family and solution operators, we refer the reader to [12].

Lemma 2.2 ([6]). *If f satisfies the uniform Hölder condition with the exponent $\beta \in (0,1]$ and A is a sectorial operator, then the unique solution of the Cauchy problem*

$$\begin{aligned} {}^cD_t^\alpha x(t) &= Ax(t) + f(t, x_t, Bx(t)), \quad t > t_0, t_0 \geq 0, 0 < \alpha < 1, \\ x(t) &= \phi(t), \quad t \leq t_0, \end{aligned} \tag{2.6}$$

is given by

$$x(t) = T_\alpha(t - t_0)(x(t_0^+)) + \int_{t_0}^t S_\alpha(t-s) f(s, x_s, Fx(s)) ds, \tag{2.7}$$

where

$$T_\alpha(t) = E_{\alpha,1}(At^\alpha) = \frac{1}{2\pi i}\int_{\widehat{B}_r} e^{\lambda t}\frac{\lambda^{\alpha-1}}{\lambda^\alpha - A} d\lambda, \tag{2.8}$$

$$S_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha) = \frac{1}{2\pi i}\int_{\widehat{B}_r} e^{\lambda t}\frac{1}{\lambda^\alpha - A} d\lambda, \tag{2.9}$$

here $\widehat{B}_r$ denotes the Bromwich path; $S_\alpha(t)$ is called the α-resolvent family and $T_\alpha(t)$ is the solution operator generated by A.

The following result on the operator $S_\alpha(t)$ appeared and proved in [23].

Theorem 2.1. *If $\alpha \in (0,1)$ and $A \in \mathcal{A}^\alpha(\theta_0, \omega_0)$ is a sectorial operator, then for any $x \in \mathcal{H}$ and $t > 0$, we have*

$$\|S_\alpha(t)\| \leq Ce^{\omega t}(1 + t^{\alpha-1}), \qquad t > 0, \omega > \omega_0,$$

where C is a constant depending only on θ and ω.

At the end of this section, we recall the fixed point theorem of Sadovskii [17] which is used to establish the existence of the mild solution to the impulsive fractional system (1.1).

Theorem 2.2 ([17]). *Let Φ be a condensing operator on a Banach space $\mathcal{H}$, that is, Φ is continuous and takes bounded sets into bounded sets, and $\mu(\Phi(B)) \leq \mu(B)$ for every bounded set B of $\mathcal{H}$ with $\mu(B) > 0$. If $\Phi(N) \subset N$ for a convex, closed and bounded set N of $\mathcal{H}$, then Φ has a fixed point in $\mathcal{H}$ (where $\mu(.)$ denotes Kuratowski's measure of noncompactness).*

3 The mild solution and existence

In this section, we consider the fractional impulsive system (1.1). We first present the definition of mild solutions for the system based on the paper [7].

Definition 3.1. *An $\mathcal{H}$-valued stochastic process $\{x(t), t \in (-\infty, T]\}$ is said to be a mild solution of the system (1.1) if $x_0 = \phi \in \mathcal{B}_h$ satisfying $x_0 \in \mathcal{L}_2^0(\Omega, \mathcal{H})$ and the following conditions hold.*

i. *$x(t)$ is $\mathcal{F}_t$ adapted and measurable, $t \geq 0$;*

ii. *x_t is $\mathcal{B}_h$-valued and the restriction of $x(.)$ to the interval $(t_k, t_{k+1}]$, $k = 1, 2, \dots, m$ is continuous;*

iii. *for each $t \in J$, $x(t)$ satisfies the following integral equation*

$$x(t) = \begin{cases} \phi(t), \quad t \in (-\infty, 0], \\ T_\alpha(t)[\phi(0) + g(0,\phi)] - g(t, x_t) + \displaystyle\int_0^t S_\alpha(t-s) f(s, x_s, B_1 x(s)) ds \\ \quad + \displaystyle\int_0^t S_\alpha(t-s)\sigma(s, x_s, B_2 x(s)) dw(s), \quad t \in [0, t_1], \\ T_\alpha(t)[\phi(0) + g(0,\phi)] + T_\alpha(t - t_1) I_1(x(t_1^-)) - g(t, x_t) \\ \quad + T_\alpha(t-t_1)[g(t_1, x_{t_1} + I_1(x_{t_1^-})) - g(t_1, x_{t_1})] \\ \quad + \displaystyle\int_0^t S_\alpha(t-s) f(s, x_s, B_1 x(s)) ds + \int_0^t S_\alpha(t-s)\sigma(s, x_s, B_2 x(s)) dw(s), \quad t \in (t_1, t_2], \\ \vdots \\ T_\alpha(t)[\phi(0) + g(0,\phi)] + \displaystyle\sum_{k=1}^m T_\alpha(t-t_k) I_k(x(t_k^-)) - g(t, x_t) \\ \quad + \displaystyle\sum_{k=1}^m T_\alpha(t-t_k)[g(t_k, x_{t_k} + I_k(x_{t_k^-})) - g(t_k, x_{t_k})] \\ \quad + \displaystyle\int_0^t S_\alpha(t-s) f(s, x_s, B_1 x(s)) ds + \int_0^t S_\alpha(t-s)\sigma(s, x_s, B_2 x(s)) dw(s), \quad t \in (t_m, T]. \end{cases} \tag{3.1}$$

iv. *$\Delta x|_{t=t_k} = I_k(x(t_k^-))$, $k = 1, 2, \dots, m$ the restriction of $x(.)$ to the interval $[0, T) \backslash \{t_1, \dots, t_m\}$ is continuous.*

In order to explain our theorem, we need the following assumptions.

$(H1)$: If $\alpha \in (0,1)$ and $A \in \mathcal{A}^\alpha(\theta_0, \omega_0)$, then for $x \in \mathcal{H}$ and $t > 0$ we have $\|T_\alpha(t)\| \leq M e^{\omega t}$ and $\|S_\alpha(t)\| \leq C e^{\omega t}(1 + t^{\alpha-1})$, $\omega > \omega_0$. Thus we have

$$\|T_\alpha(t)\| \leq \widetilde{M}_T \quad \text{and} \quad \|S_\alpha(t)\| \leq t^{\alpha-1}\widetilde{M}_S,$$

where $\widetilde{M}_T = \sup_{0 \leq t \leq T} \|T_\alpha(t)\|$, and $\widetilde{M}_S = \sup_{0 \leq t \leq T} C e^{\omega t}(1 + t^{1-\alpha})$ (fore more details, see [23]).

$(H2)$: The function $g : J \times \mathcal{B}_h \to \mathcal{H}$ is continuous and there exists some constant $M_g > 0$ such that

$$\mathbb{E}\|g(t, \psi_1) - g(t, \psi_2)\|_{\mathcal{H}}^2 \leq M_g \|\psi_1 - \psi_2\|_{\mathcal{B}_h}^2, \quad (t, \psi_i) \in J \times \mathcal{B}_h, \quad i = 1, 2,$$

$$\mathbb{E}\|g(t, \psi)\|_{\mathcal{H}}^2 \leq M_g \Big(\|\psi\|_{\mathcal{B}_h}^2 + 1\Big).$$

$(H3)$: The function $f : J \times \mathcal{B}_h \times \mathcal{H} \to \mathcal{H}$ satisfies the following properties:

i. $f(t, \cdot, \cdot) : \mathcal{B}_h \times \mathcal{H} \to \mathcal{H}$ is continuous for each $t \in J$ and for each $(\psi, x) \in \mathcal{B}_h \times \mathcal{H}$, $f(\cdot, \psi, x) : J \to \mathcal{H}$ is strongly measurable;

ii. there exist two positive integrable functions $\mu_1, \mu_2 \in L^1([0, T])$ and a continuous nondecreasing function $\Xi_f : [0, \infty) \to (0, \infty)$ such that for every $(t, \psi, x) \in J \times \mathcal{B}_h \times \mathcal{H}$, we have

$$\mathbb{E}\|f(t, \psi, x)\|_{\mathcal{H}}^2 \leq \mu_1(t)\Xi_f\Big(\|\psi\|_{\mathcal{B}_h}^2\Big) + \mu_2(t)\mathbb{E}\|x\|_{\mathcal{H}}^2, \qquad \liminf_{q \to \infty} \frac{\Xi_f(q)}{q} = \Lambda < \infty.$$

iii. there exist two positive integrable functions $\mu_1, \mu_2 \in L^1([0,T])$ such that

$$\mathbb{E}\|f(t,\psi,x) - f(t,\varphi,y)\|^2_{\mathcal{H}} \leq \mu_1(t)\|\psi - \varphi\|^2_{\mathcal{B}_h} + \mu_2(t)\mathbb{E}\|x-y\|^2_{\mathcal{H}},$$

for every (t,ψ,x) and $(t,\varphi,y) \in J \times \mathcal{B}_h \times \mathcal{H}$.

$(H4)$: The function $\sigma : J \times \mathcal{B}_h \times \mathcal{H} \to \mathcal{L}^0_2$ satisfies the following properties:

i. $\sigma(t,\cdot,\cdot) : \mathcal{B}_h \times \mathcal{H} \to \mathcal{L}^0_2$ is continuous for each $t \in J$ and for each $(\psi,x) \in \mathcal{B}_h \times \mathcal{H}$, $\sigma(\cdot,\psi,x) : J \to \mathcal{L}^0_2$ is strongly measurable;

ii. there exist two positive integrable functions $\nu_1, \nu_2 \in L^1([0,T])$ and a continuous nondecreasing function $\Xi_\sigma : [0,\infty) \to (0,\infty)$ such that for every $(t,\psi,x) \in J \times \mathcal{B}_h \times \mathcal{H}$, we have

$$\mathbb{E}\|\sigma(t,\psi,x)\|^2_{\mathcal{L}^0_2} \leq \nu_1(t)\Xi_\sigma\left(\|\psi\|^2_{\mathcal{B}_h}\right) + \nu_2(t)\mathbb{E}\|x\|^2_{\mathcal{H}}, \qquad \liminf_{q\to\infty}\frac{\Xi_\sigma(q)}{q} = \Upsilon < \infty.$$

iii. there exist two positive integrable functions $\nu_1, \nu_2 \in L^1([0,T])$ such that

$$\mathbb{E}\|\sigma(t,\psi,x) - \sigma(t,\varphi,y)\|^2_{\mathcal{L}^0_2} \leq \nu_1(t)\|\psi - \varphi\|^2_{\mathcal{B}_h} + \nu_2(t)\mathbb{E}\|x-y\|^2_{\mathcal{H}},$$

for every (t,ψ,x) and $(t,\varphi,y) \in J \times \mathcal{B}_h \times \mathcal{H}$.

$(H5)$: The function $I_k : \mathcal{H} \to \mathcal{H}$ is continuous and there exists $\Theta > 0$ such that

$$\Theta = \max_{1\leq k\leq m,\;\; x\in B_q}\{\mathbb{E}\|I_k(x)\|^2_{\mathcal{H}}\},$$

where $B_q = \{y \in \mathcal{B}^0_b, \|y\|^2_{\mathcal{B}^0_b} \leq q, \;\; q > 0\}$.
The set B_q is clearly a bounded closed convex set in $\mathcal{B}^0_b$ for each q and for each $y \in B_q$. From Lemma 2.1, we have

$$\begin{aligned}
\|y_t + \bar{z}_t\|^2_{\mathcal{B}_h} &\leq 2(\|y_t\|^2_{\mathcal{B}_h} + \|\bar{z}_t\|^2_{\mathcal{B}_h}) \\
&\leq 4\left(l^2\sup_{0\leq t\leq T}\mathbb{E}\|y(t)\|^2_{\mathcal{H}} + \|y_0\|^2_{\mathcal{B}_h}\right) + 4\left(l^2\sup_{0\leq t\leq T}\mathbb{E}\|y(t)\|^2_{\mathcal{H}} + \|\bar{z}_0\|^2_{\mathcal{B}_h}\right) \\
&\leq 4(\|\phi\|^2_{\mathcal{B}_h} + l^2 q).
\end{aligned} \tag{3.2}$$

The main object of this paper is to explain and prove the following theorem.

Theorem 3.1. *Assume that the assumptions $(H1)$-$(H5)$ hold. Then the impulsive stochastic fractional system* (1.1) *has a mild solution on $(-\infty,T]$ provided that*

$$\widetilde{C} + 16M_g l^2 + 7\widetilde{M}^2_S T^{2\alpha}\Big[\frac{\eta_1}{\alpha^2} + \frac{\eta_2}{T(2\alpha-1)}\Big] < 1 \tag{3.3}$$

and

$$l^2 M_g + \widetilde{M}^2_S T^{2\alpha}\Big[\frac{\vartheta_1}{\alpha^2} + \frac{\vartheta_2}{T(2\alpha-1)}\Big] < 1, \tag{3.4}$$

$\widetilde{C}$ is a positive constant depending only on $\widetilde{M}_T, M_g$ and l.

Proof. Consider the operator $\mathcal{P}:\mathcal{B}_b\to\mathcal{B}_b$ defined by

$$\mathcal{P}(t)=\begin{cases}\phi(t), \quad t\in(-\infty,0],\\ T_\alpha(t)[\phi(0)+g(0,\phi)]-g(t,x_t)+\int_0^t S_\alpha(t-s)f(s,x_s,B_1x(s))ds\\ \quad+\int_0^t S_\alpha(t-s)\sigma(s,x_s,B_2x(s))dw(s), \quad t\in[0,t_1],\\ T_\alpha(t)[\phi(0)+g(0,\phi)]+T_\alpha(t-t_1)I_1(x(t_1^-))-g(t,x_t)\\ \quad+T_\alpha(t-t_1)[g(t_1,x_{t_1}+I_1(x_{t_1^-}))-g(t_1,x_{t_1})]\\ \quad+\int_0^t S_\alpha(t-s)f(s,x_s,B_1x(s))ds+\int_0^t S_\alpha(t-s)\sigma(s,x_s,B_2x(s))dw(s), \quad t\in(t_1,t_2],\\ \vdots\\ T_\alpha(t)[\phi(0)+g(0,\phi)]+\sum_{k=1}^m T_\alpha(t-t_k)I_k(x(t_k^-))-g(t,x_t)\\ \quad+\sum_{k=1}^m T_\alpha(t-t_k)[g(t_k,x_{t_k}+I_k(x_{t_k^-}))-g(t_k,x_{t_k})]\\ \quad+\int_0^t S_\alpha(t-s)f(s,x_s,B_1x(s))ds+\int_0^t S_\alpha(t-s)\sigma(s,x_s,B_2x(s))dw(s), \quad t\in(t_m,T].\end{cases} \tag{3.5}$$

We shall show that $\mathcal{P}$ has a fixed point, which is then a mild solution for the impulsive system (1.1). For $\phi\in\mathcal{B}_h$, define

$$\bar{z}(t)=\begin{cases}\phi(t), & t\in(-\infty,0];\\ 0, & t\in J.\end{cases}$$

Then $\bar{z}\in\mathcal{B}_b$. Let $x(t)=y(t)+\bar{z}(t)$, $t\in(-\infty,T]$. It is easy to check that x satisfies (1.1) if and only if $y_0=0$ and

$$y(t)=\begin{cases}T_\alpha(t)[\phi(0)+g(0,\phi)]-g(t,y_t+\bar{z}_t)+\int_0^t S_\alpha(t-s)f(s,y_s+\bar{z}_s,B_1(y(s)+\bar{z}(s)))ds\\ \quad+\int_0^t S_\alpha(t-s)\sigma(s,y_s+\bar{z}_s,B_2(y(s)+\bar{z}(s)))dw(s), \quad t\in[0,t_1],\\ T_\alpha(t)[\phi(0)+g(0,\phi)]+T_\alpha(t-t_1)I_1(y(t_1^-))-g(t,y_t+\bar{z}_t)\\ \quad+T_\alpha(t-t_1)[g(t_1,y_{t_1}+\bar{z}_{t_1}+I_1(y_{t_1^-}+\bar{z}_{t_1^-}))-g(t_1,y_{t_1}+\bar{z}_{t_1})]\\ \quad+\int_0^t S_\alpha(t-s)f(s,y_s+\bar{z}_s,B_1(y(s)+\bar{z}(s)))ds\\ \quad+\int_0^t S_\alpha(t-s)\sigma(s,y_s+\bar{z}_s,B_2(y(s)+\bar{z}(s)))dw(s), \quad t\in(t_1,t_2],\\ \vdots\\ T_\alpha(t)[\phi(0)+g(0,\phi)]+\sum_{k=1}^m T_\alpha(t-t_k)I_k(y(t_k^-))-g(t,y_t+\bar{z}_t)\\ \quad+\sum_{k=1}^m T_\alpha(t-t_k)[g(t_k,y_{t_k}+\bar{z}_{t_k}+I_k(y_{t_k^-}+\bar{z}_{t_k^-}))-g(t_k,y_{t_k}+\bar{z}_{t_k})]\\ \quad+\int_0^t S_\alpha(t-s)f(s,y_s+\bar{z}_s,B_1(y(s)+\bar{z}(s)))ds\\ \quad+\int_0^t S_\alpha(t-s)\sigma(s,y_s+\bar{z}_s,B_2(y(s)+\bar{z}(s)))dw(s), \quad t\in(t_m,T].\end{cases}$$

Set

$$\mathcal{B}_b^0=\{y\in\mathcal{B}_b, y_0=0\in\mathcal{B}_h\}.$$

Thus, for any $y\in\mathcal{B}_b^0$ we have

$$\|y\|_b=\|y_0\|_{\mathcal{B}_h}+\sup_{0\le s\le T}\Big(\mathbb{E}\|y(s)\|^2\Big)^{\frac{1}{2}}=\sup_{0\le s\le T}\Big(\mathbb{E}\|y(s)\|^2\Big)^{\frac{1}{2}}.$$

Therefore, $(\mathcal{B}_b^0,\|\cdot\|_b)$ is a Banach space.

Consider the map Π on $\mathcal{B}_b^0$ defined by

$$(\Pi y)(t) = \begin{cases} T_\alpha(t)[\phi(0)+g(0,\phi)] - g(t,y_t+\bar{z}_t) + \int_0^t S_\alpha(t-s)f(s,y_s+\bar{z}_s,B_1(y(s)+\bar{z}(s)))ds \\ \quad + \int_0^t S_\alpha(t-s)\sigma(s,y_s+\bar{z}_s,B_2(y(s)+\bar{z}(s)))dw(s), \quad t\in[0,t_1], \\ T_\alpha(t)[\phi(0)+g(0,\phi)] + T_\alpha(t-t_1)I_1(y(t_1^-)) - g(t,y_t+\bar{z}_t) \\ \quad + T_\alpha(t-t_1)[g(t_1,y_{t_1}+\bar{z}_{t_1}+I_1(y_{t_1^-}+\bar{z}_{t_1^-})) - g(t_1,y_{t_1}+\bar{z}_{t_1})] \\ \quad + \int_0^t S_\alpha(t-s)f(s,y_s+\bar{z}_s,B_1(y(s)+\bar{z}(s)))ds \\ \quad + \int_0^t S_\alpha(t-s)\sigma(s,y_s+\bar{z}_s,B_2(y(s)+\bar{z}(s)))dw(s), \quad t\in(t_1,t_2], \\ \vdots \\ T_\alpha(t)[\phi(0)+g(0,\phi)] + \sum_{k=1}^m T_\alpha(t-t_k)I_k(y(t_k^-)) - g(t,y_t+\bar{z}_t) \\ \quad + \sum_{k=1}^m T_\alpha(t-t_k)[g(t_k,y_{t_k}+\bar{z}_{t_k}+I_k(y_{t_k^-}+\bar{z}_{t_k^-})) - g(t_k,y_{t_k}+\bar{z}_{t_k})] \\ \quad + \int_0^t S_\alpha(t-s)f(s,y_s+\bar{z}_s,B_1(y(s)+\bar{z}(s)))ds \\ \quad + \int_0^t S_\alpha(t-s)\sigma(s,y_s+\bar{z}_s,B_2(y(s)+\bar{z}(s)))dw(s), \quad t\in(t_m,T]. \end{cases}$$

It is clear that the operator $\mathcal{P}$ has a fixed point if and only if Π has a fixed point. So let us prove that Π has a fixed point. Now, we decompose Π as $\Pi = \Pi_1 + \Pi_2$, where

$$(\Pi_1 y)(t) = \begin{cases} 0, \quad t\in[0,t_1], \\ T_\alpha(t-t_1)I_1(y(t_1^-)) \\ \quad + T_\alpha(t-t_1)[g(t_1,y_{t_1}+\bar{z}_{t_1}+I_1(y_{t_1^-}+\bar{z}_{t_1^-})) - g(t_1,y_{t_1}+\bar{z}_{t_1})], \quad t\in(t_1,t_2], \\ \vdots \\ \sum_{k=1}^m T_\alpha(t-t_k)I_k(y(t_k^-)) \\ \quad + \sum_{k=1}^m T_\alpha(t-t_k)[g(t_k,y_{t_k}+\bar{z}_{t_k}+I_k(y_{t_k^-}+\bar{z}_{t_k^-})) - g(t_k,y_{t_k}+\bar{z}_{t_k})], \quad t\in(t_m,T], \end{cases}$$

$$\begin{aligned} (\Pi_2 y)(t) &= T_\alpha(t)g(0,\phi) - g(t,y_t+\bar{z}_t) + \int_0^t S_\alpha(t-s)f(s,y_s+\bar{z}_s,B_1(y(s)+\bar{z}(s)))ds \\ &\quad + \int_0^t S_\alpha(t-s)\sigma(s,y_s+\bar{z}_s,B_2(y(s)+\bar{z}(s)))dw(s), \quad t\in J. \end{aligned}$$

In order to use Theorem 2.2 we will verify that Π_1 is compact and continuous while Π_2 is a contraction operator. For the sake of convenience, we divide the proof into several steps.

Step1. We show that there exists a positive number q such that $\Pi(B_q)\subset B_q$. If this is not true, then for each $q>0$, there exists a function $y^q(\cdot)\in B_q$, but $\Pi(y^q)\notin B_q$, that is $\mathbb{E}\|(\Pi y^q)(t)\|_{\mathcal{H}}^2 > q$. An elementary inequality can show that, for $t\in[0,t_1]$

$$\begin{aligned} q &\le \mathbb{E}\|\Pi(y^q)(t)\|_{\mathcal{H}}^2 \\ &\le 4\mathbb{E}\|T_\alpha(t)g(0,\phi)\|_{\mathcal{H}}^2 + 4\mathbb{E}\|g(t,y_t^q+\bar{z}_t)\|_{\mathcal{H}}^2 + 4\mathbb{E}\left\|\int_0^t S_\alpha(t-s)f(s,y_s^q+\bar{z}_s,B_1(y^q(s)+\bar{z}(s)))ds\right\|_{\mathcal{H}}^2 \\ &\quad + 4\mathbb{E}\left\|\int_0^t S_\alpha(t-s)\sigma(s,y_s^q+\bar{z}_s,B_2(y^q(s)+\bar{z}(s)))dw(s)\right\|_{\mathcal{H}}^2 \\ &= 4\sum_{i=1}^4 I_i. \end{aligned} \tag{3.6}$$

Let us now estimate each term above I_i, $i = 1, \ldots, 4$. By Lemma 2.1 and assumptions $(H1)$-$(H2)$, we have

$$I_1 \leq \widetilde{M}_T^2 \mathbb{E}\|g(0,\phi)\|_{\mathcal{H}}^2 \leq \widetilde{M}_T^2 M_g(\|\phi\|_{\mathcal{B}_h}^2 + 1), \tag{3.7}$$

$$I_2 \leq M_g\Big(\|y_t^q + \bar{z}_t\|_{\mathcal{B}_h}^2 + 1\Big) \leq M_g\Big[4\Big(\|\phi\|_{\mathcal{B}_h}^2 + l^2 q\Big) + 1\Big]. \tag{3.8}$$

Together with assumption $(H3)$ and (3.2), we have

$$\begin{aligned}
I_3 &\leq \int_0^t \|S_\alpha(t-s)\|ds \int_0^t \|S_\alpha(t-s)\|\mathbb{E}\|f(s, y_s^q + \bar{z}_s, B_1(y^q(s) + \bar{z}(s)))\|_{\mathcal{H}}^2 ds \\
&\leq \widetilde{M}_S^2 \int_0^t (t-s)^{\alpha-1} ds \int_0^t (t-s)^{\alpha-1}\Big[\mu_1(s)\Xi_f\Big(\|y_s^q + \bar{z}_s\|_{\mathcal{B}_h}^2\Big) + \mu_2(s)\mathbb{E}\|B_1(y^q(s) + \bar{z}(s))\|_{\mathcal{H}}^2\Big]ds \\
&\leq \widetilde{M}_S^2 \frac{T^\alpha}{\alpha} \int_0^t (t-s)^{\alpha-1}\Big[\Xi_f\Big(4(\|\phi\|_{\mathcal{B}_h}^2 + l^2 q)\Big)\mu_1^* + B_1^*\mu_2^* \sup_{0\leq s\leq T} \mathbb{E}\|y^q(s) + \bar{z}(s)\|_{\mathcal{H}}^2\Big]ds \\
&\leq \widetilde{M}_S^2 \frac{T^{2\alpha}}{\alpha^2}\Big[\Xi_f\Big(4(\|\phi\|_{\mathcal{B}_h}^2 + l^2 q)\Big)\mu_1^* + B_1^*\mu_2^* q\Big],
\end{aligned} \tag{3.9}$$

where $B_1^* = \sup_{t\in[0,T]} \int_0^t K(t,s)ds < \infty$, $\mu_1^* = \sup_{s\in[0,t]} \mu_1(s)$, $\mu_2^* = \sup_{s\in[0,t]} \mu_2(s)$.
A similar argument involves assumption $(H4)$, we obtain

$$\begin{aligned}
I_4 &\leq \int_0^t \|S_\alpha(t-s)\|^2 \mathbb{E}\|\sigma(s, y_s^q + \bar{z}_s, B_2(y^q(s) + \bar{z}(s)))\|_{\mathcal{L}_2^0}^2 ds \\
&\leq \widetilde{M}_S^2 \int_0^t (t-s)^{2(\alpha-1)}\Big[\Xi_\sigma\Big(4(\|\phi\|_{\mathcal{B}_h}^2 + l^2 q)\Big)\nu_1^* + B_2^*\nu_2^* \sup_{0\leq s\leq T} \mathbb{E}\|y^q(s) + \bar{z}(s)\|_{\mathcal{H}}^2\Big]ds \\
&\leq \widetilde{M}_S^2 \frac{T^{2\alpha-1}}{2\alpha-1}\Big[\Xi_\sigma\Big(4(\|\phi\|_{\mathcal{B}_h}^2 + l^2 q)\Big)\nu_1^* + B_2^*\nu_2^* q\Big],
\end{aligned} \tag{3.10}$$

where $B_2^* = \sup_{t\in[0,T]} \int_0^t P(t,s)ds < \infty$, $\nu_1^* = \sup_{s\in[0,t]} \nu_1(s)$, $\nu_2^* = \sup_{s\in[0,t]} \nu_2(s)$.
Combining these estimates (3.6)-(3.10) yields

$$\begin{aligned}
q &\leq \mathbb{E}\|\Pi(y^q)(t)\|_{\mathcal{H}}^2 \\
&\leq L_0 + 16M_g l^2 q + 4\widetilde{M}_S^2 \frac{T^{2\alpha}}{\alpha^2}\Big[\Xi_f\Big(4(\|\phi\|_{\mathcal{B}_h}^2 + l^2 q)\Big)\mu_1^* + B_1^*\mu_2^* q\Big] \\
&\quad + 4\widetilde{M}_S^2 \frac{T^{2\alpha-1}}{2\alpha-1}\Big[\Xi_\sigma\Big(4(\|\phi\|_{\mathcal{B}_h}^2 + l^2 q)\Big)\nu_1^* + B_2^*\nu_2^* q\Big],
\end{aligned} \tag{3.11}$$

where

$$L_0 = 4\widetilde{M}_T^2 M_g\Big(\|\phi\|_{\mathcal{B}_h}^2 + 1\Big) + 4M_g\Big(1 + 4\|\phi\|_{\mathcal{B}_h}^2\Big).$$

Dividing both sides of (3.11) by q and taking $q \to \infty$, we obtain

$$\begin{aligned}
&16M_g l^2 + 4\widetilde{M}_S^2 \frac{T^{2\alpha}}{\alpha^2}\Big[4\Lambda\mu_1^* + B_1^*\mu_2^*\Big] + 4\widetilde{M}_S^2 \frac{T^{2\alpha-1}}{2\alpha-1}\Big[4\Upsilon\nu_1^* + B_2^*\nu_2^*\Big] \\
&= 16M_g l^2 + 4\widetilde{M}_S^2 T^{2\alpha}\Big[\frac{\eta_1}{\alpha^2} + \frac{\eta_2}{T(2\alpha-1)}\Big] \geq 1,
\end{aligned}$$

which is a contradiction to our assumption in (3.3).

For $t \in (t_1, t_2]$, we have

$$\begin{aligned}
q &\leq \mathbb{E}\|\Pi(y^q)(t)\|_{\mathcal{H}}^2 \\
&\leq 7\|T_\alpha(t-t_1)\|^2 \mathbb{E}\|I_1(y^q(t_1^-))\|_{\mathcal{H}}^2 + 7\|T_\alpha(t-t_1)\|^2 \mathbb{E}\|g(t_1, y_{t_1}^q + \bar{z}_{t_1} + I_1(y_{t_1^-}^q + \bar{z}_{t_1^-}))\|_{\mathcal{H}}^2 \\
&\quad + 7\|T_\alpha(t-t_1)\|^2 \mathbb{E}\|g(t_1, y_{t_1}^q + \bar{z}_{t_1})\|_{\mathcal{H}}^2 + 7\mathbb{E}\|T_\alpha(t)g(0,\phi)\|_{\mathcal{H}}^2 + 7\mathbb{E}\|g(t, y_t^q + \bar{z}_t\|_{\mathcal{H}}^2 \\
&\quad + 4\mathbb{E}\left\|\int_0^t S_\alpha(t-s)f(s, y_s^q + \bar{z}_s, B_1(y^q(s) + \bar{z}(s)))ds\right\|_{\mathcal{H}}^2 \\
&\quad + 7\mathbb{E}\left\|\int_0^t S_\alpha(t-s)\sigma(s, y_s^q + \bar{z}_s, B_2(y^q(s) + \bar{z}(s)))dw(s)\right\|_{\mathcal{H}}^2.
\end{aligned} \tag{3.12}$$

Using assumptions $(H1)$-$(H5)$ we obtain

$$
\begin{aligned}
& \mathbb{E}\|\Pi(y^q)(t)\|_{\mathcal{H}}^2 \\
\leq\ & L_1 + 70\widetilde{M}_T^2 M_g l^2 q + 28 M_g l^2 q + 7\widetilde{M}_S^2 \frac{T^{2\alpha}}{\alpha^2}\Big[\Xi_f\Big(4(\|\phi\|_{\mathcal{B}_h}^2 + l^2 q)\Big)\mu_1^* + B_1^*\mu_2^* q\Big] \\
& + 7\widetilde{M}_S^2 \frac{T^{2\alpha-1}}{2\alpha-1}\Big[\Xi_\sigma\Big(4(\|\phi\|_{\mathcal{B}_h}^2 + l^2 q)\Big)\nu_1^* + B_2^*\nu_2^* q\Big],
\end{aligned}
$$

where

$$
L_1 = 7\widetilde{M}_T^2\Big(\Theta + M_g\Big[1 + 6(\|\phi\|_{\mathcal{B}_h}^2 + l^2\Theta)\Big]\Big) + 7\widetilde{M}_T^2 M_g\Big(1 + \|\phi\|_{\mathcal{B}_h}^2\Big) + 7M_g\Big(1 + 4\|\phi\|_{\mathcal{B}_h}^2\Big).
$$

A Similar argument gives

$$
\begin{aligned}
& 70\widetilde{M}_T^2 M_g l^2 + 28 M_g l^2 + 7\widetilde{M}_S^2 \frac{T^{2\alpha}}{\alpha^2}\Big[4\Lambda\mu_1^* + B_1^*\mu_2^*\Big] + 7\widetilde{M}_S^2 \frac{T^{2\alpha-1}}{2\alpha-1}\Big[4\Upsilon\nu_1^* + B_2^*\nu_2^*\Big] \\
& = 70\widetilde{M}_T^2 M_g l^2 + 28 M_g l^2 + 7\widetilde{M}_S^2 T^{2\alpha}\Big[\frac{\eta_1}{\alpha^2} + \frac{\eta_2}{T(2\alpha-1)}\Big] \geq 1,
\end{aligned}
$$

which is a contradiction to our assumption in (3.3).

Similarly for $t \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots, m$, we obtain

$$
\begin{aligned}
& \widetilde{C} + 16 M_g l^2 + 7\widetilde{M}_S^2 \frac{T^{2\alpha}}{\alpha^2}\Big[4\Lambda\mu_1^* + B_1^*\mu_2^*\Big] + 7\widetilde{M}_S^2 \frac{T^{2\alpha-1}}{2\alpha-1}\Big[4\Upsilon\nu_1^* + B_2^*\nu_2^*\Big] \\
& = \widetilde{C} + 16 M_g l^2 + 7\widetilde{M}_S^2 T^{2\alpha}\Big[\frac{\eta_1}{\alpha^2} + \frac{\eta_2}{T(2\alpha-1)}\Big] \geq 1,
\end{aligned}
$$

with $\eta_1 = 4\Lambda\mu_1^* + B_1^*\mu_2^*$, $\eta_2 = 4\Upsilon\nu_1^* + B_2^*\nu_2^*$ and $\widetilde{C}$ is a positive constant depending only on $\widetilde{M}_T, M_g$ and l. This is a contradiction to our assumption in (3.3).
Thus, for some positive number q, $\Pi(B_q) \subset B_q$.

Step 2. The map Π_1 is continuous on B_q.

Let $\{y^n\}_{n=1}^{\infty}$ be a sequence in B_q with $\lim y^n \to y \in B_q$. Then for $t \in (t_i, t_{i+1}]$, we have

$$
\begin{aligned}
& \mathbb{E}\|(\Pi_1 y^n)(t) - (\Pi_1 y)(t)\| \\
\leq\ & 3\sum_{k=1}^{i}\|T_\alpha(t-t_k)\|^2\Bigg[\mathbb{E}\|I_k(y^n(t_k^-)) - I_k(y(t_k^-))\|_{\mathcal{H}}^2 + \\
& \mathbb{E}\|g(t_k, y_{t_k}^n + \bar{z}_{t_k} + I_k(y_{t_k^-}^n + \bar{z}_{t_k^-})) - g(t_k, y_{t_k} + \bar{z}_{t_k} + I_k(y_{t_k^-} + \bar{z}_{t_k^-}))\|_{\mathcal{H}}^2 \\
& + \mathbb{E}\|g(t_k, y_{t_k}^n + \bar{z}_{t_k}) - g(t_k, y_{t_k} + \bar{z}_{t_k})\|_{\mathcal{H}}^2\Bigg].
\end{aligned}
$$

Since the functions g, I_i, $i = 1, 2, \ldots, m$ are continuous, hence $\lim_{n\to\infty}\mathbb{E}\|\Pi_1 y^n - \Pi_1 y\|^2 = 0$ which implies that the mapping Π_1 is continuous on B_q.

Step 3. Π_1 maps bounded sets into bounded sets in B_q.

Let us prove that for $q > 0$ there exists a $\delta > 0$ such that for each $y \in B_q$, we have $\mathbb{E}\|(\Pi_1 y)(t)\|_{\mathcal{H}}^2 \leq \delta$ for $t \in (t_i, t_{i+1}]$, $i = 0, 1, \ldots, m$. We have

$$
\begin{aligned}
\mathbb{E}\|(\Pi_1 y)(t)\|_{\mathcal{H}}^2 \leq\ & 3\sum_{k=1}^{i}\|T_\alpha(t-t_k)\|^2\Bigg[\mathbb{E}\|I_k(y(t_k^-))\|_{\mathcal{H}}^2 + \mathbb{E}\|g(t_k, y_{t_k} + \bar{z}_{t_k})\|_{\mathcal{H}}^2 \\
& + \mathbb{E}\|g(t_k, y_{t_k} + \bar{z}_{t_k} + I_k(y_{t_k^-} + \bar{z}_{t_k^-}))\|_{\mathcal{H}}^2\Bigg] \\
\leq\ & 3m\widetilde{M}_T^2\Bigg[\Theta\Big(1 + 6M_g l^2\Big) + 2M_g + 10M_g\Big(\|\phi\|_{\mathcal{B}_h}^2 + l^2 q\Big)\Bigg] \\
:=\ & \delta,
\end{aligned}
$$

which proves the desired result.

Step 4. The set $\{\Pi_1 y, y \in B_q\}$ is an equicontinuous family of functions on J.
Let $u, v \in (t_i, t_{i+1}]$, $t_i \le u < v \le t_{i+1}$, $i = 0, 1, \dots, m$, $y \in B_q$. We have

$$\begin{aligned}
&\mathbb{E}\|(\Pi_1 y)(v) - (\Pi_1 y)(u)\|^2_{\mathcal{H}} \\
\le\ & 3\sum_{k=1}^{i} \|T_\alpha(v - t_k) - T_\alpha(u - t_k)\|^2 \Bigg[\mathbb{E}\|I_k(y(t_k^-))\|^2_{\mathcal{H}} + \mathbb{E}\|g(t_k, y_{t_k} + \bar{z}_{t_k})\|^2_{\mathcal{H}} \\
&+\mathbb{E}\|g(t_k, y_{t_k} + \bar{z}_{t_k} + I_k(y_{t_k^-} + \bar{z}_{t_k^-}))\|^2_{\mathcal{H}}\Bigg] \\
\le\ & 3\left[\Theta\Big(1 + 6M_g l^2\Big) + 2M_g + 10M_g\Big(\|\phi\|^2_{\mathcal{B}_h} + l^2 q\Big)\right]\sum_{k=1}^{i}\|T_\alpha(v - t_k) - T_\alpha(u - t_k)\|^2.
\end{aligned}$$

Since T_α is strongly continuous and it allows us to conclude that $\lim_{u\to v}\|T_\alpha(v - t_k) - T_\alpha(u - t_k)\|^2 = 0$ for all $k = 1, 2, \dots, m$, which implies that the set $\{\Pi_1 y, y \in B_q\}$ is equicontinuous. Finally, combining Step 1 to Step 4 together with Ascoli's theorem, we conclude that the operator Π_1 is compact.

Step 5. Π_2 is contractive. Let $y, y^* \in B_q$ and $t \in (t_i, t_{i+1}]$, $i = 0, 1, \dots, m$. Then

$$\begin{aligned}
&\mathbb{E}\|(\Pi_2 y)(t) - (\Pi_2 y^*)(t)\|^2_{\mathcal{H}} \\
\le\ & 3\|g(t, y_t + \bar{z}_t) - g(t, y^*_t + \bar{z}_t)\|^2_{\mathcal{H}} \\
&+3\mathbb{E}\left\|\int_0^t S_\alpha(t-s)\Big[f(s, y_s + \bar{z}_s, B_1(y(s) + \bar{z}(s))) - f(s, y^*_s + \bar{z}_s, B_1(y^*(s) + \bar{z}(s)))\Big]ds\right\|^2_{\mathcal{H}} \\
&+3\mathbb{E}\left\|\int_0^t S_\alpha(t-s)\Big[\sigma(s, y_s + \bar{z}_s, B_2(y(s) + \bar{z}(s))) - \sigma(s, y^*_s + \bar{z}_s, B_2(y^*(s) + \bar{z}(s)))\Big]dw(s)\right\|^2_{\mathcal{H}} \\
\le\ & 3\|g(t, y_t + \bar{z}_t) - g(t, y^*_t + \bar{z}_t)\|^2_{\mathcal{H}} + 3\int_0^t \|S_\alpha(t-s)\|ds\int_0^t \|S_\alpha(t-s)\| \\
&\times\mathbb{E}\|f(s, y_s + \bar{z}_s, B_1(y(s) + \bar{z}(s))) - f(s, y^*_s + \bar{z}_s, B_1(y^*(s) + \bar{z}(s)))\|^2_{\mathcal{H}}ds \\
&+3\int_0^t \|S_\alpha(t-s)\|^2\mathbb{E}\|\sigma(s, y_s + \bar{z}_s, B_2(y(s) + \bar{z}(s))) - \sigma(s, y^*_s + \bar{z}_s, B_2(y^*(s) + \bar{z}(s)))\|^2_{\mathcal{L}_2^0}ds \\
\le\ & 3M_g\|y_t - y^*_t\|^2_{\mathcal{B}_h} + 3\widetilde{M}_S^2\int_0^t (t-s)^{\alpha-1}ds\int_0^t (t-s)^{\alpha-1} \\
&\times\Big[\mu_1(s)\|y_s - y^*_s\|^2_{\mathcal{B}_h} + \mu_2(s)\mathbb{E}\|B_1(y(s) + \bar{z}(s)) - B_1(y^*(s) + \bar{z}(s))\|^2_{\mathcal{H}}\Big]ds \\
&+3\widetilde{M}_S^2\int_0^t (t-s)^{2(\alpha-1)}\Big[\nu_1(s)\|y_s - y^*_s\|^2_{\mathcal{B}_h} + \nu_2(s)\mathbb{E}\|B_2(y(s) + \bar{z}(s)) - B_2(y^*(s) + \bar{z}(s))\|^2_{\mathcal{H}}\Big]ds \\
\le\ & 3M_g\|y_t - y^*_t\|^2_{\mathcal{B}_h} + 3\widetilde{M}_S^2\frac{T^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1} \\
&\times\Big[\mu_1^* l^2 \sup \mathbb{E}\|y(s) - y(s)^*\|^2_{\mathcal{H}} + \mu_2^* B_1^* \sup \mathbb{E}\|y(s) - y(s)^*\|^2_{\mathcal{H}}\Big]ds \\
&+3\widetilde{M}_S^2\int_0^t (t-s)^{2(\alpha-1)}\Big[\nu_1^* l^2 \sup \mathbb{E}\|y(s) - y(s)^*\|^2_{\mathcal{H}} + \nu_2^* B_2^* \sup \mathbb{E}\|y(s) - y(s)^*\|^2_{\mathcal{H}}\Big]ds \\
\le\ & 3\left(l^2 M_g + \widetilde{M}_S^2 T^{2\alpha}\left[\tfrac{1}{\alpha^2}(\mu_1^* l^2 + \mu_2^* B_1^*) + \tfrac{1}{T(2\alpha-1)}(\nu_1^* l^2 + \nu_2^* B_2^*)\right]\right)\|y - y^*\|^2_{\mathcal{B}_b^0} \\
=\ & 3\left(l^2 M_g + \widetilde{M}_S^2 T^{2\alpha}\left[\tfrac{\vartheta_1}{\alpha^2} + \tfrac{\vartheta_2}{T(2\alpha-1)}\right]\right)\|y - y^*\|^2_{\mathcal{B}_b^0}.
\end{aligned}$$

So Π_2 is a contraction by our assumption in (3.4). Hence, by Sadovskii's fixed point theorem we can conclude that the problem (1.1) has at least one solution on $(-\infty, T]$. This completes the proof of the theorem. □

4 An example

In this section, we consider an example to illustrate our main theorem. We examine the existence of solutions for the following fractional stochastic partial differential equation of the form

$$
\begin{aligned}
&D_t^q[u(t,x)+\int_{-\infty}^{t} a(t,x,s-t)Q_1(u(s,x))ds] = \frac{\partial^2}{\partial x^2}[u(t,x)+\int_{-\infty}^{t} a(t,x,s-t)Q_1(u(s,x))ds]\\
&\quad + \int_{-\infty}^{t} H(t,x,s-t)Q_2(u(s,x))ds + \int_0^t k(s,t)e^{-u(s,x)}ds\\
&\quad + \left[\int_{-\infty}^{t} V(t,x,s-t)Q_3(u(s,x))ds + \int_0^t p(s,t)e^{-u(s,x)}ds\right]\frac{d\beta(t)}{dt},\\
&\quad x\in[0,\pi],\quad t\in[0,b],\quad t\neq t_k\\
&u(t,0)=0=u(t,\pi),\quad t\geq 0\\
&u(t,x)=\phi(t,x),\quad t\in(-\infty,0],\quad x\in[0,\pi],\\
&\Delta u(t_i)(x)=\int_{-\infty}^{t} q_i(t_i-s)u(s,x)ds,\quad x\in[0,\pi],
\end{aligned}
\tag{4.1}
$$

where $\beta(t)$ is a standard cylindrical Wiener process in $\mathcal{H}$ defined on a stochastic space $(\Omega,\{\mathcal{F}_t\},\mathcal{F},\mathbb{P})$; D_t^q is the Caputo fractional derivative of order $0<q<1$; $0<t_1<t_2<\ldots<t_n=T$ are prefixed numbers; a,Q_1,H,Q_2,V,Q_3 are continuous; $\phi\in\mathcal{B}_h$.

Let $\mathcal{H}=L^2([0,\pi])$ with the norm $\|\cdot\|$. Define $A:\mathcal{H}\to\mathcal{H}$ by $Ay=y''$ with the domain

$$\mathcal{D}(A)=\{y\in\mathcal{H}; y,y' \text{ are absolutely continuous, } y''\in\mathcal{H} \text{ and } y(0)=y(\pi)=0\}.$$

Then, $Ay=\sum_{n=1}^{\infty} n^2(y,y_n)y_n$, $y\in\mathcal{D}(A)$, where $y_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx)$, $n=1,2,\ldots$, is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ in $\mathcal{H}$ is given by

$$T(t)y=\sum_{n=1}^{\infty} e^{-n^2t}(y,y_n)y_n,\quad \text{for all } y\in\mathcal{H}, t>0.$$

It follows from the above expressions that $(T(t))_{t\geq 0}$ is a uniformly bounded compact semigroup, so that, $R(\lambda,A)=(\lambda-A)^{-1}$ is a compact operator for all $\lambda\in\rho(A)$.

Let $h(s)=e^{2s}$, $s<0$, then $l=\int_{-\infty}^{0} h(s)ds=\frac{1}{2}$ and define

$$\|\phi\|_{\mathcal{B}_h}=\int_{-\infty}^{0} h(s)\sup_{s\leq\theta\leq 0}\Big(\mathbb{E}|\phi(\theta)|^2\Big)^{1/2}ds.$$

Hence for $(t,\phi)\in[0,T]\times\mathcal{B}_h$, where $\phi(\theta)(y)=\phi(\theta,y)$, $(\theta,y)\in(-\infty,0]\times[0,\pi]$. Set $u(t)(x)=u(t,x)$,

$$
\begin{aligned}
g(t,\phi)(x)&=\int_{-\infty}^{0} a(t,x,\theta)Q_1(\phi(\theta)(x))d\theta,\\
f(t,\phi,B_1u(t))(x)&=\int_{-\infty}^{0} H(t,x,\theta)Q_2(\phi(\theta)(x))d\theta+B_1u(t)(x),\\
\sigma(t,\phi,B_2u(t))(x)&=\int_{-\infty}^{0} V(t,x,\theta)Q_3(\phi(\theta)(x))d\theta+B_2u(t)(x),\\
I_i(\phi)(x)&=\int_{-\infty}^{0} q_i(-\theta)\phi(\theta)(x)d\theta,
\end{aligned}
$$

where $B_1u(t)=\int_0^t k(s,t)e^{-u(s,x)}ds$ and $B_2u(t)=\int_0^t p(s,t)e^{-u(s,x)}ds$. Then with these settings the equations in (4.1) can be written in the abstract form of Eq. (1.1). All conditions of Theorem 3.1 are now fulfilled, so we deduce that the system (4.1) has a mild solution on $(-\infty,T]$.

5 Conclusion

We have studied the existence of mild solutions for a class of impulsive fractional stochastic differential equations in Hilbert spaces, which is new and allow us to develop the existence of various fractional differential equations and stochastic fractional differential equations. An example is provided to illustrate the applicability of the new result. The results presented in this paper extend and improve the corresponding ones announced by Dabas et al [6], Dabas and Chauhan [7], Shu et al [23], Sakthivel et al [18] and others.

Acknowledgement

The work of the corresponding author was supported by The National Agency of Development of University Research (ANDRU), Algeria (PNR-SMA 2011-2014).

References

[1] P. Balasubramaniam, J.Y. Park, A. Vincent Antony Kumar, Existence of solutions for semilinear neutral stochastic functional differential equations with nonlocal conditions, *Nonlinear Anal. TMA.,* 71 (2009), 1049-1058.

[2] J. Bao, Z. Hou, and C. Yuan, Stability in distribution of mild solutions to stochastic partial differential equations, *Pro. American Math. Soc.,* 138(6)(2010), 2169-2180.

[3] M. Benchohra, J. Henderson, S.K. Ntouyas, Existence results for impulsive semilinear neutral functional differential equations in Banach spaces, *Memoirs on Diff. Equ. Math. Phys.,* 25(2002), 105-120.

[4] Y.K. Chang, Z.H. Zhao, G. M. N'Guérékata, Squaremean almost automorphic mild solutions to non-autonomous stochastic differential equations in Hilbert spaces, *Computers & Mathematics with Applications,* 61(2)(2011), 384-391.

[5] J. Cui, L. Yan, Existence result for fractional neutral stochastic integro-differential equations with infinite delay, *J. Phys. A: Math. Theor.* 44 (2011) 335201 (16pp)

[6] J. Dabas, A. Chauhan, M. Kumar, Existence of the mild solutions for impulsive fractional equations with infinite delay, *Int. J. Differ. Equ.,* (2011) 20. Article ID 793023.

[7] J. Dabas, A. Chauhan, Existence and uniqueness of mild solution for an impulsive neutral fractional integro-differential equation with infinite delay, *Math. Computer Modelling,* 57(2013), 754-763.

[8] M. M. Fu and Z. X. Liu, Square-mean almost automorphic solutions for some stochastic differential equations, *Pro. American Math. Soc.,* 138(10) (2010), 3689-3701.

[9] T. Guendouzi, I. Hamada, Existence and controllability result for fractional neutral stochastic integro-differential equations with infinite delay, *AMO - Advanced Modeling and Optimization,* 15(2)(2013), 281-300.

[10] E. Hernández. M. Pierri, G. Goncalves, Existence results for an impulsive abstract partial differential equation with state-dependent delay, *Computers & Mathematics with Applications,* 52(3-4)(2006), 411-420.

[11] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Publishing, Singapore, 2000.

[12] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V, Amsterdam, 2006.

[13] A. Lin, L. Hu, Existence results for impulsive neutral stochastic functional integro-differential inclusions with nonlocal initial conditions, *Computers & Mathematics with Applications,* 59(2010), 64-73.

[14] G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*, Vol 44 Encyclopedia of Mathematics and its Applications, Cambridge university Press, Cambridge, Mass, USA, 1992.

[15] Y. Ren, R. Sakthivel, Existence, uniqueness and stability of mild solutions for second-order neutral stochastic evolution equations with infinite delay and Poisson jumps, *J. Math. Phys.,* 53(2012), 073517.

[16] Y. Ren, Q. Zhou, L. Chen, Existence, uniqueness and stability of mild solutions for time-dependent stochastic evolution equations with Poisson jumps and infinite delay, *J. Optim. Theory Appl.,* 149(2011), 315-331.

[17] B.N. Sadovskii, On a fixed point principle, *Funct. Anal. Appl.,* 1 (1967), 71-74.

[18] R. Sakthivel, P. Revathi, Yong Ren, Existence of solutions for nonlinear fractional stochastic differential equations, *Nonlinear Anal. TMA.,* 81(2013), 70-86.

[19] R. Sakthivel, P. Revathi, N. I.Mahmudov, Asymptotic stability of fractional stochastic neutral differential equations with infinite delays, *Abstract and Applied Analysis* V. 2013, Article ID 769257, 9.

[20] R. Sakthivel, P. Revathi, S. Marshal Anthoni, Existence of pseudo almost automorphic mild solutions to stochastic fractional differential equations, *Nonlinear Anal. TMA.,* 75(7)(2012), 3339-3347.

[21] R. Sakthivel, Yong Ren, Exponential stability of second-order stochastic evolution equations with Poisson jumps, *Comm. Nonlinear Sci. Numerical Simulation,* 17(2012), 4517-4523.

[22] R. Sakthivel, Yong Ren, Hyunsoo Kim, Asymptotic stability of second-order neutral stochastic differential equations, *J. Math. Phys.,* 51(2010), 052701.

[23] X.B. Shu, Y. Lai, Y. Chen, The existence of mild solutions for impulsive fractional partial differential equations, *Nonlinear Anal. TMA.,* 74 (2011), 2003-2011.

Received: June 13, 2013; *Accepted*: June 25, 2013

UNIVERSITY PRESS