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Qualitative behaviour of solutions of hybrid fractional integral equation with linear perturbation of second kind in Banach Space

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Abstract
In this paper, we present existence and qualitative behaviour of solution of hybrid fractional integral equation with linear perturbation of second kind by applying measure of noncompactness in Banach space. We established our result in the Banach space of real-valued functions defined, continuous and bounded in the right hand real axis.

Keywords
Hybrid fractional integral equation, measure of noncompactness, attractivity of solution.

AMS Subject Classification 45G10, 45M10, 47H10.

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Contents

1. Introduction

Measure of noncompactness and fixed point theorems are the most valuable and effective implements in the framework of nonlinear analysis , which acts as principle role for solvability of linear and nonlinear integral equations. Recently, the theory of such integral equations is developed effectively and emerge in the fields of mathematical, analysis, engineering, mathematical physics and nonlinear functional analysis (e.g., 1, 3-5,9, 10, 15-17, 19).

Nonlinear integral equation with bounded intervals has been studied extensively in the literature as regard various aspects of the solutions. This includes existence, uniqueness, stability and extremality of solutions. But the study of nonlinear integral equation with unbounded intervals is relatively new and exploited for the new characteristics of attractivity and asymptotic attractivity of solutions. There are two approaches for dealing with theses characteristics of solutions, namely, classical fxed point theorems involving the hypotheses from analysis and topology and the fixed point theorems involving the use of measure of noncompactness. Each one of these approaches has some advantage and disadvantages over the others was discussed in Dhage [12]. In 2005, Apell [2] discussed some measure of noncompactness in the application of nonlinear integral equations.

Let $J=[t_0,t_0+a]$ in $\mathbb{R}$ be a closed and bounded interval where $t_0\in\mathbb{R}$ and $a\in\mathbb{R}$ with $a>0$ and a given a real number $0<q<1$. Consider the hybrid fractional integral equation(HFIE) with linear perturbation of second type

$$a(t)=h(t)+f(t,a(t))+\frac{1}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}g(t,a(s))ds \quad (1.1)$$

where $t,s\in J,h:J\to\mathbb{R}$, $f:J\times\mathbb{R}\to\mathbb{R}$ is continuous and $g:J\times\mathbb{R}\to\mathbb{R}$ is locally Holder continuous.

In this paper,we are going to discuss two qualitative behavior such as global attractivity and positivity of hybrid fractional integral equation (1.1) with linear perturbation of second type using measure of noncompactness under certain conditions. We established our result in the Banach space of real-valued functions defined, continuous and bounded on the right hand real half axis $\mathbb{R}_+$.

2. Basic Definitions and Results

This section is devoted to conflict to collect some definitions and auxialiary results which will be needed in the further considerations of this paper. At the beginning we present some basic facts concerning the measure of noncompactness. We accept the following definitions of the concept of a measure of noncompactness given in Dhage [7].

Let X be a Banach space, $P(X)$ a class of subset of E with property p. $P_{cl}(X)$,$P_{bd}(X)$,$P_{cl,bd}(X)$,$P_{rcp}(X)$ denote the class of closed, bounded, closed and bounded and relatively compact subsets of X respectively.

A function $d_H : P(X)\times P(X) \to \mathbb{R}_+$ defined by

$$d_H(A,B) = max\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(b,A)\} \qquad (2.1)$$

satisfies all the conditions of a metric on $P(X)$ and is called Hausdorff Pompeiu metric on X, where $D(a,B) = \inf\{\|a-b\| : b\in B\}$. It is clear that the space $(P_{cl}(X),d_H)$ is complete metric space.

Definition 2.1. *A sequence $\{X_n\}$ of non-empty sets in $P_p(X)$ is said to be converge to a set X, called the* ***limiting set*** *if $d_H(X_n \to 0)$ as $n\to\infty$. a mapping $\mu : P_p(X)\to\mathbb{R}_+$ is called continuous if for any sequence $\{X_n\}$ in $P_p(X)$ we have*

$$d_H(X_n,X)\to 0 \Rightarrow |\ \mu(X_n)-\mu(X)\ |\to 0 \qquad as \qquad n\to\infty$$

Definition 2.2. *A mapping $\mu : P_p(X)\to\mathbb{R}_+$ is called if $X_1,X_2\in P_p(X)$ are two sets with $A\subseteq B$, then $\mu(X_1)\le\mu(X_2)$, where $\subseteq$ is a order relation by inclusion in $P_p(X)$.*

Now we define the measure of noncompactness for a bounded subset of the Banach space X.

Definition 2.3. *Let $X_1\subset X$. A function $\mu : P_{bd}(X)\to\mathbb{R}_+$ is called a* ***measure of noncompactness****, if it satisfies:*

1. $\phi \neq \mu^{-1}(0)\subset P_{rcp}(X)$,
2. $\mu(X_1) = \mu(\overline{X_1})$, *where $\overline{X_1}$ is closure of X_1,*
3. $\mu(X_1) = \mu(Conv(X_1))$, *where $Conv(X_1)$ is convex hull of X_1,*
4. *μ is nondecreasing, and*
5. *if $\{X_n\}$ is a decreasing sequence of sets in $P_{bd}(X)$ such that $\lim_{n\to\infty}\mu(X_n) = 0$, then the limiting set $X_\infty = \lim_{n\to\infty} = \cap_{n=0}^{\infty}\overline{X_n}$ is nonempty.*

Definition 2.4. *The family $ker\mu$ is said to be the kernel of measure of noncompactness where*

$$ker\mu = \{X_1\in P_{bd}(X)\mid \mu(X_1) = 0\}\subset P_{rcp}(X).$$

Definition 2.5. *A measure μ is* ***complete*** *or* ***full*** *if the kernel $ker\mu$ of μ consists of all possible relatively compact subsets of X.*

The following definition appear in Dhage[12].

Definition 2.6. *A mapping $K : X\to X$ is called $\mathscr{D}-set-Lipschitz$ if there exists a continuous nondecreasing function $\phi : \mathbb{R}_+\to\mathbb{R}_+$ such that $\mu(K(X_1))\le\phi(\mu(X_1))$ for all $X_1\in P_{bd}(X)$ with $K(X_1)\in P_{bd}(X)$, where $\phi(0) = 0$. Sometimes we call the function ϕ to be a $\mathscr{D}-function$ of K on X. In the special case, when $\phi(r) = kr, k>0$, K is called a $k-set-Lipschitz$ mapping and if $k<1$, then K is called a $k-set-contraction$ on X. If $\phi(r)<r$ for $r>0$, then K is called a nonlinear $\mathscr{D}-set-contraction$ on X.*

Theorem 2.7. *([14]) Let C be a non-empty, closed, convex and bounded subset of a Banach space X and let $K : C\to C$ be a continuous and nonlinear $\mathscr{D}-set-contraction$. Then K has a fixed point.*

Remark 2.8. *Let us write $Fix(K)$ by the set all fixed points of the operator K which belongs to C. It can be easily shown that the $Fix(K)$ existing in Theorem 2.7 belongs to family $ker\mu$. In fact if $Fix(K)\notin ker\mu$, then $\mu(Fix(K))>0$ and $K(Fix(K)) = Fix(K)$. From nonlinear $\mathscr{D}-set-contraction$ it follows that $\mu(K(Fix(K)))\le\phi(\mu(Fix(K)))$ which is a contradiction since $\phi(r)<r$ for $r>0$. Hence $Fix(K)\in ker(\mu)$.*

Let the Banach space $BC(\mathbb{R}_+,\mathbb{R})$ be consisting of all real functions $a = a(t)$ defined, continuous and bounded on $\mathbb{R}_+$. This space is equipped with the standard supremum norm

$$\|a\| = \sup\{|\ a(t)\ | : t\in\mathbb{R}_+\}$$

We will use the Hausdorff or ball measure of noncompactness in $BC(\mathbb{R}_+,\mathbb{R})$. A formula for Hausdorff measure of noncompactness useful in application is defined as follows. Let us fix a nonempty and bounded subset X_1 of the space $BC(\mathbb{R}_+,\mathbb{R})$ and a positive number T. For $x\in X_1$ and $\varepsilon\ge 0$ denote by $\omega^T(a,\varepsilon)$ the modulus of continuity of the function a on the closed and bounded interval $[0,T]$ defined by

$$\omega^T(a,\varepsilon) = \sup\{|\ a(t)-a(s)\ | : t,s\in[0,T], |\ t-s\ |\le\varepsilon\}$$

Next, let us

$$\omega^T(X_1,\varepsilon) = \sup\{\ \omega^T(a,\varepsilon) : a\in X_1\},$$

$$\omega_0^T(X_1) = \lim_{\varepsilon\to 0}\omega^T(X_1,\varepsilon).$$

It is known that ω_0^T is a measure of noncompactness in the Banach space $C([0,T],\mathbb{R})$ of real valued and continuous functions defined on a closed and bounded interval $[0,T]$ in $\mathbb{R}$ which is equivalent to Hausdorff or ball measure

$$\chi(X_1) = \frac{1}{2}\omega_0^T(X_1)$$

for any bounded subset X_1 of $C([0,T],\mathbb{R})$. We define

$$\omega_0(X_1) = \lim_{T\to\infty}\omega_0^T(X_1).$$

For a fixed number $t\in\mathbb{R}_+$ let us write

$$X_1(t) = \{a(t) : a\in X_1\},$$

$$\| X_1(t) \| = \sup\{a(t) : a \in X_1\},$$

and

$$\| X_1(t) - c \| = \sup\{a(t) - c : x \in X_1\}.$$

Let the functions $\mu' s$ be defined on the family $P_{cl,bd}(X_1)$ by the formulas

$$\mu_a(X_1) = \max\{\omega_0(X_1), \limsup_{t\to\infty} \operatorname{diam} X_1(t)\}, \tag{2.2}$$

$$\mu_b(X_1) = \max\{\omega_0(X_1), \limsup_{t\to\infty} \| X_1(t) \|\}, \tag{2.3}$$

$$\mu_c(X_1) = \max\{\omega_0(X_1), \limsup_{t\to\infty} \| X_1(t) - c \|\}. \tag{2.4}$$

Let $T > 0$ be fixed. Then for any $a \in BC(\mathbb{R}_+, \mathbb{R})$ define

$$\delta_T(a) = \sup\{\big| \mid a(t) \mid - a(t)\big| : a \in X_1\},$$

$$\delta_T(X_1) = \sup\{\delta_T(a) : a \in X_1\}$$

$$\delta(X_1) = \lim_{T\to\infty} \delta_T(X_1)$$

Define functions $\mu_{ad}, \mu_{bd}, \mu_{cd} : P_{bd}(X) \to \mathbb{R}_+$ by

$$\mu_{ad}(X_1) = \max\{\mu_a(X_1), \delta(X_1)\}, \tag{2.5}$$

$$\mu_{bd}(X_1) = \max\{\mu_b(X_1), \delta(X_1)\} \tag{2.6}$$

$$\mu_{cd}(X_1) = \max\{\mu_c(X_1), \delta(X_1)\} \tag{2.7}$$

for all $X_1 \in P_{cl,bd}(X)$.

Remark 2.9. *It is shown as in Banas and Goebel [7] that the functions $\mu_a, \mu_b, \mu_c, \mu_{ad}$, μ_{bd} and μ_{cd} are measure of noncompactness in the space $BC(\mathbb{R}_+, \mathbb{R})$. The kernels $ker\mu_a, ker\mu_b$ and $ker\mu_c$ of the measures μ_a, μ_b and μ_c consists of nonempty and bounded subsets X of $BC(\mathbb{R}_+, \mathbb{R})$ such that functions from X_1 are locally equicontinuous on $\mathbb{R}_+$ and the thickness of bundle formed by functions from X_1 tends to zero at infinity. The functions from $ker\mu_c$ come closer along a line $y(t) = c$ and the functions from $ker\mu_b$ come closer to line $y(t) = c$ as t increases to ∞ through $\mathbb{R}_+$. A similar situation is true for the kernels $ker\mu_{ad}, ker\mu_{bd}$ and $ker\mu_{cd}$. Moreover, these measure μ_{ad}, μ_{bd} and μ_{cd} characterize the ultimate positivity of the functions belonging to the kernels of $ker\mu_{ad}, ker\mu_{bd}$ and $ker\mu_{cd}$.*

The above property of $ker\mu_a, ker\mu_b, ker\mu_c$ and $ker\mu_{ad}$, $ker\mu_{bd}, ker\mu_{cd}$ permits us to characterize solutions of the integral equations considered in the sequel. In order to introduce further concepts used in this paper, let us assume that $X = BC(\mathbb{R}_+, \mathbb{R})$ and Ω be a subset of X. Let $K : X \to X$ be an operator and consider the following operator equation in X,

$$Ka(t) = a(t) \tag{2.8}$$

for all $t \in \mathbb{R}_+$. We give different characterizations of the solutions for the the operator equation (2.8) on $\mathbb{R}_+$. The following definitions appear in Dhage[13].

Definition 2.10. *We say that solutions of the equation (2.8) are locally attractive if there exists a closed ball $\overline{\mathscr{B}}_r(a_0)$ in the space $BC(\mathbb{R}_+, \mathbb{R})$ for some $a_0 \in BC(\mathbb{R}_+, \mathbb{R})$ such that arbitrary solutions $a = a(t)$ and $b = b(t)$ of the equation(2.8) belonging to $\overline{\mathscr{B}}_r(a_0) \cap \Omega$ we have that*

$$\lim_{t\to\infty}(a(t) - b(t)) = 0. \tag{2.9}$$

In this case when the limit(2.9) is uniform with respect to the set $\overline{\mathscr{B}}_r(a_0) \cap \Omega$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that

$$\mid a(t) - b(t) \mid \leq \varepsilon \tag{2.10}$$

for all $a, b \in \overline{\mathscr{B}}_r(a_0) \cap \Omega$ being solution of (2.1) and for $t \geq T$, we will say that solutions of equation(2.8) are uniformally locally attractive on $\mathbb{R}_+$.

Definition 2.11. *The solution $a = a(t)$ of equation(2.8) is said to be globally attractive if (2.9) holds for each solution $b = b(t)$ of (2.8) on ω. In other words, we may say that the solutions of the equation (2.9) are globally attractive if for arbitrary solutions $a(t)$ and $b(t)$ of (2.8) on Ω, the condition (2.9) is satisfied. In the case when the condition (2.9) is satisfied uniformly with respect to the set Ω, i.e., if for every $\varepsilon > 0$ there exists $T > 0$ such that the inequality(2.10) is satisfied for all $a, b \in \Omega$ being the solutions of (2.8) and for $t \geq T$, we will say that solutions of the equation(2.8) are uniformly globally attractive on $\mathbb{R}_+$.*

Definition 2.12. *A line $b(t) = c$, where c a real number, is called an attractor for a solution $a \in BC(\mathbb{R}_+, \mathbb{R})$ to the equation(2.8) if $\lim_{t\to\infty}[a(t) - c] = 0$ and the solution a to the equation (2.8) is also called asymptotic to the line $b(t) = c$ and the line is an asymptote for the solution a on $\mathbb{R}_+$.*

The following definitions appear in Dhage[12].

Definition 2.13. *The solutions of equations (2.8) are said to be globally asymptotic attractive if for any two solutions $a = a(t)$ and $b = b(t)$ of the equation (2.8), the condition (2.9) is satisfied and there is a line which is a common attractor to them on $\mathbb{R}_+$. When the condition (2.9) is satisfied uniformly , i.e., if for every $\varepsilon > 0$ there exists $T > 0$ such that the inequality (2.10) is satisfied for $t \geq T$ and for all a, b being the solution of (2.8) and having a line as common attractor, we will say that solutions of the equation (2.8) are uniformly globally asymptotically attractive on $\mathbb{R}_+$.*

Remark 2.14. *The notion of global attractivity of solutions are introduced in Hu and Yan [17] and concept of global and local asymptotic attractivity have been presented in Dhage[13] while concept of uniform global and local attractivity were introduced in Banas and Rzepka [6] and concept of global asymptotic attractivity of solutions are presented in Dhage[12] and local attractivity of a nonlinear quadratic fractional integral equation have been presented in [11].*

Definition 2.15. *A solution a of the equation (2.8) is called locally ultimately positive if there exists a closed ball $\overline{\mathscr{B}}_r(a_0)$ in $BC(\mathbb{R}_+,\mathbb{R})$ for some $a_0 \in BC(\mathbb{R}_+,\mathbb{R})$ such that $a \in \overline{\mathscr{B}}_r(a_0)$ and*

$$\lim_{t\to\infty}[\mid a(t) \mid -a(t)] = 0. \tag{2.11}$$

When the limit (2.11) is uniform with respect to the solution set of the operator equation (2.8),i.e., when for each $\varepsilon > 0$ there exist $T > 0$ such that

$$|| a(t) || - | a(t) | \leq \varepsilon \tag{2.12}$$

for all a being solutions of (2.8) and for $t \geq T$,we will say that solutions of equation (2.8) are uniformly locally ultimately positive on $\mathbb{R}_+$.

Definition 2.16. *A solution $a \in C(\mathbb{R}_+,\mathbb{R})$ of the equation (2.8) is called globally ultimate positive if (2.11) is satisfied. When the limit (2.11) is uniform with respect to the solution set of the operator equation (2.8) in $C(\mathbb{R}_+,\mathbb{R})$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that (2.12) is satisfied for all x being solutions of (2.8) and for $t \geq T$, we will say that solutions of equation 2.8 are uniformly globally ultimate positive on $\mathbb{R}_+$.*

Remark 2.17. *The global attractivity and global asymptotic attractivity implies the local attractivity and local asymptotic attractivity, respectively, to the solutions for the operator equation (2.8) on $\mathbb{R}_+$. Similarly, global ultimate positivity implies local ultimate positivity to the solutions for the operator equation (2.8) on unbounded intervals. The converse of the above statements may not be true.*

3. Attractivity and Positivity of Solutions

By a solution of FIE(1.1), we mean a function $a \in C(J,\mathbb{R})$ that satisfies FIE(1.1) where $C(J,\mathbb{R})$ is the space of continuous real valued functions on J. Let FIE(1.1) satisfies the following assumptions:

($\mathbf{K_0}$) The function $h : J \to \mathbb{R}$ is continuous.

($\mathbf{K_1}$) The function $f : J\times\mathbb{R} \to \mathbb{R}$ is continuous and there is bounded function $l : J \to \mathbb{R}$ with bound L and a positive constant M such that

$$\mid f(t,a_1) - f(t,a_2) \mid \leq \frac{l(t) \mid a_1 - a_2 \mid}{M + \mid a_1 - a_2 \mid}$$

for $t \in J$ and for all $a_1,a_2 \in \mathbb{R}$. Moreover assume that $L \leq M$.

($\mathbf{K_2}$) The function $t \mapsto f(t,0)$ is bounded on J with

$$F_0 = \sup\{\mid f(t,0) \mid : t \in J\}.$$

($\mathbf{K_3}$) The function $(t,s) \mapsto \frac{1}{\Gamma q}(t-s)^{q-1}$ is continuous and there is a positive real number N such that

$$\left|\frac{1}{\Gamma q}(t-s)^{q-1}\right| \leq N$$

($\mathbf{K_4}$) The function $g : J\times\mathbb{R} \to \mathbb{R}$ is continuous such that there is a continuous map $B : J\times J$ such that

$$\mid g(t,a) \mid \leq B(t)$$

for $t,s \in J$. Moreover, we assume that

$$\lim_{t\to\infty}\int_0^t B(s)ds = 0.$$

Theorem 3.1. *Under the assumptions ($\mathbf{K_0}$)- ($\mathbf{K_4}$) , the FIE(1.1) for $t_0 = 0$ has atleast one solution in the space $C(J,\mathbb{R})$. Moreover solution of FIE(1.1) are globally uniformly attractive on J.*

Let the operator K be defined on the space $C(J,\mathbb{R})$ such that

$$Ka(t) = h(t) + f(t,a(t)) + \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}g(t,a(s))ds \tag{3.1}$$

By assumptions, the function $Ka(t)$ is continuous for any function of $a \in C(J,\mathbb{R})$. For arbitrarily fixed $t \in J$,

$$\begin{aligned}
\left|Ka(t)\right| &= \left|h(t) + f(t,a(t)) + \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}g(t,a(s))ds\right| \\
&\leq \left|h(t)\right| + \left|f(t,a(t)) - f(t,0)\right| + \left|f(t,0)\right| \\
&\quad + \frac{1}{\Gamma q}\int_0^t \left|(t-s)^{q-1}\right|\left|g(t,a(s))\right|ds \\
&\leq ||h|| + \frac{L|a(t)-0|}{M+|a(t)-0|} + \left|f(t,0)\right| + N\int_0^t B(s)ds \\
&\leq ||h|| + \frac{L\left|\left|x\right|\right|}{M+\left|\left|x\right|\right|} + F_0 + Nv(t) \\
&\leq ||h|| + \frac{L\left|\left|x\right|\right|}{M+\left|\left|x\right|\right|} + F_0 + Nv(t) \\
&\leq ||h|| + L + F_0 + Nv(t) \\
\left|\left|K(x)\right|\right| &\leq ||h|| + L + F_0 + NV
\end{aligned} \tag{3.2}$$

for all $x \in C(J,\mathbb{R})$. This means that the operator K transforms the space $C(J,\mathbb{R})$ into itself. From (3.2), we obtain the operator K transforms continuously the space $C(J,\mathbb{R})$ into the closed ball $\overline{B}_r(0)$, where $r = ||h|| + L + F_0 + NV$. Therefore the existence of the solution for FIE(1.1) is global in nature. We will consider the operator $K : \overline{B}_r(0) \to \overline{B}_r(0)$. Now we will show that the operator K is continuous on ball $\overline{B}_r(0)$. Let $\varepsilon > 0$ be arbitrary and take $a,b \in \overline{B}_r(0)$ such that $\left|\left|a-b\right|\right| \leq \varepsilon$, then

$$\begin{aligned}
\left|(Ka)(t) - (Kb)(t)\right| &\leq \left|f(t,a(t)) - f(t,b(t))\right| \\
&\quad \int_0^t \left|\frac{1}{\Gamma q}(t,s)^{q-1}\right|\left|g(t,a(s)) - g(t,b(s))\right|ds \\
&\leq \frac{L|a(t)-b(t)|}{M+|a(t)-b(t)|} + \int_0^t N2B(s)ds \\
&\leq \frac{L||a-b||}{M+||x-y||} + 2Nv(t) \\
&\leq \varepsilon + 2Nv(t)
\end{aligned}$$

from assumption ($\mathbf{K_3}$), there exists $T>0$ such that $v(t)\le\varepsilon$ for $t\ge T$. Thus for $t\ge T$, we have

$$\big|(Ka)(t)-(Kb)(t)\big|\le 3\varepsilon. \tag{3.3}$$

Let us assume that $t\in[0,T]$. Then

$$\begin{aligned}\big|(Ka)(t) &- (Kb)(t)\big|\le \varepsilon+\int_0^t\left|\frac{1}{\Gamma q}(t-s)^{q-1}\right| \\ &\left|g(t,a(s))-g(t,b(s))\right|ds \\ &\le \varepsilon+N\int_0^t\omega_r^T(g,\varepsilon)ds \\ &\le \varepsilon+NT\omega_r^T(g,\varepsilon) \qquad (3.4)\end{aligned}$$

where

$$\begin{aligned}\omega_r^T(g,\varepsilon) &= \sup\{|g(t,a)-g(t,b)|:t\in[0,T], \\ & a,b\in[-r,r],|a-b|\le\varepsilon\}. \qquad (3.5)\end{aligned}$$

By uniform continuity of the functions $g(t,a)$ on the set $[0,T]\times[-r,r]$, we have $\omega_r^T(g,\varepsilon)\to 0$ as $\varepsilon\to 0$. Now, by (3.4), (3.5) and above established facts we conclude that the operator K maps continuously the closed ball $\overline{B}_r(0)$ into itself. Further, let us take a nonempty subset X_1 of the ball $\overline{B}_r(0)$. Next, fix arbitrarily $T>0$ and $\varepsilon>0$. Let us choose $a\in X_1$ and $t_1,t_2\in[0,T]$ with $|t_2-t_1|\le\varepsilon$. Without loss of generality we may assume that $t_1<t_2$. Then

$$\begin{aligned}|(K & a)(t_2)-(Ka)(t_1)|\le|h(t_2)-h(t_1)| \\ & +|f(t_2,a(t_2))-f(t_1,a(t_1))| \\ + & \left|\int_0^{t_2}\frac{1}{\Gamma q}(t_2-s)^{q-1}g(t_2,a(s))ds\right. \qquad (3.6)\\ & \left.+\int_0^{t_1}\frac{1}{\Gamma q}(t_1-s)^{q-1}g(t_1,a(s))\right| \\ \le & \ \omega^T(h,\varepsilon)+|f(t_2,a(t_2))-f(t_2,a(t_1)| \qquad (3.7)\\ & +|f(t_2,a(t_1))-f(t_1,a(t_1))| \\ & +\left|\int_0^{t_2}\frac{1}{\Gamma q}(t_2-s)^{q-1}g(t_2,a(s))ds\right. \qquad (3.8)\\ & -\int_0^{t_2}\frac{1}{\Gamma q}(t_2-s)^{q-1}g(t_1,a(s))ds \\ & +\int_0^{t_2}\frac{1}{\Gamma q}(t_2-s)^{q-1}g(t_1,a(s))ds \qquad (3.9)\\ & \left.-\int_0^{t_1}\frac{1}{\Gamma q}(t_1-s)^{q-1}g(t_1,a(s))ds\right| \\ \le & \ \omega^T(h,\varepsilon)+\frac{L|a(t_2)-a(t_1)|}{M+|x(a_2)-a(t_1)|}+\omega_r^T(f,\varepsilon) \\ + & \int_0^{t_2}\left|\frac{1}{\Gamma q}(t_2-s)^{q-1}\right|\left|g(t_2,a(s))-g(t_1,s,a(s))\right|ds \\ & +\left|\int_0^{t_2}\frac{1}{\Gamma q}(t_2-s)^{q-1}g(t_1,a(s))ds\right. \qquad (3.10)\\ & -\int_0^{t_2}\frac{1}{\Gamma q}(t_1-s)^{q-1}g(t_1,a(s))ds \\ & +\int_0^{t_2}\frac{1}{\Gamma q}(t_1-s)^{q-1}g(t_1,a(s))ds \qquad (3.11)\\ & \left.-\int_0^{t_1}\frac{1}{\Gamma q}(t_1-s)^{q-1}g(t_1,a(s))ds\right| \\ \le & \ \omega^T(h,\varepsilon)+\frac{L\omega^T(x,\varepsilon)}{M+\omega^T(a,\varepsilon)}+\omega_r^T(f,\varepsilon) \qquad (3.12)\\ & +N\int_0^T\omega_r^T(g,\varepsilon)ds \\ & +\int_0^{t_2}\left|\frac{1}{\Gamma q}(t_2-s)^{q-1}-\frac{1}{\Gamma q}(t_1-s)^{q-1}\right|\left|g(t_1,a(s))\right|ds \\ + & \int_{t_1}^{t_2}\left|\frac{1}{\Gamma q}(t_1-s)^{q-1}\right|\left|g(t_1,a(s))\right|ds \\ \le & \ \omega^T(h,\varepsilon)+\frac{L\omega^T(a,\varepsilon)}{M+\omega^T(a,\varepsilon)}+\omega_r^T(f,\varepsilon) \qquad (3.13)\\ & +N\int_0^T\omega_r^T(g,\varepsilon)ds \\ + & \int_0^T\omega^T\Big(\frac{1}{\Gamma q}(t-s)^{q-1},\varepsilon\Big)Vds+N\int_{t_1}^{t_2}G_T^r ds \\ \omega^T(KX_1,\varepsilon) \ \le & \ \omega^T(h,\varepsilon)+\frac{L\omega^T(a,\varepsilon)}{M+\omega^T(a,\varepsilon)}+\omega_r^T(f,\varepsilon) \qquad (3.14)\\ & +N\int_0^T\omega_r^T(g,\varepsilon)ds \\ + & \int_0^T\omega^T\Big(\frac{1}{\Gamma q}(t-s)^{q-1},\varepsilon\Big)Vds+N\varepsilon G_T^r\end{aligned}$$

where

$$\omega^T(h,\varepsilon) = \sup\{|h(t_2)-h(t_1)| : t_1,t_2 \in [0,T], |t_2-t_1| \le \varepsilon\},$$

$$\omega_r^T(f,\varepsilon) = \sup\{|f(t_2,a)-f(t_1,a)| : t_1,t_2 \in [0,T], |t_2-t_1| \le \varepsilon, x,y \in [-r,r]\},$$

$$\omega_r^T\left(\frac{1}{\Gamma q}(t-s)^{q-1},\varepsilon\right) = \sup\left\{\left|\frac{1}{\Gamma q}(t_2-s)^{q-1} - \frac{1}{\Gamma q}(t_1-s)^{q-1}\right| : t_1,t_2,s \in [0,T], |t_2-t_1| \le \varepsilon\right\},$$

$$\omega_r^T(g,\varepsilon) = \sup\{|g(t_2,a)-g(t_1,b)| : t_1,t_2,s \in [0,T], |t_2-t_1| \le \varepsilon, a,b \in [-r,r]\},$$

$$G_r^T = \sup\{|g(t,a)| : t,s \in [0,T], x \in [-r,r]\}.$$

Thus from the above estimate, we have

$$\omega^T(KX_1,\varepsilon) \le \omega^T(h,\varepsilon) + \frac{L\omega^T(X,\varepsilon)}{M+\omega^T(X,\varepsilon)} + \omega_r^T(f,\varepsilon) + N\int_0^T \omega_r^T(g,\varepsilon)ds + \int_0^T \omega^T\left(\frac{1}{\Gamma q}(t-s)^{q-1},\varepsilon\right)V ds + N\varepsilon G_T^r \quad (3.15)$$

By the uniform continuity of the functions f and g on the sets $[0,T]\times[-r,r]$, $[0,T]\times[-r,r]$, respectively, we have $\omega^T(h,\varepsilon) \to 0$, $\omega^T(f,\varepsilon) \to 0$, $\omega^T(\frac{1}{\Gamma q}(t-s)^{q-1},\varepsilon) \to 0$ and $\omega^T(g,\varepsilon) \to 0$. It is obvious that G_T^r is finite. Thus,

$$\omega_0^T \le \frac{L\omega_0^T(X)}{M+\omega_0^T(X)} \quad (3.16)$$

Let $t \in J$ be arbitrarily fixed. Then

$$\begin{aligned}
|(Ka)(t)-(Kb)(t)| &\le |f(t,a(t))-f(t,b(t))| \quad (3.17)\\
&\quad + \int_0^t \left|\frac{1}{q}(t-s)^{q-1}\right| \left|g(t,a(s))-g(t,b(s))\right| ds\\
&\le \frac{L|a(t)-b(t)|}{M+|a(t)-b(t)|} + \left|\frac{1}{q}(t-s)^{q-1}\right| \left|g(t,a(s))-g(t,b(s))\right| ds\\
&\le \frac{L|a(t)-b(t)|}{M+|a(t)-b(t)|} + 2v(t)N\\
diam(KX)(t) &\le \frac{L diamX(t)}{M+diamX(t)} + 2v(t)N\\
\limsup_{t\to\infty} diam(KX)(t) &\le \frac{L\limsup_{t\to\infty} diamX(t)}{M+\limsup_{t\to\infty} diamX(t)}
\end{aligned} \quad (3.18)$$

Using measure of noncompactness μ_a,

$$\begin{aligned}
\mu_a(KX_1) &= \max\{\omega_0(KX_1), \limsup_{t\to\infty} KX_1(t)\}\\
&\le \max\left\{\frac{L\omega_0(X_1)}{M+\omega_0(X_1)}, \frac{L\limsup_{t\to\infty} X_1(t)}{M+\limsup_{t\to\infty} X_1(t)}\right\}\\
&\le \frac{L\max\{\omega_0(X_1), \limsup_{t\to\infty} X_1(t)\}}{M+\max\{\omega_0(X_1), \limsup_{t\to\infty} X_1(t)\}}\\
&\le \frac{L\mu_a(X_1)}{M+\mu_a(X_1)}
\end{aligned} \quad (3.19)$$

since $L \le M$,

$$\mu_a(KX_1) = \phi(\mu_a(X_1)),$$

where $\frac{Lr}{M+r}$ for $r>0$. Now we apply Theorem 2.7 to deduce that operator K has a fixed point a in the ball $\overline{B}_r(0)$. Thus x is solution of the FIE (1.1). The image of the space $C(J,\mathbb{R})$ under the operator K is contained in the ball $\overline{B}_r(0)$ because the set $Fix(K)$ of all fixed points of K is contained $\overline{B}_r(0)$. The set $Fix(K)$ contain all solutions of the FIE (1.1) and from Remark 2.8 we conclude that the set $Fix(K)$ belongs to the family $ker\mu_a$. Now, taking account the description of sets belonging to $ker\mu_a$, we have that all solutions for the FIE (1.1) are globally uniformly attractive on J. In order to prove next result concerning the asymptotic positivity of the attractive solutions, we need following hypotheses.

($\mathbf{K_6}$) The functions f satisfies

$$\lim_{t\to\infty}[|f(t,a)| - f(t,a)] = 0$$

for all $a \in \mathbb{R}$.

Theorem 3.2. *If the FIE (3.1) satisfies the hypotheses of Theorem 3.1 and ($\mathbf{K_6}$). Then the FIE (1.1) has atleast one solution on J and solutions of the FIE(1.1) are uniformly globally attractive and ultimately positive on J.*

Proof. Let $\overline{B}_r(0)$ be a closed ball in the Banach space $C(J,\mathbb{R})$, where the real number r is given as in the proof of Theorem 3.1 and define a map $K : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ by (1.1). In proof of Theorem 3.1, we have shown that K is a continuous mapping from the space $C(J,\mathbb{R})$ from the space $\overline{B}_r(0)$. In particular, K maps $\overline{B}_r(0)$ into itself. Now we will prove that K is a nonlinear-set-contraction with respect to measure μ_{ad} of noncompactness in $C(J,\mathbb{R})$. For any $a,b \in \mathbb{R}$, we have

$$|a|+|b| \ge |a+b| \ge a+b,$$

therefore

$$||a+b|-(a+b)| \le ||a|+|b|-(a+b)| \le ||a|-a| + ||b|-b|$$

for all $a,b \in \mathbb{R}$. For any $a \in \overline{B}_r(0)$, we have

$$\begin{aligned}
\Big||Ka(t)| - Ka(t)\Big| &\le \big||f(t,a(t))| - f(t,a(t))\big|\\
&\quad + \left|\left|\int_0^t \frac{1}{\Gamma}(t-s)^{q-1} g(t,a(s))ds\right| - \int_0^t \frac{1}{\Gamma} g(t,a(s))ds\right|\\
&\le \delta_T(f) + 2Nv(t)\\
&\le \delta_T(f) + 2NV_T,
\end{aligned}$$

where $V_T = \sup_{t \geq T} v(t)$. Thus we have

$$\delta_T(X_1) \leq \delta_T(f) + 2NV_T$$

for all closed $X \subset \overline{B}_r(0)$. On taking the limit superior as $T \to \infty$, we have

$$\limsup_{T\to\infty} \delta_T(X_1) \leq \limsup_{T\to\infty} \delta_T(f) + 2\limsup_{T\to\infty} NV_T = 0 \quad (3.20)$$

for all closed $X_1 \subset \overline{B}_r(0)$. Hence,

$$\delta(KX_1) = \lim_{T\to\infty} \delta_T(X_1) = 0$$

for all closed $X_1 \subset \overline{B}_r(0)$. By the measure of noncompactness μ_a, we have

$$\begin{aligned} \mu_{ad}(KX_1) &= \max\{\mu_{ad}(KX_1), \delta(KX_1)\} \\ &\leq \max\{\frac{L\mu_a(X)}{M+\mu_a(X_1)}, 0\} \\ &= \frac{L\mu_a(X_1)}{M+\mu_a(X_1)} \\ &\leq \frac{L\mu_{ad}(X_1)}{M+\mu_{ad}(X_1)} \end{aligned} \quad (3.21)$$

Since $L \leq M$, therefore we have

$$\mu_{ad}(KX_1) \leq \phi(\mu_{ad}(X_1)),$$

where $\phi(r) = \frac{Lr}{M+r}$ for $r > 0$. By Theorem (2.7), the operator K has a fixed point a in the ball $\overline{B}_r(0)$ and x is a solution of FIE (1.1). The image of the space $C(J, \mathbb{R})$ is contained in $\overline{B}_r(0)$ under the operator K because the set $Fix(K)$ of all fixed points of K is contained $\overline{B}_r(0)$. The set $Fix(K)$ contain all solutions of the FIE (1.1) and from Remark 2.8 we conclude that the set $Fix(K)$ belongs to the family $ker\mu_{ad}$. Now, taking account the description of sets belonging to $ker\mu_{ad}$, we have that all solutions for the FIE (1.1) are globally uniformly attractive and ultimately positive on J. □

4. Conclusion

The uniformly global attractivity and utimately posivity are the main qualitative behaviour of solution of the nonlinear integral equations and we have shown existence and the above qualitative behaviour of solution of hybrid fractional integral equation with linear perturbation of second kind with the help of measure of noncompactness in our recent paper.

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